Integrable Evolution Equations: a Diophantine Approach

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Integrable Evolution Equations: a Diophantine Approach

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To clearly state a problem
and to actually solve it
are of equal importance.

Acknowledgements

A good four years ago I applied for a PhD position in pure mathematics. In my letter of application I wrote that I would like to stay at the boundary of physics and mathematics. It turned out to be a good thing that J.A. Sanders, who had a specific project in mind, gave me a phone call. This was exactly what I needed; being graduated in physics I did not quite know what mathematics was about. I remember that Jan Sanders told me mathematics is that which is invariant under the notation. Meanwhile he convinced me of the significance of 'good' notation. He lended me his books and pointed out to me the important problems. Jan, thank you for letting me jump onto your running train.

At the very last moment I asked J. Hulshof to be my promotor. Joost, thank you for your support. This thesis is based on the symbolic calculus of Gel'fand and Dikiĭ. J.P. Wang was the first to use this calculus systematically in the classification of integrable equations. Jing Ping, thank you for being my room-mate and for answering all sorts of questions. I would have been less proud of this thesis if it had been written without the fruitful ideas of F. Beukers. Frits, thank you for all the hours you spend pointing out how to prove Propositions 7.15, 7.17, 7.19, 7.21, E.3, E.6 and Theorems E.1, E.4. I had a good time with A.V. Mikhailov when I was in Cambridge attending the workshop 'What is integrability?' he organised. Sasha, thank you for reflecting on the object of my research. To know that some people do read your scientific work is very stimulating. In particular I would like to thank Thanasis Fokas and Jaap Top for their interest in the result of my research.

After all, especially with Conjecture 8.4 in mind, I can say that instead of staying at the boundary of physics and mathematics, with this thesis I crossed the border going from physics to mathematics.

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Papa en mama, ik zong het al eens: jullie gaven mij de kracht om door te gaan. Renske, ik hou van je.

Contents

Chapter 1

Introduction

In this introduction we describe the rise of the field of integrable equations. The intention is not to give a survey of the whole field. Instead we point out the developments that influenced this thesis in one way or another. Furthermore, we list what we consider to be the highlights in this thesis and give an overview of the various chapters.

1.1 Historical motivation

1.1.1 The soliton

Stena Line's 'HSS Discovery', which sails in three hours and forty minutes from Hoek van Holland to Harwich, was forced to slow down its speed [vK99]. This ultrafast ferry initiated a freak of nature that killed an innocent fisherman in the summer of 1999. Shortly after the start of the ferry service 'Big Waves', as they were called by the press, attacked the shore of Felixstowe. These waves were about four meters high and could rise from a smooth sea.

A smaller variant of such a Big Wave was already observed in August 1834 by the Scottish engineer J.S. Russell [Rus44]. He was touched by the beauty of the phenomenon, which he called the 'Wave of Translation'. His extensive wave-tank experiments established remarkable properties as stability and locality. At the time of publication these observations appeared to contradict the nonlinear shallow water wave theory of G.B. Airy [Air45]. The controversy arose because in the theory dispersion was neglected and this generally tends to prevent wave steepening. The problem was resolved by J. Boussinesq [Bou71] and, independently, by Lord Rayleigh [Ray76]. In 1895 D.J. Korteweg and G. de Vries derived a model equation, incorporating the effects of surface tension, which describes the unidirectional propagation of long waves in water of relatively shallow depth [KdV95]. This equation first appeared in [Bou] and is written:

$$u_t = u_3 + uu_1 \quad \text{(KDV)},$$

where u_i is the i-th x-derivative of $u(x,t)$. It is probably the most celebrated evolution equation. Korteweg and De Vries showed that periodic solutions, which they called 'cnoidal waves', could be found in closed form and without further approximations. Moreover, in the limit of infinite wavelength or spatial period, they found a localised solution representing Russell's Wave of Translation.

For a long time the solitary wave was considered a rather unimportant curiosity in the mathematical structure of nonlinear wave theory. The appearance of the computer changed the situation. It was in 1955 that E. Fermi, J.R. Pasta, and S.M. Ulam undertook a numerical study of the monoatomic anharmonic chain model [FPU55]. They expected the nonlinear interactions to result in energy equipartition or thermal equilibrium. However, much to their surprise, the system returned almost periodically to its originally excited state and a few nearby modes. Fortunately, this curious result was not ignored. In 1965, M.D. Kruskal and N.J. Zabusky approached the FPU problem from the continuum point of view [KZ65]. Quite amazingly they rederived the KDV equation and found its stable pulse-like wave with computer simulations. They named the reborn wave 'soliton' because it survives interaction, a feature that already had been observed by Russell.

It is interesting to note that around this time the KDV equation emerged everywhere, in fluid dynamical applications, in plasma physics, in a study of dispersive waves in elastic rods and, more generally, in wide classes of nonlinear Galilean-invariant systems where dispersion is dominant and the long wave length approximation is used [GGKM74]. There were also other equations that attracted attention for allowing solitary solutions. We mention A. Seeger, H. Donth, and A. Kochendörfer who, in a study of dislocations in solids [SDK53], obtained analytic expressions describing collision events between solitary wave solutions of what is now called the sine-Gordon equation $u_{tt} - u_{xx} = \sin(u)$ or

$$u_{xt} = \sin(u).$$

Furthermore, J.K. Perring and T.H.R. Skyrme were interested in the kink solutions of the sine-Gordon equation as a simple model of elementary particles [PS62]. Their computer experiments and analytic solutions showed that these solitary waves preserved their kink shape and velocity after having collided.

1.1.2 Conservation laws and inverse scattering

It was a most intriguing challenge to analytically describe and understand the strange behaviour of solitons. C.S. Gardner, J.M. Green, M.D. Kruskal and R.M. Miura (GGKM) presented a method for solving the KDV equation, by which any finite number of solitons can be expressed in closed form. They started with the Sturm-Liouville equation for ψ with eigenvalue λ:

$$\partial_x^2\psi + \frac{1}{6}(u-\lambda)\psi = 0 \quad \text{(SL)}.$$

In quantum mechanics, this is the time-independent one-dimensional Schrödinger equation for potential scattering, u being the potential and ψ the wave function. The standard problem of scattering theory is to solve SL for ψ, with appropriate boundary conditions and a given potential u. In the usual application, all that can be observed is the asymptotic behaviour of ψ for large $|x|$. This, taken for the whole spectrum, comprises the scattering data for u. The inverse scattering problem is to determine u from knowledge of its scattering data.

Fortunately, the inverse problem had been dealt with by many people [KM56, Lev55, Lev53, Mar55]. It was shown that $u = -2\partial_x K(x,x)$, where K satisfies the Gel'fand-Levitan-Marchenko integral equation

$$K(x,y) + B(x+y) + \int_x^\infty B(y+z)K(x,z)dz = 0 \quad \text{(GLM)}.$$

The kernel B depends on the reflection and transmission coefficients, which in turn depend on u. This seemingly vicious circle was broken in [GGKM74], where it was proven that the discrete eigenvalues of SL are time independence when the potential u evolves according to KDV. In this way the time evolution of the scattering data could be obtained and the problem of finding exact solutions to KDV was reduced to first solving the eigenvalue problem SL for the initial data $u(x,0)$ and secondly to solving the linear integral equation GLM for K. By choosing appropriate initial data, i.e., $u(x,0)$ should have zero reflection coefficient, the N-soliton solution can be obtained.

A clear interpretation of the results by GGKM has been given by P. Lax [Lax68]. He introduced the so called L-A pair, which played an important role in extending the applicability of the method. A second influential contribution to the inverse scattering method was the paper by V.E. Zakharov and A.B. Shabat [ZS71], where it was shown that inverse scattering is indeed a method and not a trick suitable for a single solution. Shortly after its introduction the method was used to obtain exact solutions of the nonlinear Schrödinger equation [ZS71]

$$\begin{cases} u_t = & v_2 \pm v(u^2+v^2) \\ v_t = & -u_2 \mp u(u^2+v^2) \end{cases}$$

and to the sine-Gordon equation [Lam71, AKNS73]. An important understanding emphasised in [AKNS74] is that the inverse scattering method can be viewed as an extension into the nonlinear realm of the Fourier transform method. In the Fourier transform method, sinusoids of various wavelengths and phase velocities are employed as basic entities for constructing a solution. In the inverse scattering method these sinusoidal components become elliptic functions.

The inverse scattering method was also applied in the theory of nonlinear lattices. The Toda chain, cf. [Tod70], was integrated independently by H. Flaschka [Fla74] and S.V. Manakov [Man74], see also [Tod81].

Exactly solvable equations are nowadays called 'integrable'. Many integrable equations were to be discovered and the 'theory of integrable equations' strongly stimulated the interaction between various areas of physics and mathematics.

Besides establishing the existence of solitons exhibiting remarkable stability, an explanation was to be given. Such an explanation came in the form of conservation laws. For the KDV equation, the expressions for the conservation of energy and momentum were classically known. G.B. Whitham found a third conserved density, which corresponds to Boussinesq's famous moment of instability [Whi65]. Three more conserved densities were found by Kruskal and Zabusky and four more by Miura. After the discoveries of conserved densities for both the KDV and the Modified KDV equation

$$v_t = v_3 + v^2 v_1 \quad \text{(MKDV)},$$

Miura discovered a transformation which takes any solution of MKDV into a solution of KDV [Miu68]. Such transformations now bear his name. A generalisation of the original Miura transformation,

$$u = v^2 + \sqrt{-6}v_1,$$

is used by C.S. Gardner, R.M. Miura and M.D. Kruskal to prove the existence of infinitely many conservation laws and constants of motion for KDV [MGK68].

The Miura transformation, viewed as a Riccati equation for v, can be linearised by the change of variables $v = \sqrt{-6}\psi_1/\psi$. By taking advantage of the Galilean invariance of KDV, u may be shifted and SL is obtained. This started the development of the inverse scattering method.

The crucial and most surprising result in the reduction of the KDV to a sequence of linear equations is that the discrete eigenvalues of SL are constant when the potential evolves according to KDV. One may now ask the question: Which other equations for the potential u assure the discrete eigenvalues of SL to be constant in time? Such equations necessarily possess the same conserved densities as KDV. An infinite set related to the sequence of conserved densities was discovered by Gardner and written down by P.D. Lax [Lax68]. An alternative recursive construction was given by Lenard [GGKM74]. This construction, now called the Lenard chain, was given a geometric meaning in terms of Hamiltonian and symplectic operators by R.M. Magri who also recognised the significance of the equations as generalised symmetries of KDV [Mag78]. He showed that KDV is in fact a bi-Hamiltonian equation. The KdV equation can be written as

$$u_t = Q(u) = P(u_2 + u^2/2),$$

where u and $u_2+u^2/2$ are gradients of conserved densities and P, Q the Hamiltonian, or cosymplectic , operators given by

$$P = D_x \quad \text{and} \quad Q = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_1.$$

The space of gradients, or cosymmetries, is in some sense dual to the space of symmetries. Infinitely many symmetries and cosymmetries can be constructed by alternating application of the Hamiltonian operator Q and the symplectic operator P^{-1}. Note that $QP^{-1}(u_1) = u_3 + uu_1$. For a good review on bi-Hamiltonian structures, see [FG93].

Around the same time P.J. Olver presented the theory of recursion operators. Such an operator generates the hierarchy of generalised symmetries directly [Olv77]. Of course, for KDV we have the recursion operator $\mathfrak{R} = QP^{-1}$ and the equations are written

$$u_t = \mathfrak{R}^n u_1,\ n \in \mathbb{N}.$$

Generalised symmetries first made their appearance in the fundamental paper of E. Noether [Noe18], in which their role in the construction of conservation laws was clearly enunciated. The difference with the classical Lie symmetries or contact symmetries is their dependence on higher derivatives of u. Due to this dependence they do not have a proper geometrical meaning. Somehow, they were neglected for many years and rediscovered several times since. For example, in differential geometry [Joh64a, Joh64b], in the calculus of variations [Ste62] and in the application to differential equations [AKW72]. A good general reference is [Olv93b].

We would like to draw attention to another historical line that intersected the KDV-line in [GD75]. This article describes results on the asymptotic behaviour of the kernel of the resolvent of the Sturm-Liouville equation (SL) in powers of λ^{-1}. The motivation for this problem came from the desire to give a meaning to traces of positive powers of differential operators and yet, the impetus for its study came from other directions. As we saw, the problem acquired a new significance. The connection between the integrability of KDV and the theory of traces was pointed out by V.E Zakharov and L.D. Faddeev [ZF71]. It became clear that the coefficients in the asymptotic expansion of the kernel of the resolvent can be taken as Hamilton functions, after which a fully integrable Hamiltonian system is obtained. In this respect we also like to mention the work of S.P. Novikov [Nov74], in which periodic solutions for integrable equations appeared. To elucidate the algebraic nature of the Hamiltonian structure, i.e., to clarify the independence of the boundary conditions for SL and of the corresponding spectral methods, I.M. Gel'fand and L.A. Dikiĭ developed a special algebra of polynomials in a function u and its derivatives, which includes elements of a formal calculus of variations and of formal Hamiltonian mechanics in the ring of such polynomials, cf. [Gel71]. This is essentially the complex of variational calculus we also describe in this thesis. It contains all the important objects like symmetries, conservation laws, symplectic operators and so on. Also of importance to the present thesis is the way Gel'fand and Dikiĭ proved some of their theorems: they introduced a symbolic calculus [GD75]. This cleared the path to using techniques like generating functions, which they did, but also to use methods and results from invariant theory, algebraic geometry, p-adic analysis and number theory. However, it took almost 25 years before this was realised. J.P. Wang was the first to use systematically this symbolic calculus in the classification of integrable equations [SW98, Wan98].

1.1.3 The symmetry approach to the classification of integrable equations

Historically the existence of higher conservation laws attracted attention to the equations that appeared to be integrable. The first attempts to recognise and classify integrable equations were based on the requirement of one polynomial conserved density of some fixed order n [Kul76]. In this way P.P. Kulish tried to classify the Klein-Gordon models and therefore he missed the Tzetseika equation

$$u_{xt} = e^u + e^{-2u}.$$

This equation has gaps in the sequence of conservation laws. Hence requesting the existence of a conservation law of certain order n does not guarantee the completeness of the list since the result depends essentially on n. Progress began after replacement of the conservation laws by symmetries of higher order. The notion of a formal symmetry was introduced. This is a formal series in inverse powers of D_x. It was proven for several classes of equations that the existence of a formal symmetry at sufficiently high order implies the existence of infinitely many generalised symmetries. However, this has not been proven generally and there are no realistic estimates of what 'sufficiently high' means.

The first complete list of nonlinear equations, the so-called Klein-Gordon models $u_{xt} = f(u)$, was obtained in [ŽS79]. It consists of the three well known equations

$$u_{xt} = e^u, \; u_{xt} = e^u + e^{-u}, \; u_{xt} = e^u + e^{-2u}.$$

Among the many different approaches to recognition and classification of integrable equations, this so-called *symmetry approach* has proven to be particularly successful [MSS90, SS84, MSY87].

When classifying equations we always have to specify the class of equations that are to be classified and the kind of transformations that are allowed. Initially, the order of the equations to be classified was fixed. For example, in [Svi85], all equations of the form $u_t = f(x, u, u_1, u_2)$ were classified. It was proven that any such equation possessing a formal symmetry of order 5 is equivalent up to certain invertible transformations to one of the equations

$$\begin{aligned} u_t &= u_2 + q(x), \\ u_t &= u_2 + 2uu_1 + p(x), \\ u_t &= D_x(u^{-2}u_1 + \alpha xu + \beta u), \\ u_t &= D_x(u^{-2}u_1 - 2x). \end{aligned}$$

Similarly vector equations of second order are classified in [Svi89]. Given an equation of a form that is classified in this way, one can check whether it is integrable by verifying the integrability conditions. If these are satisfied the equation is equivalent to an equation in the list, and this equation can be found explicitly. However, this is not an easy task if one wants to avoid the use of the conditions.

Another approach, which is also called a symmetry approach, is found in [Fok80]. A.S. Fokas argued as follows: if the equation of the form

$$u_t = u_n + f(u, u_1, \dots, u_{n-1})$$

possesses a recursion operator, then it possesses a generalised symmetry of order $2n - 1$. Subsequently all 2-nd and 3-rd order equations possessing respectively a 3-rd and 5-th order symmetry were found, together with their recursion operators and linearising Bäcklund transformations [Fok80].

However, in the symmetry approaches as sketched above, it seems to be impossible not to fix the order of the equations. Another (complementary) symmetry approach is the following: classify all *polynomial* equations with respect to the existence of symmetries up to homogeneous linear transformations. Using the symbolic calculus of Gel'fand and Dikiĭ and results from diophantine approximation theory this was performed by J.A. Sanders and J.P. Wang for the class of homogeneous scalar equations

$$u_t = u_n + f(u, u_1, \dots, u_{n-1}), \tag{1.1}$$

where u has positive weight [SW98]. Similar results of zero weight or noncommutative equations can be found in [SW00] and [OW00]. For the class of equations (1.1) an exhaustive list of ten known integrable equations was obtained and it was proven that there are no other equations in the possession of symmetries. First of all, this result explains why after the initial gold rush it was so difficult to find any new integrable equations, i.e., equations not contained in the hierarchy of a known equation. Secondly, it explains why once the first integrability conditions are satisfied the equation is integrable, i.e., the existence of one symmetry at certain order implies the existence of infinitely many.

In this respect, an observation was made at least twice in 1980 by different authors. In [Fok80] it is written:

> Another interesting fact regarding the symmetry structure of evolution equations is that in all known cases the existence of one generalised symmetry implies the existence of infinitely many.

In [IS80] the same statement is made together with the footnote:

> This is not true for systems of equations. For example, the system $u_t = u_2 + (v^2/2)$, $v_t = 2v_2$ has a nontrivial algebra symmetry algebra, but this algebra is exhausted by the one-parameter (with parameter τ) algebra of transformations: $u_\tau = u_3 + 3vv_1$, $v_\tau = 4v_3$.

However, although the remark in the footnote is true, the 'counterexample' presented turned out to be an integrable equation [Bak91]. In spite of this fact A.S. Fokas adapted his earlier remark and formulated the following important conjecture in 1987, cf. [Fok87].

> **Conjecture (Fokas).** If a scalar equation possesses at least one time-independent non-Lie point symmetry, then it possesses infinitely many. Similarly for N-component equations one needs N symmetries.

Note that for $N = 1$ the conjecture of Fokas is proven to be true for the class of equations (1.1) by the classification result of Sanders and Wang [SW98].

Four years later I.M. Bakirov presented the first candidate of a non-integrable equation in the possession of a generalised symmetry [Bak91]. This was the 4-th order 2-component equation

$$\begin{cases} u_t = & 5u_4 + v^2 \\ v_t = & v_4 \end{cases}.$$

It possesses a symmetry at order 6 and it was shown (by extensive computer algebra computations) that there are no other symmetries of order $n \leq 53$. In 1998 F. Beukers, J.A. Sanders and J.P. Wang proved, using p-adic analysis, that the equation of Bakirov does not possess another symmetry at any higher order, thereby proving that indeed one symmetry does not imply integrability [BSW98]. The existence of several other equations with finitely many symmetries, all of the form

$$\begin{cases} u_t = & au_n + v^2 \\ v_t = & v_n \end{cases},$$

was mentioned in [BSW98]. Also, it was conjectured, based upon a theorem of Lech and Mahler [Lec53], that this class of equations contains only finitely many integrable equations. In [BSW01] these equations were classified with respect to symmetry-integrability using an algorithm of C.J. Smyth [BS01].

In [TW99] it is stated that the Bakirov equation seems to be exceptional. This turned out not to be the case since there are infinitely many families of non-integrable 2-component equations in the possession of nontrivial generalised symmetries [vdKS02]. Therefore such equations are as common (or as rare) as integrable equations. We propose to call them 'almost integrable'. This terminology somehow reflects the idea of the conjecture of Fokas.

Definition 1.1. *An equation is called (symmetry-)***integrable** *if it possesses infinitely many generalised symmetries and* **almost integrable** *of* **depth** *(at least, at most)* **n** *if there are exactly (at least, at most) n generalised symmetries. When an equation is almost integrable but not integrable we say that it is almost integrable of finite depth.*

This definition is certainly not an answer to the philosophical question 'What is integrability?', but it makes it possible to state clearly what is proven and what is not! In this respect we bring up that for some authors the conjecture has served as a motivation to classify evolution equations with respect to almost integrability. However, these authors did not use the word 'almost' in their statements, cf. [OS98].

At present quite a lot of work has been devoted to the recognition and classification problems of integrable equations. From the 'big results' that were obtained, among which the classification of all integrable scalar equations at any order, one might get the impression that it should be no problem to find the symmetries of a given evolution equation. Surprisingly this is not that easy, as an equation found by Foursov has made clear. In [Fou00] a classification of third order symmetrically coupled KDV–like equations with respect to the existence of two symmetries is presented. One equation in the list appeared to be quite special:

$$\left\{\begin{array}{ll} u_t = & \frac{1}{2}u_3 + \frac{1}{2}v_3 + (2-\alpha)uu_1 + (6-\alpha)vu_1 + \alpha uv_1 + (4-\alpha)vv_1 \\ v_t = & \frac{1}{2}v_3 + \frac{1}{2}u_3 + (2-\alpha)vv_1 + (6-\alpha)uv_1 + \alpha vu_1 + (4-\alpha)uu_1 \end{array}\right. . \tag{1.2}$$

For all values of α odd order symmetries were found. Symmetries at even order were found as well, but only for some particular values of α. Foursov calculated all equations that possess symmetries of weight 2, 4, 6, 8 and 10 with the help of a computer and formulated the following conjecture.

> **Conjecture (Foursov).** The equation (1.2) has symmetries of order $2k$ and weight $2k+2n$ when $\alpha = 2(1-\frac{k}{n})$ for any nonnegative integer k and any positive integer n.

Answering the relatively simple question 'What are all the symmetries of a given evolution equation?' is quite difficult in this case.

1.2 Highlights and overview

The central goal we have in mind is to classify all x- and t-independent homogeneous integrable evolution equations where the dynamical variables have positive weight. We aim for a classification of equations at any order and with any number of components. The latter seems to be possible, at least for equations with diagonal linear part, since no new number theoretical problems need to be solved if the number of components exceeds 4. So far, we have been working on 2-component equations. We started to work around what we have called $\mathcal{B}$-equations, cf. equation 1.3. However, the number theoretical tools that were developed in the context of $\mathcal{B}$-equations apply equally well to other kinds of equations as we will show in Chapter 5 and 7. Here are what we consider to be the highlights in this thesis.

- ⋆ The classification of integrable $\mathcal{B}$-equations (Theorems 6.19, 6.20 and 6.21) and the recognition of integrable $\mathcal{B}$-equations (Theorem 6.15 and Section 6.5).
- ⋆ The method to obtain almost integrable $\mathcal{B}$-equations (Lemma 8.8) and the counterexample to Fokas' conjecture (Theorem 8.10).
- ⋆ The determination of the spectrum of integrable equations and the possible orders of their symmetries (Theorems 7.7, 7.8, 7.10, 7.11 and Propositions 7.15, 7.17, 7.19, 7.21).
- ⋆ The calculation of nonpolynomial symmetries of equation (1.2) and the verification of the statement in the conjecture of Foursov (Chapter 9).

We end our introduction with an the overview of the various chapters.

Chapter 2

We introduce the notion of a generalised symmetry of an evolution equation by deriving the condition for a formal transformation to leave the equation invariant. This transformation corresponds to a vertical vector field which is in the kernel of the Lie derivative with respect to time derivation. More abstractly this vector field is viewed as an *invariant* of the equation. Similarly a conserved density is seen as an invariant in the space of densities which is a representation space of the Lie algebra of vector fields. This Lie, or actually Leibniz , algebraic structure is used to introduce the notions of scaling and homogeneity.

Chapter 3

We consider equations that can be expressed in a formal power series and show the existence of gradings on the algebra. In a graded algebra the condition for an object to be an invariant of the equation is equivalent to a (possibly infinite) set of smaller conditions. By the implicit function theorem of Sanders and Wang, the first l conditions of this set provide a sufficient condition once one symmetry is found. To apply the theorem one has to check that the equation is nonlinear injective and relatively l-prime with this symmetry. We also show that the property of being nonlinear injective together with the nonexistence of terms of certain grading in the equation implies the vanishing of the terms of the same grading in any invariant.

Chapter 4

We describe the symbolic calculus of Gel'fand and Dikiĭ. Together with the implicit function theorem this calculus provides a very powerful tool in classifying evolution equations with respect to the existence of symmetries. It is based on a one-to-one correspondence between differential polynomials in the dependent variables and symmetrised polynomials in a number of symbols. We show how to prove nonlinear injectivity and relative l-primeness. In the symbolic calculus the first nontrivial invariant-conditions become divisibility conditions of certain polynomials. Much of the analysis is concerned with finding common divisors of certain polynomials which are called $\mathcal{G}$-functions. We introduce biunit coordinates to describe points in the complex plane as follows: suppose that $r \in \mathbb{C} \setminus \mathbb{R}$ is of the form $a\psi$ and also of the form $b\phi - 1$ with $|\psi| = |\phi| = 1$ and $a, b \in \mathbb{R}$. Then

$$r = \mathfrak{P}(\psi, \phi) = \psi^2 \frac{(\phi+1)(\phi-1)}{(\psi+\phi)(\psi-\phi)}.$$

Biunit coordinates are used in chapter 6, 7 and 8.

Chapter 5

We describe the symmetry-classification of equations of the form

$$u_t = u_n + f(u, u_1, \ldots, u_{n-1})$$

using the implicit function theorem and the symbolic calculus. Geometric arguments, using Bézout's theorem, show that if an equation without quadratic terms possesses a symmetry, it does have cubic terms and it is in a hierarchy of third order. For equations with quadratic terms we will treat the matter a little differently from how it was originally done by distinguishing the classification of integrable equations and the classification of almost integrable equations. This separates the difficult part from the easy part. The easy part is to obtain all integrable equations. The use of the Lech-Mahler theorem is crucial (and new) here. The difficult part is to show that there are no equations with finitely many symmetries. This was done using on modern techniques from diophantine approximation theory.

Chapter 6

We classify all integrable $\mathcal{B}$-equations, i.e., equations of the form

$$\begin{cases} u_t = & a_1 u_n + K(v, v_1, \ldots) \\ v_t = & a_2 v_n \end{cases} \tag{1.3}$$

where K is quadratic in derivatives of v. We first show that all equations of order $n < 4$ are integrable, which was known. We give a new method, based on resultants, to determine whether a given $\mathcal{B}$-equation of order m is in a hierarchy of order $1, 2$ or 3. The main result, the classification of the integrable $\mathcal{B}$-equations of order $n > 4$, is based on the use of biunit coordinates and on the Lech-Mahler theorem. Let Φ_n be the set of all n-th roots of unity not equal to ± 1. To any point

- ⋆ $r \in \Phi_{n-1}$,
- ⋆ $r \in \Phi_{2n}$ such that $r^n = -1$,
- ⋆ $r \in \mathfrak{P}(\Phi_{2n}, \Phi_{2n})$ such that $|r| \neq 1$.

corresponds an integrable n-th order $\mathcal{B}$-equation which is not in a hierarchy of order $m < 4$. There are exactly

$$\begin{aligned} n(n-2)/4 &\quad \text{if} \quad n \text{ even}, \\ (n+1)(n-3)/4 &\quad \text{if} \quad n \text{ odd}, \\ 4 &\quad \text{if} \quad n = 5 \end{aligned}$$

such equations. It is proven that together with the symmetries of equations of order $1, 2$ and 3 these are all integrable $\mathcal{B}$-equations. Furthermore we prove that all integrable nondegenerate $\mathcal{B}$-equations, i.e., equations with nonzero eigenvalues, are real (up to a complex scaling). We also describe all integrable n-th order $\mathcal{B}$-equations that are not in a hierarchy of order $m < n$.

Chapter 7

The classification of integrable $\mathcal{B}$-equations has immediate implications for any equation containing a term that is quadratic in v. In other words, we have obtained a condition on the spectrum of such equations and on the order of their symmetries. Another condition on the spectrum is obtained by requiring the existence of a term in the first component of the equation that is linear in both u and v. It turns out that all other conditions that can be obtained from the existence of a certain quadratic term follow from the two cases mentioned above. More stringent conditions can be obtained by requiring several different types of quadratic terms to be nonzero. All possible combinations are treated and in several cases we obtain a finite number of eigenvalues. Also we classify the cubic version of the class of $\mathcal{B}$-equations.

Chapter 8

This chapter is devoted to almost integrable $\mathcal{B}$-equations of finite depth, of which the Bakirov equation was the first and simplest. We give a short introduction to p-adic numbers and treat the method of Skolem which allows us to conclude that only a finite number of symmetries exist for a given equation. We introduce a method by which all $\mathcal{B}$-equations of order n with symmetries of order m can be obtained. We have performed extensive computer calculations and obtained all $\mathcal{B}$-equations at order $3 < n < 11$ with symmetries at order $n < m < n + 151$. By adding some refinements to the method of Skolem we were able to prove that all the nonintegrable $\mathcal{B}$-equations obtained in this way are almost integrable of depth 1 with the exception of three seventh order equations which possess generalised symmetries of order 11 and 29. These exceptions provide counterexamples to the conjecture of Fokas. We conjecture that the only integer $N > 2$ such that:

- $\star$ there exist $r, s \in \mathbb{C}$ for which the diophantine equation

$$(1 + r^m)(1 + s)^m = (1 + s^m)(1 + r)^m$$

  has exactly N solutions $m > 1$,

is $N = 3$. Moreover, if $N = 3$ the solutions $m > 1$ are given by

$$m = 7, 11, 29.$$

Chapter 9

We prove that the following generalisation of the KDV equation

$$\begin{cases} u_t = & u_3 + 3uu_1 \\ v_t = & \alpha u_1 v + uv_1 \end{cases}$$

has infinitely many polynomial symmetries of even weight only if α is a negative and rational number. Moreover we prove that, allowing multiplication with v^c where $c \in \mathbb{C}$, this equation possesses several mutually noncommuting hierarchies of non-polynomial symmetries for any value of $\alpha \in \mathbb{C}$. Since the equation is related to the equation of Foursov by a linear transformation, we have proven his conjecture to be true. However, the symmetry structure of his equation is bigger than that.

Chapter 10

We describe the complex of variational calculus. This consists of spaces of n-forms together with a coboundary operator between successive spaces. A general rule is given and used to construct the Lie derivative on all spaces. The coboundary operator is defined in terms of the Lie derivative and commutes with it. In this formalism we describe conserved densities, symmetries, cosymmetries, symplectic operators, cosymplectic operators and recursion operators, and how they are related. We present an interesting example in which symplectic operators are obtained from cosymmetries by the action of the coboundary operator.

Appendices

Here we have included: some words on imposing homogeneity; the proof of the implicit function theorem of Sanders and Wang; the definition of the resultant; some consequences of the Lech-Mahler theorem; and results on certain diophantine equations obtained in co-operation with F. Beukers.

Chapter 2

Evolution equations, symmetries and conservation laws

Many processes in nature are described by evolution equations. The characteristic feature of an evolution equation is that the state of the process can be calculated, in principle, if the state is given at an earlier moment in time. In all known cases where this can be done exactly, the equation has infinitely many symmetries. In some cases there are infinitely many conservation laws as well.

2.1 Evolution equations

We consider evolution equations in one spatial and one temporal variable. Let $\mathcal{M}$ be a two-dimensional space with coordinates x, t and let $\mathcal{C}$ be the space of smooth functions of x and t. To each point of $\mathcal{M}$ we attach a vector u with N components $u^\alpha \in \mathcal{C}$. These functions u^α are differentiated by applying the operators ∂_x and ∂_t. We write u_i^α for the i-th x-derivative of $u^\alpha(x,t)$ and consider the u_i^α as independent variables. They are called **dynamical variables**.

Notation 2.1. *The ring of smooth functions of x, t and a finite number of dynamical variables is denoted $\mathcal{A}$. We write $\mathcal{H}$ for the N-dimensional $\mathcal{A}$-module with basis*

$$\partial_{u^1}, \partial_{u^2}, \ldots, \partial_{u^N}.$$

A differential operator acts on a vector by acting on all its components, i.e., we have $(u_t)^\alpha = u_t^\alpha$. Here u_t is the t-derivative of the vector $u \in \mathcal{H}$ and u_t^α is the t-derivative of the function $u^\alpha \in \mathcal{C}$.

Definition 2.2. *A partial differential equation on $\mathcal{M}$ is an N-***component evolution equation** *if it can be written as a system of equations:*

$$u_t^\alpha = K^\alpha, \alpha = 1, \ldots N.$$

Shortly we write

$$u_t = K, \tag{2.1}$$

where $u, K \in \mathcal{H}$. *The* **order** *of this equation, or of* K, *is the highest number* $n \in \mathbb{N}$ *such that for some* α *we have* $\partial_{u_n^\alpha} K \neq 0$.

To be able to differentiate elements of $\mathcal{A}$ with respect to x, the differential operator $\partial_x : \mathcal{C} \to \mathcal{C}$ is prolonged to $D_x : \mathcal{A} \to \mathcal{A}$.

Definition 2.3. *For* $f \in \mathcal{A}$ *we define the* **total** **(**x**-)differentiation** *operator* D_x *by:*

$$D_x(f) = \partial_x f + \sum_{\alpha=1}^{N} \sum_{i=0}^{\infty} u_{i+1}^\alpha \partial_{u_i^\alpha} f.$$

Multiple differentiation is defined by $D_x^n(f) = D_x^{n-1}(D_x(f))$.

Observe that, although we take the summation from 0 to infinity, the sum is finite since $f \in \mathcal{A}$. The dynamical variables satisfy the rule

$$u_{i+1}^\alpha = D_x(u_i^\alpha).$$

Thus the n-th total x-derivative of the vector u is $D_x^n(u) = u_n$. When $N = 2$ we adapt the convention to use u and v instead of u^1 and u^2. Then the components satisfy $D_x^n(u) = u_n$ and $D_x^n(v) = v_n$.

Example 2.4 (KDV)**.** *The Korteweg-De Vries equation*

$$u_t = u_3 + uu_1$$

is a 1-component (or scalar) evolution equation of order 3.

Example 2.5 (Boussinesq)**.** *Sometimes it is not immediately clear that an equation is an evolution equation. The Boussinesq equation, cf. [Olv93a, Example 7.28],*

$$u_{tt} = u_4 + 4(uu_2 + u_1^2),$$

arose in a model for unidirectional propagation of long waves in shallow water. The equation can be written as $u_{tt} = D_x^2(u_2 + 2u^2)$. *By introducing a new function* v *such that* $D_x^2(v) = u_t$ *and integrating twice, the equation is converted into the 2-component evolution equation*

$$\begin{aligned} u_t &= v_2 \\ v_t &= u_2 + 2u^2 \end{aligned} .$$

We will use the KDV and the Boussinesq equations frequently in our examples.

2.2 Symmetries of evolution equations

The use of symmetries in the study of differential equations was initiated by S. Lie. The concept of Lie point symmetries is connected with one parameter groups of transformations on the space of dependent and independent variables that leave the solution set invariant. Such a transformation is given by a set of new variables $\tilde{x}, \tilde{t}, \tilde{u}^1, \ldots \tilde{u}^N$ which depend on the old variables and an additional parameter ϵ. Its (infinitesimal) generator is

$$T\partial_t + X\partial_x + \sum_{\alpha=1}^{N} U^\alpha \partial_{u^\alpha},$$

where (T, X, U^α) are the coefficients of the first order expansion around $\epsilon = 0$. This generator is a Lie point or classical symmetry of the equation if the transformation maps any solution of the equation to a invariant solution. This solution then contains the parameter ϵ. To be able to calculate the action of the generator on derivatives of the dependent variables, it has to be prolonged just like we did with ∂_x. The general formula can be found in [Olv93a, Theorem 2.36]. Since Lie point symmetries are not our main subject we give a small example only, cf [Olv93a, Example 2.44].

Example 2.6 (KDV)**.** *The vector field $g = t\partial_x + \partial_u$ generates the Galilean boost:*

$$u(x,t) \to e^{\epsilon g} u(x,t) = u(x + \epsilon t, t) + \epsilon, \;\; \epsilon \in \mathbb{R}.$$

which does not change the form of $u_t = u_3 + uu_1$. Therefore it transforms any solution of the KDV equation into a new solution.

The concept of symmetry was generalised by E. Noether. She allowed the symmetry to depend on derivatives of the dependent variables, cf. [Noe18]. In what follows we will work out the details of infinitesimal transformations of u that may depend on derivatives as well. After this we will turn to vector fields that generate this kind of transformations.

2.2.1 Formal transformations

Abstract. We show which formal transformations of the dependent variables leave an evolution equation invariant.

On the evolution equation (2.1) one can apply a formal transformation

$$\tilde{u} = u + \epsilon S, \quad S \in \mathcal{H}. \tag{2.2}$$

Unless the order of S is zero such a transformation will lead to a higher order equation. This prevents one from doing normal form theory, as one does for ordinary differential equations. However, an interesting problem is to find all S such that the equation is invariant up to first order in ϵ. Let us carry out the computation.

$$\begin{aligned}
\tilde{u}_t^\alpha &= u_t^\alpha + \epsilon\left(\partial_t S^\alpha + \sum_{\beta=1}^{N}\sum_{k=0}^{\infty}\partial_{u_k^\beta} S^\alpha D_x^k(u_t^\beta)\right)\\
&= K^\alpha + \epsilon\left(\partial_t S^\alpha + \sum_{\beta=1}^{N}\sum_{k=0}^{\infty}\partial_{u_k^\beta} S^\alpha D_x^k(K^\beta)\right)\\
&= \tilde{K}^\alpha + \epsilon\left(\partial_t S^\alpha + \sum_{\beta=1}^{N}\sum_{k=0}^{\infty}(D_x^k(K^\beta)\partial_{u_k^\beta} S^\alpha - D_x^k(S^\beta)\partial_{u_k^\beta} K^\alpha)\right) + O(\epsilon^2).
\end{aligned} \tag{2.3}$$

We have used the equation, substituted $u_t = K$, since we require the equation to be invariant on the solution set of the equation. Compare the following definition with [Olv93a, Definition 5.24, equation (5.32)].

Definition 2.7. *We define*

$$D_f[S] = \sum_{\alpha=1}^{N}\sum_{k=0}^{\infty}\partial_{u_k^\alpha} f D_x^k(S^\alpha),$$

to be the **Fréchet derivative** *of a function $f \in \mathcal{A}$ in the direction $S \in \mathcal{H}$.*

Observe that in the definition of $D_f[S]$ a differential operator acts on f. Therefore we know how to take the Fréchet derivative of a vector $K \in \mathcal{H}$, i.e.,

$$(D_K[S])^\alpha = D_{K^\alpha}[S]. \tag{2.4}$$

Notice that we have

$$D_x[S] \neq D_x(S) = D_{u_1}[S].$$

Using Fréchet derivatives the term of order ϵ in equation (2.3) vanishes if

$$\partial_t S + D_S[K] - D_K[S] = 0. \tag{2.5}$$

We emphasise that there is an asymmetry between the vectors $K, S \in \mathcal{H}$ in this equation. It should be realised that these vectors stand for different objects.

2.2.2 Vector fields

Abstract. We express the invariance condition in terms of the natural Lie bracket on the space of vector fields.

Vector fields act on each other as differential operators. The advantage of this is that there is a natural Lie bracket, defined by the commutator of two operators.

Definition 2.8. *Let $\mathcal{X}$ be the space of vector fields on $\mathcal{A}$. The general form of $v \in \mathcal{X}$ is*

$$v = T\partial_t + X\partial_x + \sum_{\alpha=1}^{N}\sum_{l=1}^{\infty} V_l^\alpha \partial_{u_l^\alpha},\ T, X, V_l^\alpha \in \mathcal{A}. \tag{2.6}$$

For all $v, w \in \mathcal{X}$ we define the product

$$[,] : \mathcal{X}^2 \to \mathcal{X},\ [v,w] = vw - wv,$$

called the **Lie bracket** *on $\mathcal{X}$.*

We notice that since all multiple derivations in vw and wv cancel each other, the commutator is indeed a vector field on $\mathcal{A}$. The bracket makes $\mathcal{X}$ a **Lie algebra**, i.e for all $v, w, y \in \mathcal{X}$ and $a, b \in \mathbb{C}$ the following axioms hold

1. $[av + bw, y] = a[v, y] + b[w, y]$ (linearity).
2. $[v, w] = -[w, v]$ (antisymmetry).
3. $[y, [v, w]] = [[y, v], w] + [v, [y, w]]$ (Jacobi identity).

Notice that the first two properties imply

$$[v, aw + by] = a[v, w] + b[v, y].$$

The Lie bracket is bilinear because a differential operator acts linearly. Antisymmetry follows directly from the definition of the bracket. The Jacobi identity follows from the associativity of differential operators.

A special element in $\mathcal{X}$ is the vector field D_x, whose action on functions is the act of total differentiation, cf. Definition 2.3.

Notation 2.9. *We denote the kernel of D_x in $\mathcal{A}$ by $Ker(D_x)$. With $\mathfrak{g}$ we denote the subspace of $\mathcal{X}$ that leaves D_x invariant, i.e., $v \in \mathcal{X}$ such that*

$$[v, D_x] = -\gamma(v)D_x,\ \gamma : \mathfrak{g} \to Ker(D_x).$$

The linear functional γ is a Lie algebra homomorphism, i.e., we have

$$\gamma([v, w]) = [\gamma(v), \gamma(w)]_{\mathrm{Ker}(D_x)} = 0,$$

where $[,]_{\mathrm{Ker}(D_x)}$ is the trivial Lie bracket on $\mathrm{Ker}(D_x)$. It follows that $\mathfrak{g}$ is a Lie subalgebra of $\mathcal{X}$.

Remark 2.10. *The function γ gets a nice interpretation in Section 2.5. If $\gamma(v)$ is nonzero, it is the weight of x, cf. Example 2.28.*

Let us see what the elements of $\mathfrak{g}$ look like. The commutator of the general element $v \in \mathcal{X}$, see Definition 2.6, with D_x is

$$\begin{aligned}[v, D_x] &= vD_x - D_x v \\ &= \sum_{\alpha=1}^{N}\sum_{k=0}^{\infty}(V_{k+1}^{\alpha}\partial_{u_k^{\alpha}}) - D_x(T)\partial_t - D_x(X)\partial_x - \sum_{\alpha=1}^{N}\sum_{k=0}^{\infty} D_x(V_k^{\alpha})\partial_{u_k^{\alpha}}.\end{aligned}$$

By linearity $[v, D_x] = Y D_x,\ Y \neq 0$ implies $[v/Y, D_x] = D_x$. Therefore we study the two cases $\gamma(v) = 0, 1$.

- ⋆ $\gamma(v) = 0$; from $[v, D_x] = 0$ we obtain the homogeneous system

  $$D_x T = D_x X = V_{k+1}^{\alpha} - D_x(V_k^{\alpha}) = 0,$$

  which implies that $T, X \in \mathrm{Ker}(D_x)$ and $V_k^{\alpha} = D_x^k V^{\alpha}$.

- ⋆ $\gamma(v) = 1$; from $[v, D_x] = -D_x$ we obtain the nonhomogeneous system

  $$D_x T = 0,\ D_x X = 1,\ V_{k+1}^{\alpha} - D_x(V_k^{\alpha}) = -u_{k+1}^{\alpha}.$$

  A particular solution is $T = 0, X = x, V_k^{\alpha} = x u_{k+1}^{\alpha}$. This gives us the vector field $xD_x \in \mathfrak{g}$. Indeed we have

  $$xD_x^2 - D_x x D_x = -D_x.$$

Thus the general form of $v \in \mathfrak{g}$ is

$$T\partial_t + X\partial_x + YxD_x + \sum_{\alpha=1}^{N}\sum_{k=1}^{\infty} D_x^k(V^{\alpha})\partial_{u_k^{\alpha}},\ T, X, Y \in \mathrm{Ker}(D_x),\ V \in \mathcal{H}.$$

We will define two special subspaces of $\mathfrak{g}$.

Definition 2.11. *With $\mathfrak{f}$ we denote the space of* **scalings** *spanned by the $N+2$ vector fields*

$$\begin{aligned}\sigma_t &= t\partial_t, \\ \sigma_x &= x\partial_x - \sum_{\alpha=1}^{N}\sum_{k=0}^{\infty} k u_k^{\alpha}\partial_{u_k^{\alpha}}, \\ \sigma_{u^{\alpha}} &= \sum_{k=0}^{\infty} u_k^{\alpha}\partial_{u_k^{\alpha}},\ \alpha = 1, \ldots, N.\end{aligned}$$

The action of these scalings on a single term is given by multiplication with respectively: the degree in t; the degree in x minus the total number of x-derivatives; and the number of variables u^{α}.

Scalings play an important role in section 2.5. The space $\mathfrak{f}$ is contained in $\mathfrak{g}$. This follows from Definition 2.11 and the equality

$$\sigma_x = xD_x - \sum_{\alpha=1}^{N}\sum_{k=1}^{\infty} D_x^k(xu_1^\alpha)\partial_{u_k^\alpha} \in \mathfrak{g}. \tag{2.7}$$

In fact $\mathfrak{f}$ is an abelian Lie subalgebra of $\mathfrak{g}$. This is most easily seen by observing that for all three operators the degree in t, the degree in x, the number of variables u^α and the number of x-derivatives vanish.

Definition 2.12. *A* **vertical vector field** *has the form*

$$w = \delta(W) = \sum_{\alpha=1}^{N}\sum_{k=1}^{\infty} D_x^k(W^\alpha)\partial_{u_k^\alpha}, \quad W \in \mathcal{H}.$$

Its **characteristic** *is $W = \delta^{-1}(w)$. The subspace of $\mathfrak{g}$ of all vertical vector fields is denoted $\mathfrak{h}$.*

Compare the form of our vector fields in Definitions 2.8 and 2.12 with the one in [Olv93a, Definitions 5.1, 5.4 and equations (2.22),(5.6)]. In the language of P.J. Olver the generalised vector field W is an evolutionary vector field and its prolongation $\delta(W)$ takes a particularly simple form. What he calls the characteristic (of W) is its corresponding element in $\mathcal{A}^N$. This has the disadvantage that the weight of an object differs from the weight of its characteristic, cf. Remark 2.29.

All $\delta(v) \in \mathfrak{h}$ commute with D_x. Therefore we have

$$\delta(v)(D_x^k(f)) = D_x^k(\delta(v)(f)), f \in \mathcal{A}.$$

This makes $\mathfrak{h}$ a Lie subalgebra since for $\delta(V), \delta(W) \in \mathfrak{h}$ we have

$$\begin{aligned}[\delta(V), \delta(W)] &= \sum_{\alpha=1}^{N}\sum_{k=1}^{\infty} D_x^k(\delta(V)(W^\alpha) - \delta(w)(V^\alpha))\partial_{u_k^\alpha} \\ &= \delta(\delta(V)(W) - \delta(W)(V)) \ \in \mathfrak{h}.\end{aligned}$$

Moreover, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, i.e., $[v, w] \in \mathfrak{h}$ if $v \in \mathfrak{g}$ and $w \in \mathfrak{h}$. This follows from the fact that any $v \in \mathfrak{g} \setminus \mathfrak{h}$ can be written as

$$v = T\partial_t + X\partial_x + YxD_x, \ T, X, Y \in \mathrm{Ker}(D_x)$$

and the fact that

$$[T\partial_t + X\partial_x + xD_x, \delta(W)] = \delta(T\partial_t W + X\partial_x W) \in \mathfrak{h}.$$

Note that $\delta(S)$ is the generator of the formal transformation (2.2) and that

$$\delta(S)(K) = D_K[S].$$

The invariance of $u_t = K$ under the formal transformation should hold only for solutions. Therefore $\tilde{u}$ is differentiated along the equation. This is done by the operator $D_t \in \mathfrak{g}$, which is, in analogy with $D_x = \partial_x + \delta(u_1)$, given by

Definition 2.13. *The* **total** t**-derivative** *is*

$$D_t = \partial_t + \delta(u_t) = \partial_t + \delta(K).$$

We are now able to express the invariance condition in terms of vector fields. Compare the following definition with [Olv93a, Definition 1.64].

Definition 2.14. *The* **Lie derivative** *of $w \in \mathfrak{g}$ with respect to $v \in \mathfrak{g}$ is defined by*

$$\mathcal{L}(v)w = [v, w],$$

and $w \in \mathfrak{h}$ is a **generalised symmetry***, or symmetry for short, of $u_t = K$ if*

$$\mathcal{L}(D_t)w = 0.$$

Notation 2.15. *In the sequel we adapt the short notations*

$$\begin{aligned} \mathcal{L}(v)K &= \delta^{-1}(\mathcal{L}(v)\delta(K)), \; v \in \mathfrak{g}, K \in \mathcal{H}, \\ \mathcal{L}(K)S &= \delta^{-1}(\mathcal{L}(\delta(K))\delta(S)), \; K, S \in \mathcal{H}. \end{aligned}$$

Also $S \in \mathcal{H}$ is called a symmetry if the characteristic condition $\mathcal{L}(D_t)S = 0$ holds.

The characteristic condition is written in terms of the Fréchet derivative as:

$$\mathcal{L}(D_t)S = \partial_t S + D_S[K] - D_K[S] = 0, \tag{2.8}$$

which coincides with equation (2.5). To give the reader some impression of the kind of computations involved, we present the check for some symmetries of KDV. Although this may not be particularly enlightening, it may help in appreciating the symbolic method which will be introduced in chapter 4.

Example 2.16 (KDV)**.** *The symmetries of the KDV equation of order* $1, 3, 5$ *are*

$$\begin{aligned} S_1 &= u_1, \\ S_3 &= u_3 + uu_1, \\ S_5 &= u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1. \end{aligned}$$

The Fréchet derivative of $K = u_3 + uu_1$ is

$$D_K = D_x^3 + uD_x + u_1.$$

The Fréchet derivatives of the symmetries are

$$\begin{aligned} D_{S_1} &= D_x, \\ D_{S_2} &= D_x^3 + uD_x + u_1, \\ D_{S_3} &= D_x^5 + \frac{5}{3}uD_x^3 + \frac{5}{3}u_3 + \frac{10}{3}u_1D_x^2 + \frac{10}{3}u_2D_x + \frac{5}{6}u^2D_x + \frac{5}{3}uu_1. \end{aligned}$$

Since these symmetries are time-independent we have $\mathcal{L}(D_t)S = \mathcal{L}(K)S$. Straightforward calculation gives:

$$\begin{aligned}
\mathcal{L}(K)S_1 &= D_{S_1}[K] - D_K[S_1] \\
&= (u_4 + u_1^2 + uu_2) - (u_4 + uu_2 + u_1^2) \\
&= 0, \\
\mathcal{L}(K)S_3 &= D_{S_3}[K] - D_K[S_3] \\
&= 0, \\
\mathcal{L}(K)S_5 &= D_{S_5}[K] - D_K[S_5] \\
&= u_8 + uu_6 + 6u_1u_5 + 15u_2u_4 + 10u_3^2 + \frac{5}{3}u(u_6 + uu_4 + 4u_1u_3 \\
&\quad +3u_2^2) + \frac{5}{3}u_3(u_3 + uu_1) + \frac{10}{3}u_1(u_5 + uu_3 + 3u_1u_2) + \frac{10}{3}u_2(u_4 \\
&\quad +uu_2 + u_1^2) + \frac{5}{6}u^2(u_4 + uu_2 + u_1^2) + \frac{5}{3}uu_1(u_3 + uu_1) - u_8 \\
&\quad -\frac{5}{3}(uu_6 + 3u_1u_5 + 3u_2u_3 + u_3^2) - \frac{10}{3}(u_1u_5 + 4u_2u_4 + 3u_3^2) \\
&\quad -\frac{5}{6}(u^2u_4 + 12uu_1u_3 + 24u_1^2u_2 + 6uu_2^2) - u_1(u_5 + \frac{5}{3}uu_3 \\
&\quad +\frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1) - u(u_6 + \frac{5}{3}(uu_4 + u_1u_3) + \frac{10}{3}(u_1u_3 + u_2^2) \\
&\quad +\frac{5}{6}(u^2u_2 + 2uu_1^2)) \\
&= 0.
\end{aligned}$$

For any x-independent $K \in \mathcal{H}$ we have $D_K[u_1] = D_{u_1}[K] = D_x(K)$. Hence in this case u_1 is a symmetry of the evolution equation $u_t = K$. Obviously any time-independent K is a symmetry of $u_t = K$. These symmetries u_1, K are called **trivial symmetries**.

2.3 Modules, representations and invariants

Abstract. We introduce the more abstract concepts Leibniz algebra module, representation and invariant. The Lie derivative is a representation of the Leibniz algebra $\mathfrak{g}$ on the $\mathfrak{g}$-module $\mathfrak{g}$. Also $\mathcal{A}$ is a $\mathfrak{g}$-module. A symmetry is an invariant of an evolution equation.

Notation 2.17. *The space of all linear transformations from V to W (vector space homomorphisms) is denoted $Hom(V, W)$. The space of vector space endomorphisms $Hom(V, V)$ is denoted $End(V)$.*

Definition 2.18. *A* **Leibniz algebra** *is a pair (U, P) in which U is a vector space and $P : U \to End(U)$ a linear map, satisfying*

$$P(P(x)y) = P(x)P(y) - P(y)P(x).$$

Note that the relation for P in Definition 2.18 coincides with the Jacobi identity if we take $P(x)y = [x, y]$. A Leibniz algebra differs from a Lie algebra by the axiom of antisymmetry.

Example 2.19. *An example of a Leibniz algebra is* (U, P)*, where* U *is a two-dimensional vector space with basis* $\{x, y\}$ *and*

$$P(x)x = P(x)y = 0, \;\; P(y)y = ax, \;\; P(y)x = bx$$

with $a, b \in \mathbb{C}$*. This is a trivial Lie algebra if* $a = b = 0$*.*

We do not need the antisymmetry property of the Lie derivative $\mathcal{L}$ for our considerations, so we can equally well consider $(\mathfrak{g}, \mathcal{L})$ as a Leibniz algebra. By doing so the construction of the complex of variational calculus is simplified, cf. Chapter 10. Instead of $(\mathfrak{g}, \mathcal{L})$ we just write $\mathfrak{g}$.

Definition 2.20. *Let* (U, P) *be a Leibniz algebra. A* (U, P)*-module is a pair* (V, Q) *in which* V *is a vector space and* $Q : U \to End(V)$ *is a linear map, satisfying*

$$Q(P(x)y) = Q(x)Q(y) - Q(y)Q(x).$$

We call Q *a* **representation** *of* (U, P) *on* V *if* (V, Q) *is a* (U, P)*-module.*

Because $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ the Lie algebra $\mathfrak{h}$ is a $\mathfrak{g}$-module. Also $\mathcal{A}$ is a $\mathfrak{g}$-module.

Definition 2.21. *The* **Lie derivative** $\mathcal{L} : \mathfrak{g} \to End(\mathcal{A})$ *is given by the natural action of vector fields on functions, i.e.,*

$$\mathcal{L}(v)f = v(f).$$

We see that for all $v, w \in \mathfrak{g}$ and $f \in \mathcal{A}$ we have

$$\mathcal{L}(\mathcal{L}(v)w)f = v(w(f)) - w(v(f)) = (\mathcal{L}(v)\mathcal{L}(w) - \mathcal{L}(w)\mathcal{L}(v))f.$$

Therefore $\mathcal{L}$ is a representation of $\mathfrak{g}$ on $\mathcal{A}$. We used the same symbol for the product on $\mathfrak{g}$ and the representation on the $\mathfrak{g}$-module $\mathcal{A}$. No confusion needs to arise; which operator is meant can be seen from its argument.

Definition 2.22. *Let* (V, Q) *be a* (U, P)*-module. We call* $y \in V$ *an* **invariant** *of* $x \in U$ *if* $Q(x)y = 0$*.*

According to this definition a symmetry is an invariant of D_t. In the present context we also call the symmetry an invariant of the evolution equation.

2.4 Conservation laws

Abstract. We introduce the space of densities which is a $\mathfrak{g}$-module. Its invariants are called conserved densities. These correspond to conservation laws.

Given an evolution equation, one is interested in functions that do not change in time along the flow of the equation, so-called conservation laws. These are given as one-forms on the underlying x, t space $\mathcal{M}$

$$\omega = \rho dx + \phi dt, \quad \rho, \phi \in \mathcal{A}.$$

Definition 2.23. *The one-form ω is a* **conservation law** *of the evolution equation (2.1) if its divergence vanishes, i.e., if*

$$d\omega = (D_t(\rho) - D_x(\phi))dt \wedge dx = 0.$$

When $\rho dx + \phi dt$ is a conservation law we call ρ a **conserved density** *and ϕ a* **conserved flux**.

When $\omega = d\alpha$, i.e., $\rho = D_x(\alpha)$ and $\phi = D_t(\alpha)$ this is automatically true, and such conservation laws are called trivial.

Consider now a time-independent domain X in x-space such that $\int_{\partial X} \phi dx = 0$, i.e., there is no flow through the boundary of X. Then, if ω is a conservation law, we have

$$D_t \int_X \rho dx = \int_X D_t(\rho) dx = \int_X D_x(\phi) dx = \int_{\partial X} \phi dx = 0,$$

Thus, we see that the quantity $\int_X \rho dx$ is a constant of motion.

We took $\rho \in \mathcal{A}$, but adding an expression of the form $D_x(\alpha),\ \alpha \in \mathcal{A}$ to ρ does not change the value of the integral $\int_X \rho dx$.

Notation 2.24. *We define Ω^0 to be $\mathcal{A}/Im(D_x)$ and denote the* **equivalence class** *of ρ by $\int \rho$. We call ρ a* **representative** *of $\int \rho$.*

The advantage of working with Ω^0 is that if $\int \rho \in \Omega^0$ is a nonzero conserved density it is nontrivial.

Example 2.25 (KDV)**.** *We give the three lowest order conservation laws of the KDV equation.*

- ⋆ *Conservation of momentum.*

  Take $\rho^{(1)} = u$. Then $\rho^{(1)} \neq 0$ in Ω^0 and

$$\begin{aligned} D_t \rho^{(1)} &= u_t \\ &= u_3 + u u_1 \\ &= D_x(u_2 + \frac{1}{2}u^2) \\ &= D_x \phi^{(1)}. \end{aligned}$$

  Therefore $\omega^{(1)} = \rho^{(1)} dx + \phi^{(1)} dt$ is a (nontrivial) conservation law.

⋆ *Conservation of energy.*
Take $\rho^{(2)} = \frac{1}{2}u^2$*. Then, with* $\rho^{(2)} \neq 0$ *in* Ω^0*,*

$$\begin{aligned} D_t\rho^{(2)} &= uu_t \\ &= uu_3 + u^2u_1 \\ &= D_x(uu_2 - \frac{1}{2}u_1^2 + \frac{1}{3}u^3) \\ &= D_x\phi^{(2)}. \end{aligned}$$

Therefore $\omega^{(2)} = \rho^{(2)}dx + \phi^{(2)}dt$ *is a (nontrivial) conservation law.*

⋆ *Conservation of moment of instability*
Take $\rho^{(3)} = \frac{1}{6}u^3 - \frac{1}{2}u_1^2$*. Then, with* $\rho^{(3)} \neq 0$ *in* Ω^0*,*

$$\begin{aligned} D_t\rho^{(3)} &= \frac{1}{2}u^2u_t - u_1D_xu_t \\ &= \frac{1}{2}(u^2u_3 + u^3u_1) - u_1u_4 - uu_1u_2 - u_1^3 \\ &= D_x(\frac{1}{8}u^4 + \frac{1}{2}u_2u^2 - u_1^2u - u_3u_1 + \frac{1}{2}u_2^2) \\ &= D_x\phi^{(3)}. \end{aligned}$$

Therefore $\omega^{(3)} = \rho^{(3)}dx + \phi^{(3)}dt$ *is a (nontrivial) conservation law.*

We will consider Ω^0 as a $\mathfrak{g}$-module by defining a representation.

Definition 2.26.

$$\mathcal{L}(v)\int\rho := \int\mathcal{L}(v)\rho = \int v(\rho)$$

is the **Lie derivative** *of* $\int\rho$ *with respect to* $v \in \mathfrak{g}$*.*

Since $\mathfrak{g}$ consists of elements that scale D_x, the expression does not depend on the choice of the representative in Ω^0, i.e., if $[v, D_x] = cD_x$ we have

$$\mathcal{L}(v)\int D_x(\rho) = \int v(D_x(\rho)) = \int(D_x(v(\rho)) + cD_x(\rho)) = 0.$$

The Lie derivative is a representation of $\mathfrak{g}$ on Ω^0 since for all $v, w \in \mathfrak{g}$ and $\int\rho \in \Omega^0$ we have

$$\mathcal{L}(\mathcal{L}(v)w)\int\rho = \int(v(w(\rho)) - w(v(\rho))) = (\mathcal{L}(v)\mathcal{L}(w) - \mathcal{L}(w)\mathcal{L}(v))\int\rho.$$

The Lie derivative of $\int\rho \in \Omega^0$ with respect to D_t is written in terms of the Fréchet derivative as:

$$\mathcal{L}(D_t)\int\rho = \int(\partial_t\rho + D_\rho[K]). \tag{2.9}$$

We see that a conserved density is an element in Ω^0 which is in the kernel of $\mathcal{L}(D_t)$. In other words, it is an invariant of the evolution equation.

2.5 Scalings and homogeneity

Abstract. We introduce weights and use a special scaling to define the notion of homogeneity. The distinction between $\mathcal{A}^N$ and $\mathcal{H}$ is made clear.

To each variable $z \in \{x, t, u^\alpha\}$ we assign a **weight** $\lambda(z) \in \mathbb{R}$. Furthermore, we define a special scaling $\sigma \in \mathfrak{f}$:

$$\sigma = \lambda(t)\sigma_t + \lambda(x)\sigma_x + \sum_{\alpha=1}^{N} \lambda(u^\alpha)\sigma_{u^\alpha},$$

Definition 2.27. *An expression h in a $\mathfrak{g}$-module is* **homogeneous** *of* **weight** $\lambda(h)$ *if*

$$\mathcal{L}(\sigma)h = \lambda(h)h$$

with $\lambda(h) \in \mathbb{R}$.

We have the following in mind: the weight of a product p is the sum of the weights of its factors and the weight of a fraction is the weight of the numerator minus the weight of the denominator. In this way, an expression e is homogeneous if all terms have the weight $\lambda(e)$.

Example 2.28 (D_x)**.** *Consider the total differentiation operator*

$$D_x = \partial_x + \sum_{\alpha=1}^{N} \sum_{i=0}^{\infty} u^\alpha_{i+1} \partial_{u^\alpha_i}.$$

We have $\lambda(u^\alpha_i) = \lambda(u^\alpha) - i\lambda(x)$. Therefore each term has weight $-\lambda(x)$ and D_x is homogeneous of weight $-\lambda(x)$. Indeed we have $\mathcal{L}(\sigma)D_x = -\lambda(x)D_x$, cf. remark 2.10.

Remark 2.29. *Here the difference between $\mathcal{H}$ and $\mathcal{A}^N$ becomes apparent. The element*

$$K = \sum_{\alpha=1}^{N} K^\alpha \partial_{u^\alpha} \in \mathcal{H}$$

is homogeneous of weight w if

$$\lambda(K^\alpha) - \lambda(u^\alpha) = w, \ 1 \leq \alpha \leq N,$$

while

$$K = \sum_{\alpha=1}^{N} K^\alpha \mathbf{e}^\alpha \in \mathcal{A}^N$$

is homogeneous of weight w if

$$\lambda(K^\alpha) = w, \ 1 \leq \alpha \leq N.$$

Observe that with $K \in \mathcal{H}$ we have $\lambda(K) = \lambda(\delta(K))$.

Example 2.30 (KDV). *The differential operator associated to the KDV equation is*

$$D_t = \partial_t + \delta(u_3 + uu_1).$$

Take $\lambda(u) = 2, \lambda(t) = -3$ *and* $\lambda(x) = -1$. *The vector field* $D_t \in \mathfrak{g}$ *is homogeneous with weight* 3 *since*

$$\mathcal{L}(\sigma)D_t = 3D_t.$$

However, the element $u_3 + uu_1 \in \mathcal{A}$ *is homogeneous with weight* 5 *since*

$$\mathcal{L}(\sigma)K = 5K.$$

We say that the weight of the KDV equation is 3.

Lemma 2.31. *Take* $f \in \mathfrak{f}, g \in \mathfrak{g}$ *and* h *in a* $\mathfrak{g}$*-module. Suppose that*

$$\mathcal{L}(f)g = ag, \ \mathcal{L}(f)h = bh, \quad a, b \in \mathbb{R}.$$

Then

$$\mathcal{L}(f)\mathcal{L}(g)h = (a+b)\mathcal{L}(g)h.$$

Proof. Using that $\mathcal{L}$ is a representation we obtain

$$\mathcal{L}(f)\mathcal{L}(g)h = \mathcal{L}(\mathcal{L}(f)g)h + \mathcal{L}(g)\mathcal{L}(f)h = (a+b)\mathcal{L}(g)h.$$

□

Thus we have, for example, that the Lie derivative of a homogeneous element with respect to a homogeneous vector field is homogeneous.

In Appendix A we will show that the problem of finding homogeneous equations with homogeneous invariants is part of the problem of finding nonhomogeneous equations with nonhomogeneous invariants. Also, we show it suffices to find all homogeneous invariants of a homogeneous equation.

However, homogeneity need not be imposed from the start; much of the analysis can be done without! In the classification of $\mathcal{B}$-equations, cf. Chapter 6, and the determination of the spectrum of eigenvalues, cf. Chapter 7, homogeneity was not imposed at all. Having determined the linear part of the integrable equations and of its first nontrivial symmetry equations, one may want to work with homogeneous equations when writing down candidate equations possessing candidate symmetries, cf. Chapter 5.

Chapter 3

Polynomial evolution equations

We restrict ourselves to polynomial evolution equations and invariants as follows: first we change the basic space $\mathcal{A}$ to formal power series. Then, we assume that the equation is homogeneous and that the weights of the variables u^α are positive. In another case we assume that the linear part is homogeneous and that the nonlinear part is polynomial.

Notation 3.1. *With $\mathcal{A}$ we denote the space of formal power series in finitely many u_n^α with coefficients in $\mathcal{C}$ where $f \in \mathcal{C}$ implies $f \notin \mathcal{A}$, i.e., for all $f \in \mathcal{A}$ we have $f(0,0,\ldots) = 0$. The space $\mathfrak{g}$ is the space of vector fields on $\mathcal{A}$ that leave D_x invariant. Compare this with notations 2.1 and 2.9. Similarly we change the meaning of the symbols we use to denote the spaces of vertical vector fields and densities.*

3.1 An implicit function theorem

Abstract. We introduce the notion of a graded module. This is used to divide the condition for the existence of an invariant into a number of smaller conditions. An implicit function theorem states that, under certain assumptions, once the first few conditions hold the others do as well.

Notation 3.2. *To denote the* **direct sum** *of an infinite set of modulus the symbol $\prod$ is used, cf. [Eis95].*

Definition 3.3. *A Leibniz algebra (U, P) is a* $\mathbb{N}$**-graded Leibniz algebra** *if it can be written as*

$$U = \prod_{i \geq 0} U^{(i)},$$

where $x \in U^{(i)}$ has grading $i \in \mathbb{N}$ and

$$P(U^{(i)})U^{(j)} \subset U^{(i+j)}.$$

The mapping $Q : U \to End(V)$ is an $\mathbb{N}$-**graded representation** *if*

$$V = \prod_{i \geq 0} V^{(i)},$$

and

$$Q(U^{(i)})V^{(j)} \subset V^{(i+j)}.$$

Then (V, Q) is called an $\mathbb{N}$-**graded** (U, P)**-module**.

Lemma 3.4. *Let (V, Q) be an $\mathbb{N}$-graded (U, P)-module. Write*

$$x = \sum_{i=0}^{\infty} x^i \quad \text{and} \quad y = \sum_{i=0}^{\infty} y^i,$$

with $x^i \in U^{(i)}$ and $y^i \in V^{(i)}$. Then the equation

$$Q(x)y = 0$$

is equivalent to the set of equations

$$\sum_{j=0}^{i} Q(x^j)y^{i-j} = 0, \tag{3.1}$$

with $i = 0, 1, 2, \ldots$.

Proof. Since (V, Q) is an $\mathbb{N}$-graded (U, P)-module, $Q(x)y \in V$ can be written as

$$Q(x)y = \sum_{i=0}^{\infty} z^i.$$

The element z^i of grading i has the form

$$z^i = \sum_j Q(x^j)y^{i-j}.$$

Since both j and $i - j$ are nonnegative, j runs from 0 to i. □

The following lemma states that certain of the y^i are zero when certain x^j are. It is based on the notion of 'nonlinear injectivity', to be introduced in Definition 3.5.

Definition 3.5. *We call x^0* **nonlinear injective** *if*

$$Q(x^0)y = 0$$

implies that y has grading 0.

Lemma 3.6. *Let (V, Q) be an $\mathbb{N}$-graded (U, P)-module. Suppose that $Q(x)y = 0$. Write*

$$x = x^0 + \sum_{i=j}^{\infty} x^i \quad \textit{and} \quad y = \sum_{i=0}^{\infty} y^i,$$

with $x^i \in U^{(i)}$ and $y^i \in V^{(i)}$. If x^0 is nonlinear injective then $y^i = 0$ for $0 < i < j$. Furthermore we have

$$Q(x^0)y^j + Q(x^j)y^0 = 0.$$

Proof. By Lemma 3.4 we have to solve the set of equations (3.1). Since $x^i = 0$ for $0 < i < j$, the equations (3.1) with $0 < i < j$ reduce to $Q(x^0)y^i = 0$. By nonlinear injectiveness $y^i = 0$. The last statement follows from taking $i = j$ in equation (3.1). □

The following implicit function theorem states that under certain conditions only the first few of equations (3.1) have to be solved in order to prove the existence of $y \in V$ such that $Q(x)y = 0$.

We introduce one other notion, 'relatively l-primeness', which may seem a little odd and hard to verify. However, for our application it is perfectly natural in the context of the symbolic calculus; it can be verified by checking relatively primeness between certain polynomials, cf. 4.3.

Definition 3.7. *We call x* **relatively l-prime** *with respect to y if*

$$Q(y)z \in Im(Q(x))$$

implies $z \in Im(Q(x))$ for all z with grading equal to or bigger than l.

Theorem 3.8 (Sanders, Wang)**.** *Let (V, Q) be an $\mathbb{N}$-graded (U, P)-module. Suppose that for $x, z \in U$ and $y^i \in V^{(i)}$*

- ⋆ $P(x)z = 0$,
- ⋆ x^0 *is nonlinear injective,*
- ⋆ x *is relatively $(l + 1)$-prime with respect to z,*
- ⋆ $\sum_{i=0}^{k} Q(x^i)y^{k-i} = 0$ *for $k = 0, 1, \ldots, l$,*
- ⋆ $Q(z^0)y^0 = 0$.

Then a unique $y = \sum_{i=0}^{\infty} y^i \in V$ exists such that

- ⋆ $\mathcal{L}(x)y = 0$,
- ⋆ $\mathcal{L}(z)y = 0$.

The proof of this theorem is quite cumbersome in a graded setting. We have included the proof in the more general setting of filtered modules in Appendix B.

Note that Theorem 3.8 can be used to prove the existence of an invariant in any (U, P)-module, but that the invariant $z \in U$ plays a special role and that in the application we have in mind this role is played by a *symmetry* of the evolution equation.

The total number of variables u_k^α gives us an $\mathbb{N}$-grading on $\mathfrak{g}$. This number is obtained by taking the Lie derivative in the direction of the scaling

$$\sigma_u = \sum_{\alpha=1}^{N} \sigma_{u^\alpha}.$$

Lemma 3.9. *The space $\mathfrak{g}$ is an $\mathbb{N}$-graded Leibniz algebra. The spaces $\mathfrak{h}, \mathcal{A}, \Omega$ are $\mathbb{N}$-graded $\mathfrak{g}$-modules.*

Proof. Since $\mathcal{A}$ consists of formal power series in the u_n^α and $f \in \mathcal{C}$ implies $f \notin \mathcal{A}$, we can write

$$\mathfrak{g} = \prod_{i \geq 0} \mathfrak{g}^{(i)},$$

where $\mathcal{L}(\sigma_u)v = i$ if $v \in \mathfrak{g}^{(i)}$. The spaces $\mathfrak{h}, \mathcal{A}, \Omega$ can be written similarly. Suppose that $v \in \mathfrak{g}^{(i)}$ and q in some $\mathfrak{g}$-module with $\mathcal{L}(\sigma_u)q = jq$. By Lemma 2.31 we have

$$\mathcal{L}(\sigma_u)\mathcal{L}(v)q = (i + j)\mathcal{L}(v)q.$$

□

Consider some homogeneous equation $u_t = K$ and a homogeneous invariant Q. By Lemma 3.9 we can write

$$\begin{aligned} K &= K^0 + K^1 + \cdots + K^n, \\ Q &= Q^0 + Q^1 + \cdots + Q^{m-n}, \end{aligned}$$

where $\mathcal{L}(\sigma_u)K^i = iK^i$ and $\mathcal{L}(\sigma_u)Q^j = jQ^j$. By Lemma 3.4 solving the equation $\mathcal{L}(K)Q = 0$ consists of solving the $m + 1$ equations

$$\sum_{i=0}^{j} \partial_t Q^j + \mathcal{L}(K^i)Q^{j-i} = 0, \tag{3.2}$$

where $j = 0, 1, \ldots, m$ and $Q^j = 0$ for all $j > m - n$. Note that m can be arbitrarily large, think of the invariants of infinite order that do exist if the equation is integrable.

In Chapter 4 we will introduce the symbolic calculus. There the strength of Theorem 3.8 can really be appreciated. Using the symbolic calculus both nonlinear injectivity and relatively $(l+1)$-primeness can be verified. For almost all systems of evolution equations, the integer l turns out to be 1 or 2, cf. Theorem 5.2. Moreover, using the symbolic calculus the first few equations can be solved for infinitely many orders at once. Therefore, Theorem 3.8 can be used to prove integrability.

3.2 Bigraded modules

Abstract. The more gradings the smaller the problems to solve. Here we introduce the notion of a bigraded module. This will be useful when treating 2-component equations.

Definition 3.10. *A Leibniz algebra (U,P) is a* **bigraded Leibniz algebra** *if it can be written*

$$U = \prod_{i,j\geq -1, i+j>0} U^{(i,j)},$$

where $x \in U^{(i,j)}$ has bigrading (i,j) and

$$P(U^{(i,j)})U^{(k,l)} \subset U^{(i+k,j+l)}.$$

In particular we have

$$P(U^{-1,k})U^{(-1,l)} = P(U^{(k,-1)})U^{(l,-1)} = 0.$$

The mapping $Q : U \to End(V)$ is a **bigraded representation** *if*

$$V = \prod_{i,j\geq -1, i+j>0} V^{(i,j)},$$

and

$$Q(U^{(i,j)})V^{(k,l)} \subset V^{(i+k,j+l)}.$$

Then (V,Q) is called a **bigraded** (U,P)**-module** .

Note that the sum $i+j$ of this bigrading (i,j) is a nonnegative integer. Therefore it defines a $\mathbb{N}$-grading.

Definition 3.11. *The sum $i+j$ of a bigrading (i,j) is called the* **total grading**.

The total grading makes it possible to apply the implicit function theorem.

Lemma 3.12. *Let (V,Q) be a bigraded (U,P)-module . Write*

$$x = \sum_{j=0}^{\infty}\sum_{i=-1}^{j+1} x^{i,j-i} \quad \textit{and} \quad y = \sum_{j=0}^{\infty}\sum_{i=-1}^{j+1} y^{i,j-i},$$

with $x^{i,j} \in U^{(i,j)}$ and $y^{i,j} \in V^{(i,j)}$. Then the equation $Q(x)y = 0$ is equivalent to the set of equations

$$\sum_{k=max(-1,i-j-1)}^{min(j+1,i+1)} \; \sum_{l=max(-1,-k)}^{min(j-i+1,j-k)} \mathcal{L}(x^{k,l})y^{i-k,j-l-i} = 0,$$

with $j = 0,1,2,\ldots$ and $i = -1,0,\ldots,j+1$.

Proof. Since (V, Q) is a bigraded (U, P)-module, $Q(x)y \in V$ can be written

$$Q(x)y = \sum_{j=0}^{\infty} \sum_{i=-1}^{j+1} z^{i,j-i}.$$

The element $z^{i,j-i} \in V$ of bigrading $(i, j-i)$ has the form

$$z^{i,j-i} = \sum_{k,l} Q(x^{k,l}) y^{i-k,j-l-i}.$$

We have $k \geq -1$ and $l \geq -1$. Also we have $i-k \geq -1$ and $j-l-1 \geq -1$, implying that $k \leq i+1$ and $l \leq j-i+1$. Since the total grading of $y^{i-k,j-l-i}$ is nonnegative and the total grading of z is j, we have $k+l \leq j$. Together with $l \geq -1$ this implies $k \leq j+1$. Similarly we get $i-k \leq j+1$. Now fix k. The total grading of $x^{k,l}$ satisfies $0 \leq k+l \leq j$. Therefore $l \geq -k$ and $l \leq j-k$. Taking this all together we get

$$\begin{aligned} \max(-1, i-j-1) &\leq k \leq \min(j+1, i+1), \\ \max(-1, -k) &\leq l \leq \min(j-i+1, j-k). \end{aligned}$$

□

3.3 Symmetries of 2-component equations

Abstract. We explicitly write down the conditions for a 2-component equation to possess a symmetry using vector calculus and bigrading.

The 2-component equation

$$\begin{cases} u_t = & K_1 \\ v_t = & K_2 \end{cases}$$

is written in vector notation

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = K.$$

This corresponds to the vector field

$$D_t = \partial_t + \sum_{i=1}^{\infty} \left(D_x^i(K_1)\partial_{u_i} + D_x^i(K_2)\partial_{v_i} \right) \in \mathfrak{g}.$$

The equation has a symmetry $S = (S_1, S_2)$ if

$$\mathcal{L}(D_t)S = 0.$$

In all applications we consider equations and symmetries that are x, t-independent. We refer to [Ser01, Ser98] for related results on x, t-dependent equations.

Assumption 3.13. *We assume that S does not depend on x, t.*

Under this assumption $D_t = \delta(K)$ and S is a symmetry if

$$\mathcal{L}(K)S = D_S[K] - D_K[S] = 0.$$

According to equation (2.4) we have

$$\begin{aligned} 0 = \mathcal{L}(K)S &= \begin{bmatrix} D^u_{S_1} & D^v_{S_1} \\ D^u_{S_2} & D^v_{S_2} \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} - \begin{bmatrix} D^u_{K_1} & D^v_{K_1} \\ D^u_{K_2} & D^v_{K_2} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \\ &= \begin{bmatrix} D^u_{S_1}[K_1] - D^u_{K_1}[S_1] + D^v_{S_1}[K_2] - D^v_{K_1}[S_2] \\ D^u_{S_2}[K_1] - D^u_{K_2}[S_1] + D^v_{S_2}[K_2] - D^v_{K_2}[S_2] \end{bmatrix}, \end{aligned} \tag{3.3}$$

where we used the notation, with $f, g \in \mathcal{A}$,

$$D^u_f[g] = \sum_{k=0}^{\infty} \partial_{u_k} f D^k_x(g) \text{ and } D^v_f[g] = \sum_{k=0}^{\infty} \partial_{v_k} f D^k_x(g).$$

Lemma 3.14. *$\mathfrak{h}$ is a bigraded $\mathfrak{g}$-module.*

Proof. Since $\mathcal{A}$ consists of formal power series in the u_k and v_l, we can write

$$\mathfrak{g} = \prod_{i,j \geq -1, i+j>0} \mathfrak{g}^{(i,j)},$$

where $\mathcal{L}(\sigma_u)K = i,\ \mathcal{L}(\sigma_v)K = j$ if $K \in \mathfrak{g}^{(i,j)}$. The space $\mathfrak{h}$ can be written in the same way. Suppose that $K \in \mathfrak{g}^{(i,j)}$ and $S \in \mathfrak{h}^{(k,l)}$. By Lemma 2.31 we have

$$\mathcal{L}(\sigma_u)\mathcal{L}(K)S = (i+k)\mathcal{L}(K)S \text{ and } \mathcal{L}(\sigma_v)\mathcal{L}(K)S = (j+l)\mathcal{L}(K)S.$$

The spaces $\mathfrak{g}^{(-1,k)}$ (and $\mathfrak{h}^{(-1,k)}$) contain elements K of the form $K = (K_1, 0)$ only. We have for example

$$\sum_{k=0}^{\infty} D^k_x(v^2)\partial_{u_k} \in \mathfrak{g}^{(-1,2)}.$$

When $K \in \mathfrak{g}^{(-1,k)}$ and $S \in \mathfrak{h}^{(-1,l)}$ we have

$$\begin{bmatrix} 0 & D^v_{S_1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & D^v_{K_1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 \\ 0 \end{bmatrix} = 0.$$

Similarly, we see that $\mathcal{L}(\mathfrak{g}^{(l,-1)})\mathfrak{h}^{(k,-1)} = 0$. □

By Lemmas 3.14 and 3.12 the equation $\mathcal{L}(K)S = 0$ is equivalent to the set of equations

$$\sum_{k=\max(-1,i-j-1)}^{\min(j+1,i+1)} \sum_{l=\max(-1,-k)}^{\min(j-i+1,j-k)} \mathcal{L}(K^{k,l})S^{i-k,j-l-i} = 0, \tag{3.4}$$

with $j = 0, 1, 2, \ldots$ and $i = -1, 0, \ldots, j+1$. In most applications we restrict ourselves to equations and symmetries that have a diagonal linear part.

Assumption 3.15. *We assume that* $K^{\pm 1,\mp 1} = S^{\pm 1,\mp 1} = 0$.

When the weights of u and v are equal, this is not a stringent restriction.

Lemma 3.16 (Jordan form)**.** *Suppose that the weight of u equals the weight of v. By a linear transformation any homogeneous system of two evolution equations can be put in one of the forms*

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} a_1 u_n + K_1 \\ a_2 v_n + K_2 \end{bmatrix} \quad or \quad \begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} a_1 u_n + v_n + K_1 \\ a_1 v_n + K_2 \end{bmatrix},$$

where K_i contains nonlinear terms only. In the first case, one of the eigenvalues a_1, a_2 can be scaled to 1 *if it is nonzero. In the second case, the eigenvalue a_1 can be scaled to* 1 *if it is nonzero.*

Proof. The linear part K^0 of a homogeneous system of weight n has the form

$$K^0 = K^{-1,1} + K^{0,0} + K^{1,-1}.$$

In vector notation $u_t = K^0$ becomes

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = A \begin{bmatrix} u_n \\ v_n \end{bmatrix} \text{ with } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

We can apply the following linear homogeneous transformations to the system.

$$\begin{bmatrix} u \\ v \end{bmatrix} \to M \begin{bmatrix} u \\ v \end{bmatrix} \text{ with } M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

After the transformation we write the system in evolutionary form and get

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = M^{-1} A M \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \cdots .$$

The matrix A of our linear equation will be put as diagonal as possible. This is the Jordan form of A. The columns of M consist of 'generalised' eigenvectors of A, cf. [Str80]. In this way two different cases are obtained. The matrix $M^{-1}AM$ is either

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \text{ or } \begin{bmatrix} a_1 & 1 \\ 0 & a_1 \end{bmatrix}.$$

Suppose we are in the first case and $a_i \neq 0,\ i = 1, 2$. The transformation

$$t \to \frac{t}{a_i}$$

sets the i-th eigenvalue to 1. Suppose we are in the second case and $a_1 \neq 0$. The transformations

$$t \to \frac{t}{a_1},\ v \to a_1 v$$

set the eigenvalues to 1. □

Example 3.17 (Boussinesq)**.** *Consider the 2-component evolution equation*

$$\begin{cases} u_t = & v_2 \\ v_t = & u_2 + 2u^2 \end{cases}.$$

A matrix of eigenvectors of

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ is } M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The corresponding linear homogeneous transformation of u, v puts the Boussinesq system in Jordan form:

$$\begin{cases} u_t = & u_2 + (u+v)^2 \\ v_t = & -v_2 - (u+v)^2 \end{cases}.$$

Under Assumption 3.15 the symmetry conditions of total grading 0, 1 and 2 become

$$\mathcal{L}(K^{0,0})S^{0,0} = 0. \tag{3.5}$$

$$\begin{aligned} \mathcal{L}(K^{-1,2})S^{0,0} + \mathcal{L}(K^{0,0})S^{-1,2} &= 0, \\ \mathcal{L}(K^{0,0})S^{0,1} + \mathcal{L}(K^{0,1})S^{0,0} &= 0, \\ \mathcal{L}(K^{0,0})S^{1,0} + \mathcal{L}(K^{1,0})S^{0,0} &= 0, \\ \mathcal{L}(K^{0,0})S^{2,-1} + \mathcal{L}(K^{2,-1})S^{0,0} &= 0. \end{aligned} \tag{3.6}$$

$$\begin{aligned} \mathcal{L}(K^{-1,2})S^{0,1} + \mathcal{L}(K^{-1,3})S^{0,0} + \mathcal{L}(K^{0,0})S^{-1,3} & \\ +\mathcal{L}(K^{0,1})S^{-1,2} &= 0, \\ \mathcal{L}(K^{-1,2})S^{1,0} + \mathcal{L}(K^{0,0})S^{0,2} + \mathcal{L}(K^{0,1})S^{0,1} & \\ +\mathcal{L}(K^{0,2})S^{0,0} + \mathcal{L}(K^{1,0})S^{-1,2} &= 0, \\ \mathcal{L}(K^{-1,2})S^{2,-1} + \mathcal{L}(K^{0,0})S^{1,1} + \mathcal{L}(K^{0,1})S^{1,0} + \mathcal{L}(K^{1,0})S^{0,1} & \\ +\mathcal{L}(K^{1,1})S^{0,0} + \mathcal{L}(K^{2,-1})S^{-1,2} &= 0, \\ \mathcal{L}(K^{0,0})S^{2,0} + \mathcal{L}(K^{0,1})S^{2,-1} + \mathcal{L}(K^{1,0})S^{1,0} & \\ +\mathcal{L}(K^{2,-1})S^{0,1,} + \mathcal{L}(K^{2,0})S^{0,0} &= 0, \\ \mathcal{L}(K^{0,0})S^{3,-1} + \mathcal{L}(K^{1,0})S^{2,-1} + \mathcal{L}(K^{2,-1})S^{1,0} & \\ +\mathcal{L}(K^{3,-1})S^{0,0} &= 0. \end{aligned} \tag{3.7}$$

The higher the grading the bigger the size of the symmetry conditions. Luckily, most equations are relatively 2- or 3-prime with respect to their symmetries. Therefore by the implicit function theorem the above equations of grading 0, 1 and 2 are the only equations that have to be solved at arbitrary order.

If in an equation terms of certain grading vanish, nonlinear injectiveness can be used to conclude the vanishing of the terms with this grading in any invariant.

Lemma 3.18. *Suppose that $K^{0,0}$ is nonlinear injective. Under Assumption 3.13 and 3.15, for any positive integer k, if $K^{i,j} = 0$ for all $0 < i + j < k$, we have the following symmetry conditions*

$$\mathcal{L}(K^{0,0})S^{i,k-i} + \mathcal{L}(K^{i,k-i})S^{0,0} = 0, \ i = -1, 0, \dots k + 1. \tag{3.8}$$

Moreover we have $S^{i,j} = 0$, for all $0 < i + j < k$.

Proof. The equations of total grading k have the form

$$\sum_{m,n} \mathcal{L}(K^{m,n})S^{i-m,k-n-i} = 0, \ i = -1, 0, \dots k + 1. \tag{3.9}$$

Since $K^{m,n} = 0$ for all $0 < m + n < k$ and $K^{\pm 1,\mp 1} = S^{\pm 1,\mp 1} = 0$ the sum contains only two terms, i.e. $m = n = 0$ and $m = i, n = k - i$. A similar argument shows that for all $0 < l < k$ the equations of total grading l are

$$\mathcal{L}(K^{0,0})S^{i,l-i} = 0.$$

By the nonlinear injectivity of $K^{0,0}$ it follows that $S^{i,k-i} = 0$. $\square$

The components of the first term in the left hand side of equation (3.8) are

$$\mathcal{L}(K^{0,0})S^{i,j} = \begin{bmatrix} D^u_{S_1^{i,j}}[K_1^{0,0}] + D^v_{S_1^{i,j}}[K_2^{0,0}] - D^u_{K_1^{0,0}}[S_1^{i,j}] \\ D^u_{S_2^{i,j}}[K_1^{0,0}] + D^v_{S_2^{i,j}}[K_2^{0,0}] - D^v_{K_2^{0,0}}[S_2^{i,j}] \end{bmatrix}. \tag{3.10}$$

This follows from equation (3.3) and

$$D^v_{K_1^{0,0}} = D^u_{K_2^{0,0}} = 0.$$

Chapters 6 and 8 are devoted to '$\mathcal{B}$-equations'. These are equations of the form $u_t = K$ where

$$K = K^{0,0} + K^{-1,2}.$$

Here we write out the symmetry conditions for such equations.

Lemma 3.19. *The vector S is a symmetry of $u_t = K^{0,0} + K^{-1,2}$ if, for any non-negative integer k, the following equations are satisfied.*

- $\star$ $\mathcal{L}(K^{0,0})S^{i,k-i} = 0$ *if* $i = k, k + 1$ *and* $k = 0, i = -1$.
- $\star$ $\mathcal{L}(K^{0,0})S^{i,k-i} + \mathcal{L}(K^{-1,2})S^{i+1,k-i-2} = 0$ *if* $i = 0, \dots, k - 1$ *and* $i = -1, k > 0$.

Proof. The equations of total grading k have the form

$$\sum_{m,n} \mathcal{L}(K^{m,n})S^{i-m,k-n-i} = 0, \tag{3.11}$$

with $i = -1, 0, \dots k+1$. Due to the form of K each sum contains at most two terms, $(m, n) = (0, 0), (-1, 2)$. We have $k - n - i \geq -1$. Hence the terms with $n = 2, \ i < k$ do not contribute. When $k = 0$ the term $S^{0,k-1}$ does not exist. $\square$

Once more we prove that, under the assumption of nonlinear injectiveness, the non-existence of certain terms puts stringent conditions on the symmetry.

Lemma 3.20. *Suppose that $K^{0,0}$ is nonlinear injective. Under the assumption that $S^{1,-1} = 0$, any symmetry S of $u_t = K^{0,0} + K^{-1,2}$ has the form*

$$S = S^{0,0} + S^{-1,1} + S^{-1,2}.$$

Proof. By induction on k. Take $k > 0$. As induction-hypothesis we suppose that $S^{i,j} = 0$ for all i, j such that $i + j = k$ and $j \neq 2$ if $i = -1$. For $k = 1$ this follows from the assumption. For $k > 1$, the equations in Lemma 3.19 of total grading $k+1$ reduce to

$$\mathcal{L}(K^{0,0})S^{i,k+1-i},$$

with $i = -1, 0, \ldots, k+2$, because the term $\mathcal{L}(K^{-1,2})S^{i+1,k-i-1}$ does not contribute. This follows from the fact that the total grading of $S^{i+1,k-i-1}$ is k and its u-weight nonnegative. By the nonlinear injectivity of $K^{0,0}$ it follows that $S^{i,j} = 0$ if $i + j = k + 1$. □

Chapter 4

Symbolic calculus and proving integrability

The basic idea of the symbolic calculus is very old, probably dating from the time when the position of index and power were not as fixed as they are today. The following formulas illustrate that differentiating a product is similar to taking the power of a sum. From the Leibniz rule we have

$$(uv)_n = \sum_{i=0}^{n} \binom{n}{i} u_i v_{n-i},$$

while Newton's binomial formula reads

$$(u+v)^n = \sum_{i=0}^{n} \binom{n}{i} u^i v^{n-i}.$$

It is seen that in these formulas the index, counting the number of derivatives, could be interchanged with the power. Of course, with expressions containing both indices and powers, one has to be more careful. In this case we perform the Gel'fand-Dikiĭ transformation, cf. [GD75]. This can be seen as short notation for the Fourier transform, as was clearly pointed out in [MN02].

4.1 Symbolic calculus for scalar equations

Abstract. We introduce the symbolic calculus for scalar equations and show how to use this calculus to prove nonlinear injectivity or relatively l-primeness.

The 'symbolic method' consists of a rule to translate any polynomial P in the variables $u, u_1, u_2, \ldots$ into a polynomial $\widehat{P}$ in the variables $u, \xi_1, \xi_2, \ldots$.

Notation 4.1. *The* **symbolic expression** *for $P \in \mathcal{A}$ is denoted by $\widehat{P}$. The Gel'fand-Dikiĭ transformation is written $P \trianglerighteq \widehat{P}$.*

A differential monomial M with m variables u_k is transformed as:

$$\begin{aligned} M &= \prod_{j=1}^{m} u_{i_j} \\ &\trianglerighteq \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \prod_{j=1}^{m} \xi_{\sigma(j)}^{i_j} u^m \\ &= \widehat{M}, \end{aligned}$$

where $\sum_{\sigma \in \mathfrak{S}_m}$ means one has to sum over all different permutations of the integers $1, \ldots, m$. This symmetrising is done to ensure that, for example, $\widehat{u_i u_j} = \widehat{u_j u_i}$. By linearity the mapping extends to polynomials.

When multiplying two polynomials the total u-grading increases. Therefore symbolic polynomials can not simply be multiplied. Suppose that $\widehat{P}$ depends on p symbols. Number the symbols in $\widehat{M}$ from 1 to m and the ones in $\widehat{P}$ from $m+1$ to $m+p$, then multiply and symmetrise:

$$\begin{aligned} MP &\trianglerighteq \widehat{M} \circ \widehat{P} \\ &= \frac{1}{(m+p)!} \sum_{\sigma \in \mathfrak{S}_{m+p}} \widehat{M}(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(m)}) \widehat{P}(\xi_{\sigma(m+1)}, \cdots, \xi_{\sigma(m+p)}) u^{m+p} \\ &= \widehat{MP}. \end{aligned}$$

The operation of taking a total derivative turns into multiplication with the sum of all symbols involved. We have

$$\begin{aligned} D_x M &= \sum_{k=1}^{m} u_{i_k+1} \prod_{j \neq k}^{m} u_{i_j} \\ &\trianglerighteq \sum_{k=1}^{m} \xi_k^{i_k+1} \circ \frac{1}{(m-1)!} \sum_{\sigma \in \mathfrak{S}_{m-1}} \prod_{j \neq k}^{m} \xi_{\sigma(j)}^{i_j} u^m \\ &= \sum_{i=1}^{m} \xi_i \widehat{M}(\xi) \\ &= \widehat{D_x M}, \end{aligned}$$

which follows from the fact that taking the sum over all permutations of a symmetric polynomial in $m-1$ symbols equals multiplying with $(m-1)!$.

The Fréchet derivative becomes

$$\begin{aligned} D_M &= \sum_{k=1}^{m} \prod_{j\neq k}^{m} u_{i_j} D_x^{i_k} \\ &\trianglerighteq \sum_{k=1}^{m} \frac{1}{(m-1)!} \sum_{\sigma\in\mathfrak{S}_{m-1}} \prod_{j\neq k}^{m} \xi_{\sigma(j)}^{i_j} \widehat{D_x}^{i_k} u^{m-1} \\ &= m\widehat{M}(\xi_1,\ldots,\xi_{m-1},\widehat{D_x})/u \\ &= \widehat{D_M}, \end{aligned}$$

where ξ_n is replaced by the symbol $\widehat{D_x}$, representing the sum of all symbols in the monomial the Fréchet derivative is acting on. Symmetrising over these symbols has to be done after multiplication. As was noticed by Mikhailov and Novikov, cf. [MN02], an operator is a Fréchet derivative if it is symmetric in the symbols

$$D_x, \xi_1, \xi_2, \ldots.$$

Note that in the symbolic calculus taking the Fréchet derivative in the direction of a linear term becomes

$$D_M[u_n] \trianglerighteq (\sum_{i=1}^{m} \xi_i^n)\widehat{M}.$$

Thus, the Lie derivative of $S \in \mathcal{H}^{(i)}$, i.e., of grading i, in the direction of u_n is symbolic multiplication with so-called $\mathcal{G}$-functions. We have

$$\mathcal{L}(u_n)S \trianglerighteq \mathcal{G}_n^i \widehat{S}$$

where the $\mathcal{G}$-functions are

$$\mathcal{G}_n^i = \sum_{j=1}^{i+1} \xi_j^n - (\sum_{j=1}^{i+1} \xi_j)^n. \tag{4.1}$$

We are going to show how to verify the conditions 'nonlinear injectiveness' and 'relative l-primeness', see Definitions 3.5 and 3.7. We need this when applying the implicit function theorem (Theorem 3.8).

Lemma 4.2. *Take $n \in \mathbb{N}$ and $n > 1$. Then u_n is nonlinear injective.*

Proof. Let $S \in \mathcal{H}^{(i)}$ be nonzero. According to Definition 3.5, u_n is nonlinear injective if

$$\mathcal{L}(u_n)S = 0$$

implies that $i = 0$. Turning to the symbolic language we get

$$\mathcal{G}_n^i \widehat{S} = 0.$$

Hence we have $\mathcal{G}_n^i = 0$ and together with $n > 1$ this implies $i = 0$. □

Lemma 4.3. *Let u_n be the linear part of K and u_m be the linear part of S. Then K is relatively l-prime with respect to S if $\mathcal{G}_n^l$ and $\mathcal{G}_m^l$ are relatively prime.*

Proof. K is relatively l-prime with respect to S if u_n is relatively l-prime with respect to u_m. According to Definition 3.7, u_n is relatively l-prime with respect to u_m if

$$\mathcal{L}(u_m)S \in \mathrm{Im}(\mathcal{L}(u_n))$$

implies

$$S \in \mathrm{Im}(\mathcal{L}(u_n))$$

for all $S \in \mathcal{H}^{(k)}$ where $k \geq l$. In symbolic language, for $k \geq l$ the equality

$$\mathcal{G}_m^k \widehat{S} = \mathcal{G}_n^k R u^{k+1}$$

should imply that

$$\widehat{S} = \mathcal{G}_n^k T u^{k+1}$$

for some $R, T \in \mathbb{C}[\xi_1, \ldots, \xi_{k+1}]$. This is the case, with $T = R/\mathcal{G}_m^k$, whenever $\mathcal{G}_n^k$ and $\mathcal{G}_m^k$ are relatively prime. On the other hand, suppose that $\mathcal{G}_n^l$ and $\mathcal{G}_m^l$ are relatively prime. Then, since

$$\mathcal{G}_n^{l+1} |_{\xi_{l+2}=0} = \mathcal{G}_n^l, \quad \mathcal{G}_m^{l+1} |_{\xi_{l+2}=0} = \mathcal{G}_m^l$$

the polynomials $\mathcal{G}_n^k$ and $\mathcal{G}_m^k$ are relatively prime for all $k \geq l$. □

Now, the question arises how to check relative primeness of two polynomials. This can be done by calculating their resultant. The definition of the resultant is given in Appendix C. Resultants can also be used for nonlinear elimination, see Theorem C.3. The origin of the resultant lies with Sylvester's criterion for determining when two polynomials have a nontrivial common factor. This criterion simply states that two polynomials, $A(x)$ and $B(x)$, have a nontrivial common factor if and only if $\mathrm{res}_x(A, B) = 0$.

Remark 4.4. *K can be relatively l-prime with S without $\mathcal{G}_n^l$ being relatively prime with $\mathcal{G}_m^l$, see Theorem 5.9.*

4.2 The Korteweg-De Vries equation

Abstract. We prove the existence of infinitely many commuting symmetries for the KDV equation. using the symbolic calculus and the implicit function theorem.

We prove the integrability of the KDV equation $u_t = K^0 + K^1 = u_3 + uu_1$ by verifying the conditions in Theorem 3.8.

- ⋆ **One symmetry**
  S_5 is a symmetry of the KDV equation, see Example 2.16.

- $\star$ **Nonlinear injectiveness**
  By Lemma 4.2, the linear part u_3 is nonlinear injective.

- $\star$ **Relatively l–primeness**
  We verify that $\mathcal{G}_3^2$ and $\mathcal{G}_5^2$ are relatively prime. Their resultant is

$$\operatorname{res}_{\xi_4}(\mathcal{G}_3^2, \mathcal{G}_5^2) = (45\xi_1\xi_2\xi_3(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)(\xi_1+\xi_2+\xi_3))^2 \tag{4.2}$$

  which is nonzero. Therefore, using Lemma 4.3, K is relatively 3-prime with respect to S_5.

- $\star$ **Infinitely many approximate symmetries**
  The n-th order symmetry is written

$$S_n = S_n^0 + S_n^1 + S_n^2 + \cdots,$$

  with $S_n^0 = u_n$. We start by solving $\mathcal{L}(K^0)S^1 + \mathcal{L}(K^1)S^0 = 0$. In the symbolic calculus

$$\widehat{S_n^1} = \frac{\mathcal{G}_n^1}{\mathcal{G}_3^1}\widehat{K^1} = \frac{(\xi_1+\xi_2)^n - \xi_1^n - \xi_2^n}{6\xi_1\xi_2}.$$

  Let us go one step further and show that we can calculate the cubic terms of infinitely many symmetries. The equation

$$\mathcal{L}(K^0)S_n^2 + \mathcal{L}(K^1)S_n^1 + \mathcal{L}(K^2)S_n^0 = 0$$

  can be solved for S_n^2 if

$$\widehat{\mathcal{L}(K_1^1)S_n^1} = \frac{2}{3!}\sum_{\mathfrak{S}_3}\Big(\widehat{K^1}(\xi_1,\xi_2+\xi_3)\widehat{S_n^1}(\xi_2,\xi_3) - \widehat{S_n^1}(\xi_1,\xi_2+\xi_3)\widehat{K^1}(\xi_2,\xi_3)\Big)$$

  is divisible by $\mathcal{G}_3^2 = 3(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)$. This is the case if n is odd since the expression is symmetric and substitution of $\xi_1 = -\xi_2$ gives

$$\frac{\xi_3(\xi_2^n + (-\xi_2)^n)}{18\xi_2^2}.$$

- $\star$ **The approximate symmetries commute with the symmetry in lowest grading**
  We have $\mathcal{L}(u_5)u_n = 0$.

By Theorem 3.8 the Korteweg-De Vries equation has infinitely many odd order symmetries. Observe that these have finitely many terms since they are homogeneous.

4.3 Symbolic calculus for 2-component equations

The Gel'fand-Dikiĭ transformation is easily extended to differential monomials in more variables by introducing other symbols. We describe the $N = 2$ case. For u we use symbols ξ and for v we use symbols η. Any monomial in u_k, v_l is a product of a monomial M in the u_k and a monomial Q in the v_l. Such a product transforms by

$$M(u)Q(v) \trianglerighteq \widehat{M}(\xi)\widehat{Q}(\eta).$$

Symmetrising is only done in the symbols with the same name since $u_i u_j = u_j u_i$ and $u_i v_j \neq u_j v_i$. By linearity the transformation extends to polynomials. Let P be a polynomial in m variables u_k and n variables v_l. Thus total differentiation becomes

$$D_x P \trianglerighteq (\sum_{i=1}^{m} \xi_i + \sum_{j=1}^{n} \eta_j)\widehat{P}. \tag{4.3}$$

and taking a Fréchet derivative is done using

$$\begin{array}{rcl} D_P^u & \trianglerighteq & m\widehat{P}(\xi_1, \ldots, \xi_{m-1}, \widehat{D_x}, \eta)/u, \\ D_P^v & \trianglerighteq & n\widehat{P}(\xi, \eta_1, \ldots, \eta_{n-1}, \widehat{D_x})/v. \end{array} \tag{4.4}$$

In the symbolic calculus taking the Lie bracket with a diagonal linear part is a semisimple operation.

Lemma 4.5. *Suppose that the linear part of the equation is given by*

$$K^{0,0} = a_1 u_n \partial_u + a_2 v_n \partial_v$$

Then we have

$$\mathcal{L}(K^{0,0})S^{i,j} \trianglerighteq \left[\begin{array}{cc} \mathcal{G}_{1;n}^{i,j}[a_1, a_2](\xi, \eta) & 0 \\ 0 & \mathcal{G}_{2;n}^{i,j}[a_1, a_2](\xi, \eta) \end{array} \right] \widehat{S^{i,j}},$$

where

$$\begin{array}{rcl} \mathcal{G}_{1;n}^{i,j}[a_1, a_2](\xi, \eta) & = & a_1(\xi_1^n + \cdots + \xi_{i+1}^n) + a_2(\eta_1^n + \cdots + \eta_j^n) \\ & & -a_1(\xi_1 + \cdots + \xi_{i+1} + \eta_1 + \cdots + \eta_j)^n \end{array}$$

and

$$\mathcal{G}_{2;n}^{i,j}[a_1, a_2](\xi, \eta) = \mathcal{G}_{1;n}^{j,i}[a_2, a_1](\eta, \xi). \tag{4.5}$$

Proof. By equations (3.10), (4.3) and (4.4) the result follows. □

4.3.1 Nonlinear injectiveness and relatively l-primeness

Lemma 4.6. *Let a_1 and a_2 be nonzero. Take*

$$K = a_1 u_n \partial_u + a_2 v_n \partial_v$$

with $n > 0$ and $a_1 \neq a_2$ if $n = 1$. Then K is nonlinear injective.

Proof. K is nonlinear injective if for all $S \in \mathfrak{h}^{(i,j)}$ the equation $\mathcal{L}(K)S = 0$ implies $i + j = 0$. In symbolics this translates to

$$\begin{aligned} \mathcal{G}^{i,j}_{1;n}[a_1, a_2] = 0 \quad &\text{when} \quad S_1 \neq 0, \\ \mathcal{G}^{i,j}_{2;n}[a_1, a_2] = 0 \quad &\text{when} \quad S_2 \neq 0. \end{aligned}$$

Note that $S_1 = 0$ if $j = -1$ and that $S_2 = 0$ if $i = -1$. We distinguish two cases:

- ⋆ Suppose $a_1 \neq a_2$. For $k = 1, 2$ we have $\mathcal{G}^{i,j}_{k;n}[a_1, a_2] = 0$ if $i = j = 0$.
- ⋆ Suppose $a_1 = a_2$. With $n > 1$ we have $\mathcal{G}^{i,j}_{1;n}[a_1, a_2] = 0$ if $i = j = 0$ or $-i = j = 1$. Also $\mathcal{G}^{i,j}_{2;n}[a_1, a_2] = 0$ if $i = j = 0$ or $i = -j = 1$.

In both cases we have $i + j = 0$. □

Lemma 4.7. *Take $K = a_1 u_n \partial_u + a_2 v_n \partial_v$ and $S = b_1 u_m \partial_u + b_2 v_m \partial_v$. Then K is relatively l-prime with respect to S if the $\mathcal{G}$-functions $\mathcal{G}^{i,l-i}_{k,n}$ and $\mathcal{G}^{i,l-i}_{k,m}$ are relatively prime for $i = -1, 0, \ldots l + 1$ and $k = 1, 2$.*

Proof. According to Definition 3.7, K is relatively l-prime with respect to S if

$$\mathcal{L}(S)Q \in \mathrm{Im}(\mathcal{L}(K))$$

implies

$$Q \in \mathrm{Im}(\mathcal{L}(K))$$

for all $Q \in \mathfrak{h}^{(i,j)}$ where $i + j \geq l$. In symbolic language we need, for all $i + j \geq l$ and $k = 1, 2$,

$$\mathcal{G}^{i,j}_{k,m} \widehat{Q_k} = \mathcal{G}^{i,j}_{k,n} \widehat{R_k}$$

implying

$$\widehat{Q_k} = \mathcal{G}^{i,j}_{k,n} \widehat{T_k}$$

for some $R, T \in \mathfrak{h}^{(i,j)}$. This is the case, with

$$\widehat{T_k} = \widehat{R_k} / \mathcal{G}^{i,j}_{k,m},$$

whenever $\mathcal{G}^{i,j}_{k,n}$ and $\mathcal{G}^{i,j}_{k,m}$ are relatively prime. On the other hand, suppose that $\mathcal{G}^{i,l-i}_{k,n}$ and $\mathcal{G}^{i,l-i}_{k,m}$ are relatively prime for $i = -1, 0, \ldots, l + 1$ and $k = 1, 2$. Since

$$\mathcal{G}^{i+1,j}_{1,k} = \mathcal{G}^{i,j}_{1,k} \mid_{\xi_{i+2}=0}, \quad \mathcal{G}^{i,j+1}_{1,k} = \mathcal{G}^{i,j}_{1,k} \mid_{\eta_{j+1}=0}$$

the $\mathcal{G}$-functions $\mathcal{G}^{i,j}_{1,n}$ and $\mathcal{G}^{i,j}_{1,m}$ are relatively prime for all $i + j \geq l$. □

4.4 An integrable 2-component equation with a continuous spectrum

Using the symbolic calculus and the implicit function theorem we prove the existence of infinitely many commuting symmetries for the 2-component equation:

$$\left\{ \begin{array}{ll} u_t = & u_2 + 2u_1v + (1-\alpha)uv_1 + (1-2\alpha)uv^2 - u^3 \\ v_t = & \alpha v_2 + 4\alpha vv_1 + 2uu_1 \end{array} \right. , \tag{4.6}$$

where $\alpha \notin \{0, \frac{1}{3}\}$. This equation appeared in [SW01], where the classification of homogeneous second order equations with two components was performed.

⋆ **One symmetry**
A symmetry S of equation (4.6) is given by $S = (S_1, S_2)$ with

$$\begin{aligned} S_1 =& u_3 + 3v_1u_1 + 3u_2v + 3\frac{1-\alpha}{2}uv_2 + 3u_1v^2 \\ &+ (-9\alpha+6)uv_1v - 3u_1u^2 - 3u^3v + (3-6\alpha)uv^3 \end{aligned}$$

and

$$\begin{aligned} S_2 =& \frac{3\alpha-1}{2}v_3 + 3u_1^2 + 3u_2u + (-3+9\alpha)v_2v \\ &+ (-3+9\alpha)v_1^2 + 6uu_1v + 3u^2v_1 + (-6+18\alpha)v_1v^2 \end{aligned}$$

⋆ **Nonlinear injectiveness**
We have

$$K^{0,0} = u_2\partial_u + \alpha v_2\partial_v.$$

By Lemma 4.6, $K^{0,0}$ is nonlinear injective if $\alpha \neq 0$.

⋆ **Relatively l–primeness**
We calculate some resultants:

$$\begin{aligned} &\mathrm{res}_{\eta_3}(\mathcal{G}_{1;2}^{-1,3}[1,\alpha], \mathcal{G}_{1;3}^{-1,3}[1, \frac{3\alpha-1}{2}]) = \\ &\quad \frac{9}{4}(\alpha-1)^2\left(2\alpha(\alpha-1)^2\eta_1^6 + (3\alpha^3-3\alpha^2+9\alpha-1)\eta_1^4\eta_2^2 + \cdots\right), \\ &\mathrm{res}_{\eta_2}(\mathcal{G}_{1;2}^{0,2}[1,\alpha], \mathcal{G}_{1;3}^{0,2}[1, \frac{3\alpha-1}{2}]) = 18\alpha(\alpha-1)^2\xi_1^2\eta_1^4 + \cdots, \\ &\mathrm{res}_{\eta_1}(\mathcal{G}_{1;2}^{1,1}[1,\alpha], \mathcal{G}_{1;3}^{1,1}[1, \frac{3\alpha-1}{2}]) = \\ &\quad 9(\alpha-1)^2\xi_1^2\xi_2^2\left((\alpha+1)(\xi_1^2+\xi_2^2) + 2\alpha\xi_1\xi_2\right), \\ &\mathrm{res}_{\xi_3}(\mathcal{G}_{1;2}^{2,0}[1,\alpha], \mathcal{G}_{1;3}^{2,0}[1, \frac{3\alpha-1}{2}]) = -12(\xi_1+\xi_2)\xi_1^2\xi_2^2, \\ &\mathrm{res}_{\xi_3}(\mathcal{G}_{2;2}^{3,-1}[1,\alpha], \mathcal{G}_{2;3}^{3,-1}[1, \frac{3\alpha-1}{2}]) = \\ &\quad 9(\alpha-1)^2\xi_1^2\xi_2^2\left((1-\alpha)(\xi_1^2+\xi_2^2) - 2\alpha\xi_1\xi_2\right), \end{aligned}$$

$$
\begin{aligned}
&\mathrm{res}_{\eta_1}(\mathcal{G}^{2,0}_{2;2}[1,\alpha],\mathcal{G}^{2,0}_{2;3}[1,\frac{3\alpha-1}{2}]) = \\
&\qquad -3/2(\xi_1+\xi_2)\left((\alpha-1)^3\xi_2^4+2(3\alpha-1)(\alpha^2+1)\xi_1^2\xi_2^2+\cdots\right), \\
&\mathrm{res}_{\eta_2}(\mathcal{G}^{1,1}_{2;2}[1,\alpha],\mathcal{G}^{1,1}_{2;3}[1,\frac{3\alpha-1}{2}]) = \\
&\qquad -3/2\xi_1^2(\eta_1+\xi_1)\left((\alpha-1)^3\xi_1^2+2\alpha(\alpha+1)(3\alpha-1)\eta_1^2+\cdots\right), \\
&\mathrm{res}_{\eta_3}(\mathcal{G}^{0,2}_{2;2}[1,\alpha],\mathcal{G}^{0,2}_{2;3}[1,\frac{3\alpha-1}{2}]) = -6\alpha^2(3\alpha-1)(\eta_1+\eta_2)\eta_1^2\eta_2^2.
\end{aligned}
$$

If $\alpha = 1$, all $\mathcal{G}$-functions are relatively prime. Note that we are in case 3 of Theorem C.3. Therefore, $K^{0,0}$ is relatively 2-prime with $S^{0,0}$ when $\alpha \notin \{0, \frac{1}{3}\}$.

If $\alpha = 0$, the $\mathcal{G}$-functions $\mathcal{G}^{0,2}_{1;2}[1,\alpha]$ and $\mathcal{G}^{0,2}_{1;3}[1,\frac{3\alpha-1}{2}]$ share the divisor $\eta_1+\eta_2$. Moreover, if $\alpha = 0$ we have $\mathcal{G}^{0,2}_{2;2}[1,\alpha] = 0$. If $\alpha = \frac{1}{3}$, we have $\mathcal{G}^{0,2}_{2;3}[1,\frac{3\alpha-1}{2}] = 0$.

⋆ **Infinitely many approximate symmetries**

We solve the equations (3.6) symbolically. The calculation is done for arbitrary order m, i.e., we set $Q^{0,0} = u_m\partial_u + \beta v_m\partial_v$.

$$
\begin{aligned}
\widehat{Q^{0,1}_1} &= \frac{\mathcal{G}^{0,1}_{1;m}[1,\beta]}{\mathcal{G}^{0,1}_{1;2}[1,\alpha]}\widehat{K^{0,1}_1} = \frac{(\xi_1+\eta_1)^m-\xi_1^m-\beta\eta_1^m}{\eta_1} \\
\widehat{Q^{0,1}_2} &= \frac{\mathcal{G}^{0,1}_{2;m}[1,\beta]}{\mathcal{G}^{0,1}_{2;2}[1,\alpha]}\widehat{K^{0,1}_2} = \beta(\eta_1+\eta_2)\frac{(\eta_1+\eta_2)^m-\eta_1^m-\eta_2^m}{\eta_1\eta_2} \\
\widehat{Q^{2,-1}_2} &= \frac{\mathcal{G}^{2,-1}_{2;m}[1,\beta]}{\mathcal{G}^{2,-1}_{2;2}[1,\alpha]}\widehat{K^{2,-1}_2} = (\xi_1+\xi_2)\frac{\beta(\xi_1+\xi_2)^m-\xi_1^m-\xi_2^m}{\alpha(\xi_1+\xi_2)^2-\xi_1^2-\xi_2^2}
\end{aligned}
$$

The fact that the first two expressions are polynomial can easily be seen by substituting $\eta_1 = 0$ or $\eta_2 = 0$ in the numerator. Demanding the latter expression to be polynomial gives us a restriction on the eigenvalue. Suppose that $(r, 1)$ is a projective zero to $\mathcal{G}^{2,-1}_{2;2}[1,\alpha]$. The other zero is given by $(1, r)$. These should be zeros of $\mathcal{G}^{2,-1}_{2;m}[1,\beta]$. This is the case if

$$\beta = \frac{1+r^m}{(1+r)^m}.$$

By nonlinear injectiveness Q has no other quadratic parts.

⋆ **The approximate symmetries commute with the symmetry in lowest grading**
We have $\mathcal{L}(S^{0,0})Q^{0,0} = 0$.

By the implicit function theorem (Theorem 3.8) the equation (4.6) has infinitely many symmetries.

4.5 Biunit coordinates and anharmonic ratios

Abstract. We introduce an uncommon way to describe complex numbers which will be especially convenient when describing the solutions to $\mathcal{G}$-functions that correspond to the integrable equations. Also we introduce the group of anharmonic ratios since the symmetry properties of $\mathcal{G}$-functions can easily be expressed in term of these ratios.

The most familiar way to describe a point r in the complex plane is probably

$$r = \Re(r) + \Im(r)i,$$

where $\Re(r) \in \mathbb{R}$ is the real part of r, $\Im(r) \in \mathbb{R}$ is the imaginary part of r and $i^2 = -1$. A second way to describe $r \in \mathbb{C}$ is

$$r = |r|e^{\arg(r)i},$$

where $|r| > 0$ is the absolute value of r and $0 \leq \arg(r) < 2\pi$ the argument of r. Yet, we would like to give a third description.

Definition 4.8. *We call* (ψ, ϕ)*, where* $|\psi| = |\phi| = 1$*,* $\psi, \phi \neq \pm 1$*,* **biunit coordinates** *of the point* $r \in \mathbb{C} \setminus \mathbb{R}$*, which is the intersection of the lines* $\psi\mathbb{R}$ *and* $\phi\mathbb{R} - 1$*, cf. Figure 4.1.*

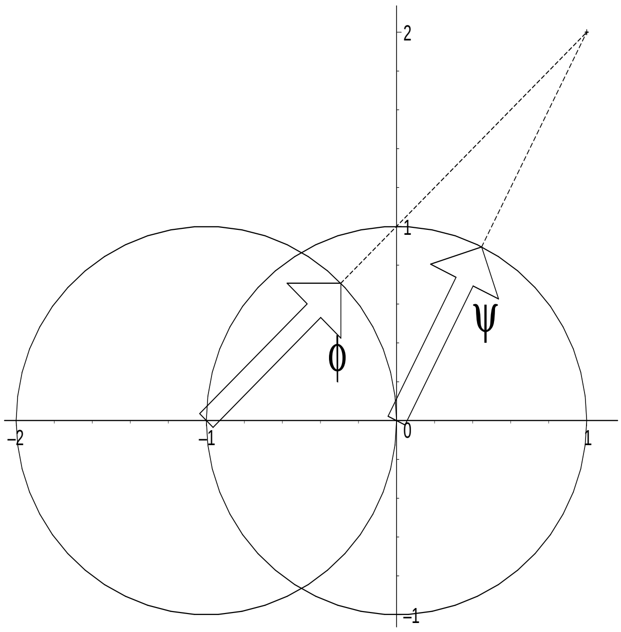

Figure 4.1: The point r in biunit coordinates (ψ, ϕ).

Lemma 4.9. *If (ψ, ϕ) are biunit coordinates of r, we have*

$$r = \mathfrak{P}(\psi, \phi) = \psi^2 \frac{(\phi+1)(\phi-1)}{(\psi+\phi)(\psi-\phi)}. \tag{4.7}$$

Proof. We solve the system of linear equations in $|r|$ and $|r+1|$:

$$\begin{aligned} |r|\sin(\arg(r)) &= |r+1|\sin(\arg(r+1)), \\ |r+1|\cos(\arg(r+1)) &= |r|\cos(\arg(r)) + 1. \end{aligned}$$

This gives

$$|r| = \frac{\sin(\arg(r+1))}{\cos(\arg(r+1))\sin(\arg(r)) - \sin(\arg(r+1))\cos(\arg(r))}.$$

Using the identities

$$\begin{aligned} \sin(\arg(r)) &= \frac{\psi - \psi^{-1}}{2i}, \ \cos(\arg(r)) = \frac{\psi + \psi^{-1}}{2}, \\ \sin(\arg(r+1)) &= \frac{\phi - \phi^{-1}}{2i}, \ \cos(\arg(r+1)) = \frac{\phi + \phi^{-1}}{2}, \end{aligned}$$

we express $|r|$ in terms of ψ, ϕ. Multiplying $|r|(\psi, \phi)$ by ψ gives expression (4.7). $\square$

Notation 4.10. *The set of points*

$$\{\mathfrak{P}(a, b))|a \in A, b \in B\}$$

is denoted with $\mathfrak{P}(A, B)$.

From Definition 4.8 and from expression (4.7) it is clear that if (ψ, ϕ) are biunit coordinates of r, then $(-\psi, \phi)$ and $(\psi, -\phi)$ are biunit coordinates of r as well. Note that we have $\psi \neq \pm\phi$, i.e., there is no point $r \in \mathbb{C}$ with biunit coordinates $(\psi, \pm\psi)$.

A $\mathcal{G}$-function can be invariant under interchanging certain symbols. For example we have

$$\mathcal{G}_{1,n}^{-1,2}(\eta_1, \eta_2) = \mathcal{G}_{1,n}^{-1,2}(\eta_2, \eta_1). \tag{4.8}$$

Another way of expressing this fact is the following: if $(1, r)$ is a projective zero to $\mathcal{G}_{1,n}^{-1,2}$, then $(1, \frac{1}{r})$ is a projective zero of $\mathcal{G}_{1,n}^{-1,2}$ as well, i.e., the set of zeros of $\mathcal{G}_{1,n}^{-1,2}$ is invariant under the anharmonic transformation $r \to \frac{1}{r}$.

Definition 4.11. *The group generated by the transformations*

$$f_2 : r \mapsto \frac{1}{r}, \ f_3 : r \mapsto -1 - r$$

is called the **group of anharmonic ratios** *and denoted by $\mathfrak{A}$.*

The group of anharmonic ratios is a representation of the permutation group $\mathfrak{S}_3$, cf. [MM97]. We number the elements $f_i \in \mathfrak{A}$ in the following way.

$$f_1 = f_2 \circ f_2,\ f_4 = f_2 \circ f_3 \circ f_2,\ f_5 = f_2 \circ f_3,\ f_6 = f_3 \circ f_2$$

Explicitly we have

$$\begin{aligned} f_1 : r \to r \quad &, \quad f_2 : r \to \frac{1}{r}, \\ f_3 : r \to -1 - r \quad &, \quad f_4 : r \to -\frac{r}{1+r}, \\ f_5 : r \to -\frac{1+r}{r} \quad &, \quad f_6 : r \to -\frac{1}{1+r}. \end{aligned} \tag{4.9}$$

We consider the image of r under the group of anharmonic ratios in terms of biunit coordinates.

$$\begin{aligned} f_1 : (\psi, \phi) \to (\psi, \phi) \quad &, \quad f_2 : (\psi, \phi) \to (\psi^{-1}, \phi\psi^{-1}), \\ f_3 : (\psi, \phi) \to (\phi, \psi) \quad &, \quad f_4 : (\psi, \phi) \to (\psi\phi^{-1}, \phi^{-1}), \\ f_5 : (\psi, \phi) \to (\psi^{-1}\phi, \psi^{-1}) \quad &, \quad f_6 : (\psi, \phi) \to (\phi^{-1}, \psi\phi^{-1}). \end{aligned} \tag{4.10}$$

These are just algebraic identities and simple to check. However, they can be given a geometrical meaning. For example the second identity is equivalent to the following proposition.

Proposition 4.12. *Consider a triangle ABC. Let D be the point on the line BC such that*

$$\angle(AB, AD) = \angle(CA, CB).$$

Then the length of BD times the length of BC equals the square length of AB.

Proof. The proposition can also be proven by using the cosine rule twice. □

Conjugation is also a simple operation in biunit coordinates. If the biunit coordinates of r are given by (ψ, ϕ) we have (ψ^{-1}, ϕ^{-1}) as biunit coordinates of $\bar{r}$, i.e.,

$$\bar{r} = \mathfrak{P}(\psi^{-1}, \phi^{-1}) = \frac{(\psi + 1)(\phi - 1)}{(\psi - \phi)(\psi + \phi)}, \tag{4.11}$$

since $\bar{\psi} = \psi^{-1}$ whenever $|\psi| = 1$.

Chapter 5

The classification of scalar equations

In this chapter we review the classification of scalar equations, with respect to the existence of symmetries, obtained by Sanders and Wang in [SW98]. Moreover, we elucidate that the classification with respect to integrability can be performed without diophantine approximation theory.

5.1 Divisibility conditions

Using the symbolic method and the implicit function theorem, the paper [SW98] classifies all homogeneous integrable and almost integrable equations of the form

$$u_t = u_n + f(u, \ldots, u_{n-1}), \quad n > 1, \tag{5.1}$$

with (purely) nonlinear $f \in \mathcal{A}$ and $\lambda(u) > 0$. The result heavily depends on divisibility properties of the functions $\mathcal{G}_n^i$, which are given by equation (4.1).

Note that we consider the right hand side of equation 5.1 to be an element in $\mathcal{H}$:

$$K = (u_n + f(u, \ldots, u_{n-1}))\partial_u \in \mathcal{H},$$

see Definition 2.2. We write

$$K = K^0 + K^1 + \cdots,$$

where

$$\mathcal{L}(\sigma_u)K^i = iK^i, i \geq 0.$$

By abuse of notation, in what follows we omit ∂_u.

By Lemma 4.2 the linear part $K^0 = u_n$ is nonlinear injective. Therefore, if the equation has a nonzero symmetry

$$S = S^0 + S^1 + \cdots ,$$

we have $S^0 \neq 0$. Let S^0 be homogeneous of weight m. Then

$$S^0 = u_m$$

and the first symmetry condition is satisfied, i.e.,

$$\mathcal{L}(K^0)S^0 \trianglerighteq \mathcal{G}_n^0 S^0 = 0.$$

Suppose that the first $i-1$ nonlinear terms of K vanish, i.e., $K^i \neq 0,\ K^j = 0$ when $0 < j < i$. Then, by Lemma 3.6, we have $S^j = 0$ with $0 < j < i$ and

$$\mathcal{L}(K^0)S^i + \mathcal{L}(K^i)S^0 = 0,$$

leading to the symbolic divisibility condition that

$$\widehat{S^i} = \frac{\mathcal{G}_m^i}{\mathcal{G}_n^i}\widehat{K^i}$$

is polynomial. Since $\lambda(u)$ is positive, the degree of $\widehat{K^i}$ is smaller than n which is the degree of $\mathcal{G}_n^i$. Therefore the greatest common divisor of $\mathcal{G}_n^i$ and $\mathcal{G}_m^i$ should have positive degree. We now distinguish three cases: $i = 1, i = 2$ and $i > 2$. These cases correspond to equations with quadratic terms, equations with cubic lowest nonlinear part and equations with neither quadratic nor cubic terms.

For the cases $i = 2$ and $i > 2$ proving the relative l-primeness condition in the implicit function theorem (Theorem 3.8) consists of showing irreducibility of the $\mathcal{G}$-functions. This is done by using the following theorem.

Theorem 5.1 (Bézout's theorem)**.** *If C and D are two projective curves of degrees n and m in $\mathbb{CP}^2$ which have no common component, they have precisely nm points of intersection counting multiplicities;*

$$\sum_{p \in C \cap D} I_p(C, D) = nm.$$

A proof based on resultants is given in [Kir92], an other proof is found in [Har77].

It was shown that if an equation without quadratic terms possesses a symmetry it does have cubic terms and it is in a hierarchy of order three, cf. Sections 5.2 and 5.3. For equations with quadratic terms we will treat the matter a little differently from how it was originally done by distinguishing the classification of integrable equations, cf. Section 5.4 and the classification of almost integrable equations, cf. Section 5.5. This serves to separate the difficult from the easy part. The easy part is to obtain all integrable equations. The use of the Lech-Mahler theorem is crucial (and new) here. The difficult part is to show that there are no equations with finitely many symmetries. This is where the results obtained by F. Beukers using diophantine approximation theory are really needed, cf. Section 5.5.

5.2 Equations with neither quadratic nor cubic terms

Abstract. Using Bézout's theorem we show that any scalar equation with neither quadratic terms nor cubic terms does not possess a symmetry. More generally this is true for diagonalisable N-component equations with nonzero eigenvalues.

Theorem 5.2 (Beukers)**.** *Let $i > 2, i \in \mathbb{N}$. For any positive integer n the function*

$$\mathcal{G} = a_1^{n-1}\xi_1^n + \cdots + a_{i+1}^{n-1}\xi_{i+1}^n - a_0(\xi_1 + \cdots \xi_{i+1})^n,$$

where $a_j \neq 0, 1 \leq j \leq i+1$, is irreducible over $\mathbb{C}$.

Proof. If $\mathcal{G}$ is reducible the projective hypersurface H given by $\mathcal{G} = 0$ consists of two components. These components intersect in an infinite number of points, which should be singularities of H. Thus it suffices to show that H has finitely many singular points. When $a_0 = 0$ it is easy to see that there is no singularity. When $a_0 \neq 0$ there are the singularities

$$\xi_j = \frac{\zeta_j}{a_j},\ j = 1, \cdots, n,$$

where

$$\zeta_j^{n-1} = 1, \quad \sum_{j=1}^{i} \frac{\zeta_j}{a_j} = 1.$$

In particular there are only finitely many of them. □

Corollary 5.3. *As a special case of this theorem all $\mathcal{G}_n^i$ with $i > 2$ are irreducible over $\mathbb{C}$. In particular the greatest common divisor (gcd) of $\mathcal{G}_n^i$ and $\mathcal{G}_m^i$ is constant if $i > 2, n \neq m$.*

This immediately implies that if an equation has neither quadratic terms nor cubic terms it has no symmetry. At the same time it implies the following.

Corollary 5.4. *K is relatively 3-prime with respect to S.*

Remark 5.5. *Theorem 5.2 is applicable to equations with any number of components. If an equation with diagonal linear part and nonzero eigenvalues has neither quadratic nor cubic terms, it has no symmetry. If the equation has a symmetry, it is relatively* 3*-prime with respect to this symmetry.*

5.3 Equations with cubic lowest nonlinear terms

Abstract. Using Bézout's theorem we show that if a scalar equation with cubic lowest nonlinear part possesses a symmetry, it is contained in a 3-rd order hierarchy. More explicitly it is a symmetry of the modified Korteweg-De Vries equation or of the Ibragimov-Shabat equation.

Theorem 5.6 (Beukers)**.** *The $\mathcal{G}$-function*

$$\mathcal{G}_n^2 = \xi_1^n + \xi_2^n + \xi_3^n - (\xi_1 + \xi_2 + \xi_3)^n$$

is irreducible over $\mathbb{C}$ when n is even. When n is odd it factorises as

$$(\xi_1 + \xi_2)(\xi_2 + \xi_3)(\xi_1 + \xi_3)H_n^2,$$

where H_n^2 is irreducible over $\mathbb{C}$.

Proof. The singular points of the projective curve C given by $\mathcal{G}_n^2 = 0$ are given by the solutions of the simultaneous equations.

$$\mathcal{G}_n^2 = \partial_{\xi_1}\mathcal{G}_n^2 = \partial_{\xi_2}\mathcal{G}_n^2 = \partial_{\xi_3}\mathcal{G}_n^2 = 0.$$

This leads to

$$\xi_1^{n-1} = \xi_2^{n-1} = \xi_3^{n-1} = \xi_0^{n-1}, \quad \text{with } \xi_0 + \xi_1 + \xi_2 + \xi_3 = 0.$$

Since we work in projective space $\mathbb{CP}^2$, we may take $\xi_3 = 1$. The singularities are the points $(\xi_1, \xi_2, 1)$ such that

$$\xi_0 + \xi_1 + \xi_2 + 1 = 0, \quad \xi_i^{n-1} = 1, \; i = 0, 1, 2.$$

Four complex numbers of the same absolute value add up to zero if they form the sides of a parallelogram with equal sides. Hence one of ξ_i equals -1 and the others are opposite. But

$$(-\xi_j)^{n-1} \neq 1$$

if $\xi_j^{n-1} = 1$ and n is even. Therefore, when n is even, the projective curve is nonsingular. When n is odd there are $3n - 6$ singularities given by

$$(\zeta, -\zeta, 1), \; (\zeta, -1, 1), \; (-1, \zeta, 1),$$

where $\zeta^{n-1} = 1, \zeta \neq \pm 1$ and the 3 points

$$(1, -1, 1), \; (-1, 1, 1), \; (-1, -1, 1).$$

Consider such a singular point, say $(\zeta, -\zeta, 1)$. We study the singular point locally by introducing coordinates

$$u' = \zeta + u, \; v' = -\zeta + v.$$

Up to 3-rd degree terms we find the local equation

$$(\zeta(u+v)+v-u)(u+v)=0.$$

Since the quadratic part consists of two distinct factors, the singularity is simple, i.e., there are two distinct tangent lines through the singular point. The projective curve

$$(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)=0$$

has 3 singular points and intersects $H_n^2=0$ in $3(n-3)$ points. This accounts for the $3n-6$ points we found. Hence $H_n^2=0$ is nonsingular and in particular H_n^2 is irreducible. $\square$

Corollary 5.7. *If n and m are even then K is relatively 2-prime with respect to S.*

Suppose that an equation of order n is integrable. Then n is odd and the symmetries appear at odd orders. The cubic part of each symmetry satisfies

$$\widehat{S^2}=\frac{\mathcal{G}_m^2}{\mathcal{G}_n^2}\widehat{K^2}.$$

We have

$$\frac{\mathcal{G}_n^2}{(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)}$$

dividing $\widehat{K^2}$. Therefore we can take $m=3$ as well and view K as a symmetry of some 3-rd order equation. We check which homogeneous 3-rd order equations have a symmetry on order 5. The possible values for the weight $\lambda(u)$ are $3/2, 1$ and $1/2$, given by the solutions of

$$\text{degree}(\widehat{K^2})+2\lambda(u)=3, \quad \text{where } 0\le \text{degree}(\widehat{K^2})<3.$$

When $\lambda(u)=1$ we find the modified Korteweg-De Vries equation

$$u_t=u_3+u^2u_1.$$

When $\lambda(u)=1/2$ we find the Ibragimov-Shabat equation

$$u_t=u_3+3u^2u_2+9uu_1^2+3u^4u_1.$$

By Lemma 4.2, u_3 is nonlinear injective. By Corollary 5.4 the 3-rd order equations are relatively 3-prime with respect to their 5-th order symmetries, another proof is based on resultants, cf. 4.2. Therefore, with the implicit function theorem (Theorem 3.8) we conclude that these two equations are integrable.

5.4 Integrable scalar equations with a quadratic part

Abstract. Using the theorem of Lech and Mahler we show that if a scalar equation with quadratic terms is integrable, it is contained in a hierarchy of order 2, 3, 5 or 7. There are eight such hierarchies.

If the scalar equation (5.1) is integrable the $\mathcal{G}$-function

$$\mathcal{G}^1_n = \xi_1^n + \xi_2^n - (\xi_1 + \xi_2)^n$$

should have a common divisor with $\mathcal{G}^1_m$ for infinitely many m. We look at zeros (ξ_1, ξ_2) of the $\mathcal{G}$-functions in projective space $\mathbb{CP}^1$, i.e., we may scale ξ_2 to 1. We call the point r a zero of $\mathcal{G}^1_n$ if

$$\mathcal{G}^1_n(r, 1) = 0.$$

The problem translates into: find all zeros $r \in \mathbb{C}$ such that

$$\mathcal{G}^1_m(r, 1) = (r+1)^m - r^m - 1 = 0$$

has infinitely many integer solutions m. The lowest solution m is the order n of our equation, the starting point of the hierarchy. We solve this problem using the theorem of Lech-Mahler, see Appendix D. By Corollary D.2 it follows that if $r \neq 0, -1$ the triple

$$r+1, r, \frac{r}{1+r}$$

consists of roots of unity. It is easy to see that this implies that r is a primitive 3-rd root of unity. We now look at the orders of the symmetries. We have

$$\begin{aligned}
\mathcal{G}^1_m(0,1) &= 0 \text{ for all } m,\\
\mathcal{G}^1_m(-1,1) &= \begin{cases} 0 & \text{if } m \equiv 1 \bmod 2\\ -2 & \text{if } m \equiv 0 \bmod 2, \end{cases}\\
\mathcal{G}^1_m(\zeta,1) &= \begin{cases} 0 & \text{if } m \equiv 1,5 \bmod 6\\ -1 & \text{if } m \equiv 0 \bmod 6\\ 2\zeta & \text{if } m \equiv 2 \bmod 6\\ -3 & \text{if } m \equiv 3 \bmod 6\\ 2\zeta^2 & \text{if } m \equiv 4 \bmod 6. \end{cases}
\end{aligned}$$

We solve the simultaneous equations

$$\mathcal{G}^1_m(r,1) = \partial_r \mathcal{G}^1_m(r,1) = 0.$$

to find that the multiple zeros at order m are given by $(m-1)$-th roots of unity. Since at these points the second derivative of $\mathcal{G}^1_n$ is nonzero, they are actually double zeros.

Translating the above to common divisors of $\mathcal{G}$-functions proves Theorem 5.8.

Theorem 5.8. *All greatest common divisors of $\mathcal{G}_n^1$ with infinitely many $\mathcal{G}_m^i$ are:*

$$\begin{aligned}
n = 2, m \equiv 0 \ mod\ 1 : &\quad \xi_1\xi_2, \\
n = 3, m \equiv 1 \ mod\ 2 : &\quad \xi_1\xi_2(\xi_1+\xi_2), \\
n = 5, m \equiv 5 \ mod\ 6 : &\quad \xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2), \\
n = 7, m \equiv 1 \ mod\ 6 : &\quad \xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2)^2.
\end{aligned}$$

We now state an important result obtained in [SW98, Theorem 5.5].

Theorem 5.9 (Wang). *If S is a symmetry of a scalar equation 5.1, K is relatively 2-prime with S.*

Proof. This is obvious if the order of K or the order of S is even, i.e. then it follows from Theorem 5.6 and Lemma 4.3. Wang first proved that if S is an odd order symmetry of an odd order equation $u_t = K$ then $\xi_1 + \xi_2$ or $\xi_1\xi_2$ is a divisor of $\widehat{K^1}$. Using this result, she proved that in this case K is relatively 2-prime with S. □

Now we can prove an result, which is weaker version of Theorem 5.12. Nevertheless, it is much easier to prove and almost as useful.

Theorem 5.10. *Suppose that a scalar equation $u_t = K$ with nonzero linear part is integrable. Then it is contained in a hierarchy starting at order* 2, 3, 5 *or* 7.

Proof. The case where $K^1 = 0$ has been treated in the previous sections. Assume that the equation $u_t = K$ of order n with quadratic part $K^1 \neq 0$ is integrable. By assumption we have one symmetry S, say of order m. By Lemma 4.2, u_n is nonlinear injective with u_m. By Theorem 5.9, K is relatively 2-prime with S. By assumption, infinitely many symmetries Q_k exist. In particular, the first integrability condition

$$\mathcal{L}(u_n)Q_k^1 + \mathcal{L}(K^1)u_k = 0$$

can be solved for infinitely many integer values of k. By Theorem 5.8, we may take $k \in \{2, 3, 5, 7\}$. By the implicit function theorem $Q = u_k + \sum_{i>0} Q_k^i$ exists such that $\mathcal{L}(K)Q_k = 0$. □

By now, the classification of integrable scalar equations has become a finite problem. A rather extensive computer algebra computation, based on generating functions, shows that if a given 7-th order equation has a nontrivial symmetry, then the symbolic expression of its quadratic part is divisible by $(\xi_1^2 + \xi_1\xi_2 + \xi_2^2)$. This means that the equation is in the hierarchy of some 5-th order equation, cf. [SW98, Section 6]. What is left, is finding the homogeneous equations of order $2, 3$ and 5 that have a symmetry of order 3, 5 and 7 respectively.

For 2-nd order equations, the possible values for the weight $\lambda(u)$ are 2 or 1 given by the solutions of

$$\text{degree}(\widehat{K^1}) + \lambda(u) = 2, \quad \text{where } 0 \leq \text{degree}(\widehat{K^1}) < 2.$$

When $\lambda(u) = 1$ we find Burgers' equation

$$u_t = u_2 + uu_1.$$

The 3-rd order homogeneous equations with a 5-th order symmetry are the potential Korteweg-De Vries equation

$$u_t = u_3 + u_1^2$$

and the Korteweg-De Vries equation

$$u_t = u_3 + uu_1.$$

The 5-th order equations with a 7-th order symmetry are the potential Sawada-Kotera equation

$$u_t = u_5 + u_1u_3 + \frac{1}{15}u_1^3,$$

the potential Kaup-Kupershmidt equation

$$u_t = u_5 + 10u_1u_3 + \frac{15}{2}u_2^2 + \frac{20}{3}u_1^3,$$

the Kupershmidt equation

$$u_t = u_5 + 5u_1u_3 + 5u_2^2 - 5u^2u_3 - 20u_2u_1u - 5u_1^3 + 5u_1u^4,$$

the Kaup-Kupershmidt equation

$$u_t = u_5 + 10uu_3 + 25u_2u_1 + 20u^2u_1$$

and the Sawada-Kotera equation

$$u_t = u_5 + 5u_3u + 5u_2u_1 + 5u_1u^2.$$

5.5 Almost integrable scalar equations

Abstract. Based upon a result obtained by using diophantine approximation theory it is shown that there are no almost integrable scalar equations of finite depth.

Suppose the existence of an almost integrable scalar equation of the form (5.1). Then, for some i, there are n and $m \neq n$ such that $H = \gcd(G_n^i, G_m^i)$ has positive degree and does not divide G_k^i for infinitely many k. By Theorems 5.2 and 5.6 this is not possible if $i > 1$. The case $i = 1$ is treated by F. Beukers, cf. [Beu97, Theorem 4.1], who used modern techniques from diophantine approximation theory to prove the following.

Theorem 5.11 (Beukers)**.** *Let $r \in \mathbb{C}$ such that $r(r+1)(r^2+r+1) \neq 0$. Then at most one integer $n > 1$ exists such that $G_n^1(r,1) = 0$.*

This result was used in [SW98, Theorem 5.7] to prove the following theorem.

Theorem 5.12 (Wang)**.** *A nontrivial symmetry of a homogeneous equation is part of a hierarchy starting at order* 2, 3, 5 *or* 7.

Chapter 6

Classification of integrable $\mathcal{B}$-equations

We classify integrable equations of the form

$$\left\{ \begin{array}{rl} u_t = & a_1 u_n + K(v_0, v_1, \ldots) \\ v_t = & a_2 v_n \end{array} \right. ,$$

where $a_1, a_2 \in \mathbb{C},\ n \in \mathbb{N}$ and K a quadratic polynomial in derivatives of v. This is done using biunit coordinates and the Lech-Mahler theorem. Furthermore we present a new method, based on resultants, to determine whether an equation is in a hierarchy of lower order.

6.1 Introduction to $\mathcal{B}$-equations

Abstract. We define a special class of (triangular) equations, of which the symmetry structure is extremely rich. This class has the nice property that it contains all symmetries of (almost) all its equations. We review some important results and give the relatively simple symmetry condition using the symbolic method.

I.M. Bakirov devoted the article [Bak91] to the description of local symmetries of the following evolution equations with parameter a:

$$\left\{ \begin{array}{rl} u_t = & u_n + v^2 \\ v_t = & a v_n \end{array} \right. , \tag{6.1}$$

for $n \geq 2$. This class of equations is interesting since it contains both integrable and almost integrable equations.

We list the main results that were obtained in [Bak91].

1. It is proved that any symmetry of equation (6.1) with $a \neq 0, 1$ is a linear combination of
$$\begin{bmatrix} a_1 u_m + \sum_{i=0}^{\lfloor\frac{m-n}{2}\rfloor} \alpha_i v_i v_{m-n-i} \\ a_2 v_m \end{bmatrix}.$$

2. In the case $n = 2$, for all a one symmetry exists at each order. For $n = 3$ odd symmetries exist for all a. Explicit calculation, performed by a computer, showed that if $n = 3, a \neq 0$, equation 6.1 does not possess symmetries at order $4, 6, 8, 10, 12$.

3. A candidate equation that possesses only one higher order symmetry is
$$\begin{cases} u_t = & u_4 + v^2 \\ v_t = & \frac{1}{5} v_4 \end{cases},$$
which possesses a symmetry at order 6. By means of computer algebra it was shown, that the equation does not possess other symmetries of order $n \leq 53$.

For a long time it was not known if this fourth order equation of Bakirov had other symmetries than the one at order 6.

The paper [BSW98], which was also devoted to equations of type (6.1), changed this situation. In this article the symbolic method was used and, to our knowledge, both the Lech-Mahler theorem and p-adic analysis first appeared in the literature in connection with symmetries of evolution equations. We give a list of the main results obtained in [BSW98].

1. It was proven, by using p-adic analysis, that the equation of Bakirov does not have generalised symmetries at any order but at order 6, i.e., it was shown beyond doubt that 'one symmetry does not imply integrability'.

2. [BSW98, Theorem 2.2]. It was proven that under one of the conditions

   ⋆ $n \geq 6$

   ⋆ $n = 4, 5$ and $\mathcal{G}_{1,n}^{-1,2}[c, 1](x, 1)$ has two zeros $r, s \neq 0, -1$ such that
$$\frac{r}{s}, \frac{1+r}{1+s} \text{ or } rs, \frac{(1+r)s}{1+s} \tag{6.2}$$
   are not simultaneously roots of unity.

   the equation (6.1) with $a = \frac{1}{c}$ has finitely many symmetries ($c \notin \{0, 1\}$).

The treatment of almost integrable equations, of which the Bakirov equation is the first and simplest example, is postponed to chapter 8. We will include the proof of the second result, since it is of great importance for the analysis presented in this chapter.

The authors of [BSW98] actually considered equations of the form

$$\begin{cases} u_t = & au_n + v^2 \\ v_t = & v_n \end{cases}, \tag{6.3}$$

with $n > 1$. Notice that by rescaling t, v, i.e.,

$$t \to t/a, \ v \to \sqrt{a}v,$$

any such an equation, with $a \neq 0$, is transformed into an equation of the form (6.1).

In [BSW98] it was implicitly assumed that $a \neq 0, 1$, cf. [BSW01, Remark 2.2]. Therefore, by the first result of Bakirov, any symmetry of equation (6.3) is a linear combination of symmetries of the form

$$\begin{bmatrix} bu_n + S \\ v_n \end{bmatrix},$$

where S is quadratic in the variables v_i. We have to solve the equation

$$\mathcal{L}(K^{0,0})S^{-1,2} + \mathcal{L}(K^{-1,2})S^{0,0} = 0.$$

This problem translates into determining whether $\mathcal{G}^{-1,2}_{1;n}[a, 1]$ is a divisor of $\mathcal{G}^{-1,2}_{1;m}[b, 1]$, cf. Lemma 4.5. If this is true for infinitely many (m, b), the equation is integrable. In [BSW98, Appendix], the cases $n = 2, 3$ with arbitrary $a \in \mathbb{C}$ were treated by expressing the $\mathcal{G}$-functions in terms of the $\mathfrak{S}_2$-invariants

$$\eta_1 + \eta_2, \ \eta_1\eta_2.$$

It was shown that all 2-nd order equations are integrable with symmetries at every order. It was mistakingly remarked that the same was true for $n = 3$. For $n > 3$ the polynomials

$$f_{a,n} = a(1 + r)^n - r^n - 1$$

were used. Observe that the divisibility condition on $\mathcal{G}$-functions is equivalent with the question [BSW98, question 1.1]:

> given a, n, for which $b \in \mathbb{C}$ and $m \in \mathbb{N}$, does $f_{a,n}$ divide $f_{b,m}$?

In the consideration it is important to realise that $f_{a,n}$ has double zeros for some values of a. After the following lemma, cf. [BSW98, Lemma 3.1], we can understand the proof of [BSW98, Theorem 2.2] as it was given by Beukers, Sanders and Wang.

Lemma 6.1. *Suppose that ζ is a multiple zero of $f_{a,n}$. Then ζ is an $(n-1)$-th root of unity and $a = 1/(\zeta + 1)^{m-1}$. Together with $1/\zeta$ these are the only multiple zeros and they have multiplicity* 2 .

Proof. (of [BSW98, Theorem 2.2].) The first step of the proof consists of showing that, when $n > 3$, there exist zeros $r, s \neq 0, -1$ such that the pairs (6.2) are not simultaneously roots of unity. For $n = 4, 5$ this follows from the assumption. For $n \geq 6$ we will show a contradiction. If one of the pairs (6.2) would be roots of unity, this implies in particular one of the relations

$$s = r, \ s = \frac{1}{r}, \ s = \bar{r}, \ s = \frac{1}{\bar{r}}.$$

If $f_{a,n}$ has only simple zeros we can certainly choose zeros r, s such that

$$s \notin \{r, \frac{1}{r}, \bar{r}, \frac{1}{\bar{r}}\}.$$

In case of double zeros we can take for r such a double zero and for s a simple zero.

Next, suppose that r, s are zeros of $f_{b,m}$. Then, we have

$$b = \frac{1 + r^m}{(1 + r)^m} = \frac{1 + s^m}{(1 + s)^m}$$

yielding

$$U_m(r, s) = (1 + r)^m + (s(1 + r))^m - (1 + s)^m - (r(1 + s))^m = 0. \tag{6.4}$$

Suppose that equation (6.4) holds for infinitely many m, including $m = n$. Then, according to Corollary D.4, if both $r \neq 0, -1$ and $s \neq 0, -1$, at least one of the pairs (6.2) consists of root of unity. This was excluded by the assumptions. □

It was conjectured, cf. [BSW98, Conjecture 2.3], that there are only finitely many integrable equations of the form (6.3). This conjecture became a theorem in [BSW01, Theorem 2.1], where the following list was proven to be exhaustive:

$$\begin{array}{ll}
\begin{cases} u_t = & au_2 + v^2 \\ v_t = & v_2 \end{cases}, & \begin{cases} u_t = & au_3 + v^2 \\ v_t = & v_3 \end{cases}, \\
\begin{cases} u_t = & -u_4 + v^2 \\ v_t = & v_4 \end{cases}, & \begin{cases} u_t = & -3u_4 + v^2 \\ v_t = & v_4 \end{cases}, \\
\begin{cases} u_t = & -\frac{1}{4}u_5 + v^2 \\ v_t = & v_5 \end{cases}, & \begin{cases} u_t = & \frac{-13\pm5\sqrt{5}}{2}u_5 + v^2 \\ v_t = & v_5 \end{cases}, \\
\begin{cases} u_t = & u_5 + v^2 \\ v_t = & v_5 \end{cases}, & \begin{cases} u_t = & u_7 + v^2 \\ v_t = & v_7 \end{cases},
\end{array} \tag{6.5}$$

where $a \in \mathbb{C}, a \neq 0$.

Of great importance was the use of the algorithm of Smyth, cf. [BS01], that solves polynomial equations for roots of unity. Since roots of unity play an crucial role in the classification of integrable equations and the points ± 1 often are exceptional cases, we like to introduce the following notation:

Notation 6.2.

$$\Phi_n = \{\zeta \in \mathbb{C} | \ \zeta^n = 1, \ \zeta \neq \pm 1\}.$$

We will now sketch how all integrable equations (6.3) were found in [BSW01], and elucidate how it can be done more efficiently. We treat the 4-th order case. By [BSW98, Theorem 2.2],

$$(x-r)(x-\frac{1}{r})(x-\bar{r})(x-\frac{1}{\bar{r}})$$

should be a divisor of $f_{a,4}$. The authors of [BSW01] obtain the necessary condition

$$|r + \frac{1}{r} + 3/2| = 1/2 \tag{6.6}$$

by comparing coefficients. Note that the points

$$x = \frac{r}{\bar{r}}, \ y = \frac{1+r}{1+\bar{r}}$$

are roots of unity. Substituting

$$r = \frac{x(1-y)}{y-x}$$

in equation (6.6) gives, using $\bar{x} = x^{-1}, \ \bar{y} = y^{-1}$,

$$2y^2x^2 - yx^2 + x^2 - xy - 2y^2x - y^3x + y^4 + 2y^2 - y^3 = 0. \tag{6.7}$$

Applying the algorithm of Smyth yields

$$(x,y) = (1, \pm\zeta_4), (\pm\zeta_4, -1), (\pm\zeta_4, \mp\zeta_4), (\zeta_3, \zeta_3^2), (\zeta_3^2, \zeta_3),$$

where $\zeta_n \in \Phi_n$. From these solutions we conclude

$$r = -1, -1 \pm \zeta_4, \frac{-1 \pm \zeta_4}{2}, \zeta_3, \zeta_3^2.$$

By applying the map $r \to (1+r^4)/(1+r)^4$ the ratios of eigenvalues are found,

$$\frac{a_1}{a_2} = -1, -3.$$

Method 6.3. *There is a more efficient way of obtaining equation (6.7): substitute*

$$n = 4, r = \frac{x(1-y)}{y-x}, \bar{r} = \frac{1-y}{y-x}$$

in

$$U_n(r, \bar{r}) = 0$$

This idea works for any value of n. Thus, by applying the algorithm of Smyth, all points r such that $U_m(r, \bar{r}) = 0$ for infinitely many m, including $m = n$, can be found.

In [BSW01] a recursive formula that produces all symmetries is explicitly given to confirm that an equation is integrable. Here, we directly show that for the points r, found by Method 6.3, the equation

$$U_m(r, \bar{r}) = 0$$

holds for infinitely many m, thereby proving the integrability of the equations. By working with n-th roots of unity it becomes possible to prove the integrability of both a 4-th and a 5-th order equation (and more) with only one calculation. To illustrate this idea we we will now explicitly prove that all equations of order $n > 3$ in the list (6.5) are integrable and, meanwhile, give the order of their symmetries.

The multiple zeros are given by Lemma 6.1. Note that

$$\Phi_{n-1} \subset \Phi_m$$

whenever $m \equiv 0 \bmod (n-1)$ and hence each root in Φ_{n-1} is a double root of some $\mathcal{G}_m$-function with $m \equiv 1 \bmod (n-1)$. This implies the following:

- ⋆ the 4-th order equation with $a = -1$ corresponds to the double zeros Φ_3, it has symmetries at order $m \equiv 1 \bmod 3$.
- ⋆ the 5-th order equation with $a = -1/4$ corresponds to the double zeros Φ_4, it has symmetries at order $m \equiv 1 \bmod 4$.

Lemma 6.4. *The point $r = -1 - \zeta_n$ satisfies $U_m(r, \bar{r}) = 0$ for all $m \equiv 0 \bmod n$.*

Proof. By substitution of $r = -1 - \zeta_n$ we get

$$U_m(r, \bar{r}) = (-\zeta_n)^m + ((1 + \bar{\zeta}_n)\zeta_n)^m - (-\bar{\zeta}_n)^m - ((1 + \zeta_n)\bar{\zeta}_n)^m$$

When $m \equiv 0 \bmod n$ we have ${\zeta_n}^m = \bar{\zeta_n}^m$ which makes the expression vanish. □

Lemma 6.4 has the following applications:

- ⋆ The 4-th order equation with $a = -3$ correspond to the set of zeros
$$\{-1 \pm \zeta_4, -\frac{1 \pm \zeta_4}{2}\}.$$
It has symmetries at order $m \equiv 0 \bmod 4$.
- ⋆ The 5-th order equations with $a = (-13 \pm 5\sqrt{5})/2$ correspond to a set of zeros of the form
$$\{-1, -1 - \zeta_5, -1 - \zeta_5 - \zeta_5^3, \zeta_5 + \zeta_5^3, \zeta_5 + \zeta_5^2 + \zeta_5^3\}.$$
Since the zero -1 appears at odd orders only and the other four appear at $m \equiv 0 \bmod 5$, the equations have symmetries at order $m \equiv 5 \bmod 10$.

Note that the integrability of the equations with equal eigenvalues follows from Theorem 5.8.

Thus, the classification of integrable equations of the form 6.3 (or 6.1) is completed. However, there is still a question that could be asked:

do the equations in the list (6.5) have other symmetries?

For the equations that correspond to double zeros of $\mathcal{G}$-functions the answer is 'no', since by Lemma 6.1 we know all double zeros. For the other equations the answer might be 'yes'.

A corollary, cf. [BSW01, Corollary 2.1], says that each of the equations (6.5) with arbitrary quadratic part (in derivatives of v) is integrable as well. It was remarked that the list is not necessarily complete in this more general class of equations that is the object of research in this chapter.

Definition 6.5. *A* **$\mathcal{B}$-equation***, after I.M. $\mathcal{B}$akirov, is an equation of the form*

$$\mathcal{B}_n[a_1, a_2](K) : \begin{cases} u_t = & a_1 u_n + K(v_0, v_1, \ldots) \\ v_t = & a_2 v_n \end{cases}, \tag{6.8}$$

where $a_1, a_2 \in \mathbb{C},\ n \in \mathbb{N}$ and K a quadratic polynomial in derivatives of v.

The quadratic part of a $\mathcal{B}$-equation may contain some v_i with i higher than the order of the linear part. It might look more natural to restrict to equations where this is not the case. Although this could be done, it would be a more stringent condition than what we have in mind. Moreover, the theorems concerning the recognition of integrable $\mathcal{B}$-equations would be harder to formulate. Therefore we rather adapt the definition of order.

Definition 6.6. *The* **order of a $\mathcal{B}$-equation** *is the order of its linear part.*

$\mathcal{B}$-equations at order 0 have weird symmetries symmetries outside the class of $\mathcal{B}$-symmetries; any linear combination of

$$\begin{bmatrix} \sqrt[n]{(v_{i_1} v_{i_2} \cdots v_{i_n})^a} \\ 0 \end{bmatrix}$$

is a symmetry of the zeroth order equation

$$\begin{cases} u_t = au + F(v, v_1, \ldots) \\ v_t = v \end{cases},$$

where F is an arbitrary, i.e., not necessarily quadratic, function that depends on derivatives of v.

Assumption 6.7. *For any $\mathcal{B}_n[a_1, a_2](K)$ we assume that its $\mathcal{G}$-function $\mathcal{G}^{-1,2}_{1;n}[a_1, a_2]$ does not divide its quadratic part $\widehat{K}$. This rules out all $\mathcal{B}$-equations of order 0 as well as their symmetries. We also assume that the $\mathcal{G}^{0,1}_{1;n}[a_1, a_2](\eta_1, \eta_2)$ does not divide the quadratic part $\widehat{K}$.*

Proposition 6.8. *Under Assumption 6.7 any symmetry of a $\mathcal{B}$-equation with non-equal and nonzero eigenvalues, is a linear combination of* $\mathcal{B}$**-symmetries***:*

$$\left[\begin{array}{c} b_1 u_m + S(v, v_1, \ldots) \\ b_2 v_m \end{array}\right],$$

where $b_1, b_2 \in \mathbb{C}$, $m \in \mathbb{N}$ and S a quadratic polynomial in derivatives of v. If the eigenvalues are equal and the order of the $\mathcal{B}$-equation is higher than 1, any symmetry is a linear combination of $\mathcal{B}$-symmetries and the linear symmetry $(v_m, 0)$.

Proof. In our framework, at least in the setting of formal power series, the statement almost follows from the nonlinear injectiveness of the linear part, cf. Lemma 4.6 and Lemma 3.20. We only have to prove that $S^{1,-1} = 0$. As we will show, this follows from

$$\mathcal{L}(K^{0,0})S^{0,1} + \mathcal{L}(K^{-1,2})S^{1,-1} = 0.$$

Since the eigenvalues are nonzero we may scale a_1 to 1. We take

$$K^{0,0} = (u_n, av_n), \; K^{-1,2} = (K, 0).$$

Furthermore, we choose the linear part of the symmetry to be homogeneous

$$S^{1,-1} = (0, bu_m).$$

By equations (3.3) and (3.10), we see that

$$\left[\begin{array}{c} D^u_{S_1^{0,1}}[u_n] + D^v_{S_1^{0,1}}[av_n] - D^u_{u_n}[S_1^{0,1}] - D^v_K[bu_m] \\ D^v_{S_2^{0,1}}[av_n] - D^v_{av_n}[S_2^{0,1}] + D^u_{bu_m}[K] \end{array}\right] = 0.$$

By equations (4.3) and (4.4), this translates into

$$\begin{array}{rcl} (\xi_1^n + a\eta_1^n - (\xi_1 + \eta_1)^n)\,\widehat{S_1^{0,1}} & = & b\xi_1^m \widehat{K}(\xi_1, \eta_1), \\ a\,(\eta_1^n + \eta_2^n - (\eta_1 + \eta_2)^n)\,\widehat{S_2^{0,1}} & = & -b(\eta_1 + \eta_2)^m \widehat{K}(\eta_1, \eta_2). \end{array}$$

By Assumption 6.7 and the first equation, η_1 divides $\mathcal{G}^{0,1}_{1;n}[1, a]$. This implies that $a = 1$. Therefore

$$\widehat{S_1^{0,1}}(\eta_1, \eta_2) = b\frac{\eta_1^m}{\mathcal{G}_n^1}\widehat{K}, \quad \widehat{S_2^{0,1}}(\eta_1, \eta_2) = b\frac{(\eta_1 + \eta_2)^m}{\mathcal{G}_n^1}\widehat{K}.$$

Since for $a = 1$, by Assumption 6.7, $\mathcal{G}_n^1$ does not divide $\widehat{K}$ this implies that $b = 0$. Because of nonlinear injectiveness, cf. Lemma 4.6, the statement follows from Lemma 3.20. □

We do consider $\mathcal{B}$-equations with zero eigenvalues. In this case we solve the restricted problem of finding all $\mathcal{B}$-equations that commute with them.

Notation 6.9. *In this chapter we omit all redundant indices, i.e., we write*

$$K = K_1^{-1,2}, \quad \mathcal{G}_n[a_1, a_2] = \mathcal{G}_{1;n}^{-1,2}[a_1, a_2].$$

Thus, we have

$$\mathcal{G}_n[a_1, a_2](\eta_1, \eta_2) = a_2(\eta_1^n + \eta_2^n) - a_1(\eta_1 + \eta_2)^n.$$

Note that if r is a zero of both $\mathcal{G}_n[a_1, a_2]$ and $\mathcal{G}_m[b_1, b_2]$ then these have a common factor $(\eta_1 - r\eta_2)$.

In the symbolic calculus we have $\widehat{K} \in \mathbb{C}[\eta_1, \eta_2]$. The equation $\mathcal{B}_m[b_1, b_2](S)$ with $\widehat{S} \in \mathbb{C}[\eta_1, \eta_2]$ is a symmetry of $\mathcal{B}_n[a_1, a_2](K)$ when

$$\widehat{S} = \frac{\mathcal{G}_m[b_1, b_2]}{\mathcal{G}_n[a_1, a_2]} \widehat{K}$$

is polynomial. By Assumption 6.7 a necessary condition for the existence of a symmetry is: $\mathcal{G}_n[a_1, a_2]$ has a common factor with $\mathcal{G}_m[b_1, b_2]$. Moreover, it is a sufficient condition. Suppose we have $a_1, a_2, b_1, b_2 \in \mathbb{C}$ such that, with $F, L, T \in \mathbb{C}[\eta_1, \eta_2]$,

$$\mathcal{G}_n[a_1, a_2] = FL, \quad \mathcal{G}_m[b_1, b_2] = FT.$$

Then, if we take $\widehat{K} = LMv^2$ and $\widehat{S} = MTv^2$, where $M \in \mathbb{C}[\eta_1, \eta_2]$ can be chosen freely, the Lie derivative of $\mathcal{B}_m[b_1, b_2](S)$ with respect to $\mathcal{B}_n[a_1, a_2](K)$ vanishes.

In what follows we will solve the classification and recognition problems for $\mathcal{B}$-equations by answering the following (related) questions:

- ⋆ What are all hierarchies of $\mathcal{B}$-equations?
- ⋆ What are all symmetries of a given integrable $\mathcal{B}$-equation?
- ⋆ What are all integrable n-th order $\mathcal{B}$-equations that are not in a lower hierarchy?
- ⋆ Given an n-th order $\mathcal{B}$-equation, how to efficiently determine whether it is integrable?

A finite number of integrable $\mathcal{B}$-equations exist at any order $n > 4$. We will present a formula for the number of n-th order integrable $\mathcal{B}$-equations, as well as a formula for the number of n-th order integrable $\mathcal{B}$-equations that are not in a lower hierarchy. Also we will prove that all these $\mathcal{B}$-equations are real, up to complex scalings. The results described in this chapter are taken from [vdK02a].

6.2 $\mathcal{B}$-equations of order 1, 2 or 3 and their symmetries

Abstract. We prove that all $\mathcal{B}$-equations of order 1, 2 or 3 are integrable and we show how to efficiently calculate their symmetries.

Proposition 6.10. *All* 1*-st order* $\mathcal{B}$*-equations are integrable.*

This proposition is easy to prove and an explicit formula for all the symmetries of $\mathcal{B}_1[a_1, a_2](a_3 v^2)$ can be given.

Proof. To find all symmetries of $\mathcal{B}_n[a_1, a_2](K)$ we have to find for arbitrary m all (b_1, b_2) such that $\mathcal{G}_1[a_1, a_2]$ divides $\mathcal{G}_m[b_1, b_2]$. This can be done by substitution. Take $a_1 \neq a_2$. The $\mathcal{G}$-function

$$\mathcal{G}_1[a_1, a_2] = (a_2 - a_1)(\eta_1 + \eta_2)$$

has a common factor with $\mathcal{G}_m[b_1, b_2]$ if

$$b_2(\eta_1^m + (-\eta_1)^m) - b_1(\eta_1 - \eta_1)^m = 0.$$

The infinitely many solutions are $b_2 = 0$ or m is odd.

Exceptional case:

- $\star$ Take $a_1 = a_2$. For any S the symmetry condition becomes

  $$\mathcal{G}_m[b_1, b_2]\widehat{K} = 0.$$

  Equality holds when $b_1 = b_2 = 0$. The symmetries (at any order) have arbitrary nonlinear part but no linear part.

$\square$

Example 6.11. *We explicitly write down the symmetries of*

$$\begin{cases} u_t = & a_1 u_1 + a_3 v^2 \\ v_t = & a_2 v_1 \end{cases}$$

Its quadratic part is calculated as follows:

$$\widehat{S} = \frac{\mathcal{G}_m[b_1, b_2]}{\mathcal{G}_1[a_1, a_2]}\widehat{K} = \frac{a_3}{a_2 - a_1}(b_2 \frac{1}{\eta_1 + \eta_2}(\eta_1^m + \eta_2^m) - b_1(\eta_1 + \eta_2)^{m-1}).$$

By applying the inverse Gel'fand and Dikiĭ transformation, at even order m we obtain the symmetry

$$\begin{bmatrix} b_1 u_m + \frac{a_3 b_1}{a_1 - a_2} D_x^{m-1} v^2 \\ 0 \end{bmatrix}$$

and at odd order m we obtain the symmetry

$$\begin{bmatrix} b_1 u_m + \frac{a_3 b_1}{a_1 - a_2} D_x^{m-1} v^2 + \frac{a_3 b_2}{a_1 - a_2} D_x^{-1} v v_m \\ b_2 v_m \end{bmatrix}.$$

It is only here that we can describe the whole hierarchy in differential language. For higher order $\mathcal{B}$-equations we have to do the computation of a particular symmetry symbolically and translate the result to obtain its differential expression.

Proposition 6.12. *All 2-nd order $\mathcal{B}$-equations are integrable.*

Proof. They have symmetries at all orders. The ratio of eigenvalues (and quadratic part) of the symmetries are fixed. Take $a_1 \neq a_2$ again and $a_2 \neq 0$, i.e.,$r \neq 0, -1$. The $\mathcal{G}$-function

$$\mathcal{G}_2[a_1, a_2] = \frac{a_2 - a_1}{r}(\eta_1 - r\eta_2)(r\eta_1 - \eta_2) \text{ with } r^2 + \frac{2a_1}{a_1 - a_2}r + 1 = 0$$

has a factor $(\eta_1 - r\eta_2)$ in common with $\mathcal{G}_m[b_1, b_2]$ when

$$\mathcal{G}_m[b_1, b_2] \mid_{\eta_1 = r\eta_2} = 0 \Rightarrow \frac{b_1}{b_2} = \frac{1 + r^m}{(1 + r)^m}.$$

For this ratio $(r\eta_1 - \eta_2)$ is a factor as well because the fraction

$$\frac{1 + r^m}{(1 + r)^m}$$

is invariant under $r \to 1/r$, i.e., the $\mathcal{G}$-function is symmetric in η_1, η_2, cf. 4.8.

Exceptional cases:

- $\star$ When $a_1 = a_2$ the equation is integrable; we have $\mathcal{G}_2[a_1, a_1] = -2a_1\eta_1\eta_2$ divides $\mathcal{G}_m[b_1, b_2]$ for arbitrary $m > 2$ if $b_1 = b_2$.
- $\star$ When $a_2 = 0$ the equation is integrable; we have $\mathcal{G}_2[a_1, 0] = -a_1(\eta_1 + \eta_2)^2$ divides $\mathcal{G}_m[b_1, b_2]$ for arbitrary $m > 2$ if $b_2 = 0$.

□

We demonstrate the method by calculating a symmetry of some nonhomogeneous second order equation.

Example 6.13. *We calculate the 3-rd order symmetry $\mathcal{B}_m[b_1, b_2](S)$ of*

$$\begin{cases} u_t = & a_1 u_2 + a_3 v^2 + a_4 v v_1 + a_5 v_1^2 \\ v_t = & a_2 v_2 \end{cases}.$$

The ratio of eigenvalues of the symmetry is

$$\frac{1 + r^3}{(1 + r)^3} = \frac{3a_1 - a_2}{2a_2}.$$

We take $b_1 = 3a_1 - a_2$ and $b_2 = 2a_2$. The $\mathcal{G}$-function of the symmetry is

$$\mathcal{G}_3[3a_1 - a_2, 2a_2] = 3(\eta_1 + \eta_2)\mathcal{G}_2[a_1, a_2].$$

The quadratic part S is obtained by multiplying

$$K = a_3 + a_4 \frac{\eta_1 + \eta_2}{2} + a_5 \eta_1 \eta_2$$

with the ratio of $\mathcal{G}$-functions $3(\eta_1+\eta_2)$:

$$S = 6a_3\frac{\eta_1+\eta_2}{2} + 3a_4(\frac{\eta_1^2+\eta_2^2}{2} + \eta_1\eta_2) + 6a_5\frac{\eta_1\eta_2^2+\eta_1^2\eta_2}{2}.$$

By applying the inverse Gel'fand and Dikiĭ transformation we obtain the 3-rd order symmetry of the above equation

$$\left[\begin{array}{c} (3a_1-a_2)u_3 + 6a_3vv_1 + 3a_4(vv_2+v_1^2) + 6a_5v_1v_2 \\ 2a_2v_3 \end{array}\right].$$

The procedure works for symmetries of any order.

Proposition 6.14. *All 3-rd order $\mathcal{B}$-equations are integrable*

Proof. All 3-rd order equations have infinitely many symmetries but unlike the 2-nd order equations not all of them have symmetries at odd order. The reason is that $(\eta_1+\eta_2)$ is a divisor of $\mathcal{G}_m[b_1,b_2]$ only when m is odd or when $b_2=0$. Therefore, unless the 3-rd order equation is in a lower hierarchy, its first symmetry appears at order 5.

Take $a_2 \neq 0, a_1$ again. The 3-rd order $\mathcal{G}$-function factorises like

$$\mathcal{G}_3[a_1,a_2] = \frac{a_1-a_2}{r}(\eta_1+\eta_2)(\eta_1-r\eta_2)(r\eta_1-\eta_2),$$

with

$$r^2 + \frac{2a_1+a_2}{a_1-a_2}r + 1 = 0.$$

This can be used to calculate all higher order $\mathcal{G}$-functions in the same way we did for 2-nd order equations.

Exceptional cases:

- ⋆ When $a_1=a_2$ the equation is integrable, $\mathcal{G}_2[a_1,a_1] = -3a_1\eta_1\eta_2(\eta_1+\eta_2)$ divides $\mathcal{G}_m[b_1,b_2]$ for arbitrary odd $m>3$ if $b_1=b_2$.
- ⋆ When $a_2=0$ the equation is integrable, $\mathcal{G}_2[a_1,0] = -a_1(\eta_1+\eta_2)^3$ divides $\mathcal{G}_m[b_1,b_2]$ for arbitrary $m>3$ if $b_2=0$.

□

We have now proven that all $\mathcal{B}$-equations of order smaller than 4 are integrable.

6.3 $\mathcal{B}$-equations in a hierarchy of order 1, 2 or 3

Abstract. In this section we turn from the classification problem to the recognition problem. We present an efficient way to determine whether a $\mathcal{B}$-equation is in a hierarchy of order 1, 2 or 3 using resultants.

Theorem 6.15. *$\mathcal{B}_m[b_1, b_2](S)$ is in a hierarchy of order n, where n is 1, 2 or 3, if the degree of the greatest common divisor of $\mathcal{G}_m[b_1, b_2]$ and S equals $m - n$.*

Proof. Since every symmetric factor of degree n, where n is 1,2 or 3, is a multiple of $\mathcal{G}_n[a_1, a_2]$ for some (calculable) a_1, a_2, the quadratic part can be written

$$S = \frac{\mathcal{G}_m[b_1, b_2]}{\mathcal{G}_n[a_1, a_2]} K$$

such that gcd($K, \mathcal{G}_n[a_1, a_2]$)=1. □

The use of resultants is very effective here, as we will show in the following two examples. Recall that if the greatest common divisor of two polynomials has positive degree, then their resultant vanishes.

Example 6.16. *The equation*

$$\begin{cases} u_t = & b_1 u_3 + b_3 v_2 v + b_4 v_1^2 \\ v_t = & b_2 v_3 \end{cases}$$

can be in a hierarchy of order 1 *or* 2*. The η_1-resultant of $\mathcal{G}_3[b_1, b_2]$ and $\widehat{S}$ is:*

$$\frac{\eta_2^6}{4}(b_3 - b_4)(2b_3 b_1 + b_3 b_2 - 2b_4 b_1 + 2b_4 b_2)^2.$$

There are two special cases.

- ⋆ *When $b_3 = b_4$ the quadratic part is $\widehat{S} = \frac{b_3}{2}(\eta_1 + \eta_2)^2$. The greatest common divisor of $\widehat{S}$ and $\mathcal{G}_3[b_1, b_2]$ has degree 1, so the order of the hierarchy is 2. The ratio of eigenvalues can be calculated using the above factorising of the $\mathcal{G}_3$-function and the map $r \rightarrow \frac{1+r^2}{(1+r)^2}$. The equation commutes with*

$$\begin{cases} u_t = & (2b_1 + b_2)u_2 + 2b_3 v v_1 \\ v_t = & 3b_2 v_2 \end{cases}.$$

- ⋆ *When $b_3 = 2(b_1 - b_2)a_1, b_4 = (2b_1 + b_2)a_1$ the equation is in the hierarchy of*

$$\begin{cases} u_t = & b_1 u_1 + a_1(b_1 - b_2)v^2 \\ v_t = & b_2 v_1 \end{cases}.$$

All other cases are not in an other hierarchy.

The method works for any order in principle. However it depends on the order and the number of parameters in the equation whether we can actually solve the resultant. This is illustrated by the following example.

Example 6.17. *When an equation of the form*

$$\begin{cases} u_t = & au_7 + cvv_4 + dv_1v_3 + ev_2^2 \\ v_t = & bv_7 \end{cases}$$

is a symmetry of a lower equation, its parameters make the resultant of the $\mathcal{G}$-function and the quadratic part vanish, i.e.,

$$\begin{aligned} 0 \;=\;& a^2(12d^2e - 24dc^2 - 24dec - 6de^2 + 24d^2c + 6e^2c + 12ec^2 + 8c^3 + e^3 - 8d^3) \\ &+b^2(de^2 + c^3 - 3cde + e^3 - d^3 + 4c^2d - 2c^2e + 3cd^2 - ce^2 - 2d^2e) + ab(9d^3 \\ &-2e^3 + 40c^3 - 10d^2e + 5de^2 + 44ce^2 - 108c^2e - 71cde + 118c^2d + 22cd^2). \end{aligned}$$

For some specific 7-th order $\mathcal{B}$-equation (which may contain free parameters) we can check whether it is contained in a hierarchy by substituting the coefficients into the above equation. However, to describe all such equations we really have to know where they could come from. Here we have the following possibilities:

- ⋆ *The equation is a symmetry of the integrable equation*

$$\begin{cases} u_t = & fu_3 + hv^2 \\ v_t = & gv_3 \end{cases}$$

if $a = 7f^3 + 21gf^2 - g^3$, $b = 27g^3$, $c = 14h(f^2 + 4fg + 4g^2)$, $d = 14h(4f^2 + 13fg + g^2)$ and $e = 21h(2f^2 + 6fg + g^2)$.

- ⋆ *The equation is a symmetry of the integrable equation ($g \neq 0$)*

$$\begin{cases} u_t = & u_4 + \frac{f}{g}vv_1 \\ v_t = & -v_4 \end{cases}$$

if $a = g$, $b = g$, $c = 0$, $d = \frac{7f}{2}$ and $e = \frac{7f}{2}$.

- ⋆ *The equation is a symmetry of the integrable equation*

$$\begin{cases} u_t = & u_5 + \frac{f}{g}vv_2 + \frac{h}{g}v_1^2 \\ v_t = & v_5 \end{cases}$$

if $a = g$, $b = g$, $c = \frac{7f}{5}$, $d = \frac{7f+14h}{10}$ and $e = \frac{7(f+h)}{5}$.

- ⋆ *The equation is the symmetry of the almost integrable equation*

$$\begin{cases} u_t = & 88u_6 + \frac{37f}{g}vv_3 + \frac{101f}{g}v_1v_2 \\ v_t = & 125v_6 \end{cases}$$

if $a = 83g$, $b = 125g$, $c = 42f$, $d = 133f$ and $e = 91f$.

As can easily be verified, all these parametric representations of the coefficients a, b, c, d, e make the resultant vanish.

How did we find the order $n < 7$ equations in this example? In Chapter 8 we give a general procedure, which is also based on resultants, to determine all eigenvalues of n-th order $\mathcal{B}$-equations possessing a symmetry of order m, cf. Lemma 8.8. Once the eigenvalues are known, the quadratic part of the 7-th order symmetry can be calculated and by equating coefficients the equations obtained.

6.4 Integrable $\mathcal{B}$-equations of order higher than 3 and their symmetries

Abstract. Using biunit coordinates we prove the existence of integrable $\mathcal{B}$-equations at order n with symmetries of order $m \equiv 0 \bmod n$. Based on the Lech–Mahler theorem we prove that every nondegenerate $\mathcal{B}$-symmetry is contained in such a hierarchy, or in a hierarchy with orders $m \equiv 1 \bmod n-1$, or in a hierarchy of order $2,\ 3,\ 5$ or 7. Moreover, using biunit coordinates together with number theoretical arguments provided by F. Beukers, we prove that the integrable $\mathcal{B}$-equations do not have any other symmetries.

As we are now interested in equations that are not in a 1-st , 2-nd or 3-rd order hierarchy, we need to consider common factors of $\mathcal{G}$-functions of degree at least 4, cf. Theorem 6.15. The case where $a_2 = 0$ is almost trivial, the equation is integrable since $\mathcal{G}_n[a_1, 0] = a_1(\eta_1 + \eta_2)^n$ divides $\mathcal{G}_m[b_1, b_2]$ for arbitrary $m > n$ if $b_2 = 0$. In the following we assume $a_2 \neq 0$.

Lemma 6.18. *The function $\mathcal{G}_n[1 + r^n, (1 + r)^n](\eta_1, \eta_2)$ has a factor of the form*

$$(\eta_1 - r\eta_2)(r\eta_1 - \eta_2)(\eta_1 - s\eta_2)(s\eta_1 - \eta_2),\ \ s \neq r, r^{-1},$$

whenever

$$U_n(r, s) = \mathcal{G}_n[1 + r^n, (1 + r)^n](s, 1) = 0.$$

Proof. The condition $U_n(r, s) = 0$ expresses the fact that the ratio of eigenvalues of the $\mathcal{G}$-function containing zero r equals the ratio of eigenvalues of the $\mathcal{G}$-function containing zero s, cf. expression (6.4). □

As proven in [BSW98] the only factors of $\mathcal{G}$-functions (with nonzero eigenvalue) which appear on infinitely many orders have zeros forming a subset of a set of the form

$$\{0, -1, r, \frac{1}{r}, \bar{r}, \frac{1}{\bar{r}}\}. \tag{6.9}$$

Therefore, to find all hierarchies of $\mathcal{B}$-equations is to find all points r such that $U_m(r, \bar{r}) = 0$ for infinitely many integers m. At fixed order Method 6.3 can be used. By means of computer algebra, we raised the order up to 23. We did observe quite some structure in the minimal polynomials of all the points we calculated. However, a clear picture did not arose until we plotted the points in the complex plane. We have included the plot for order 23, cf. figure 6.1. Note that the upper half unit disc may be taken as a fundamental domain.

The inspection of the patterns formed by the values r obtained in this way, can be described as a form of experimental mathematics. At every fixed order n the calculated points formed a similar pattern, which inspired us to use biunit coordinates, cf. Definition 4.8 and equation (4.7).

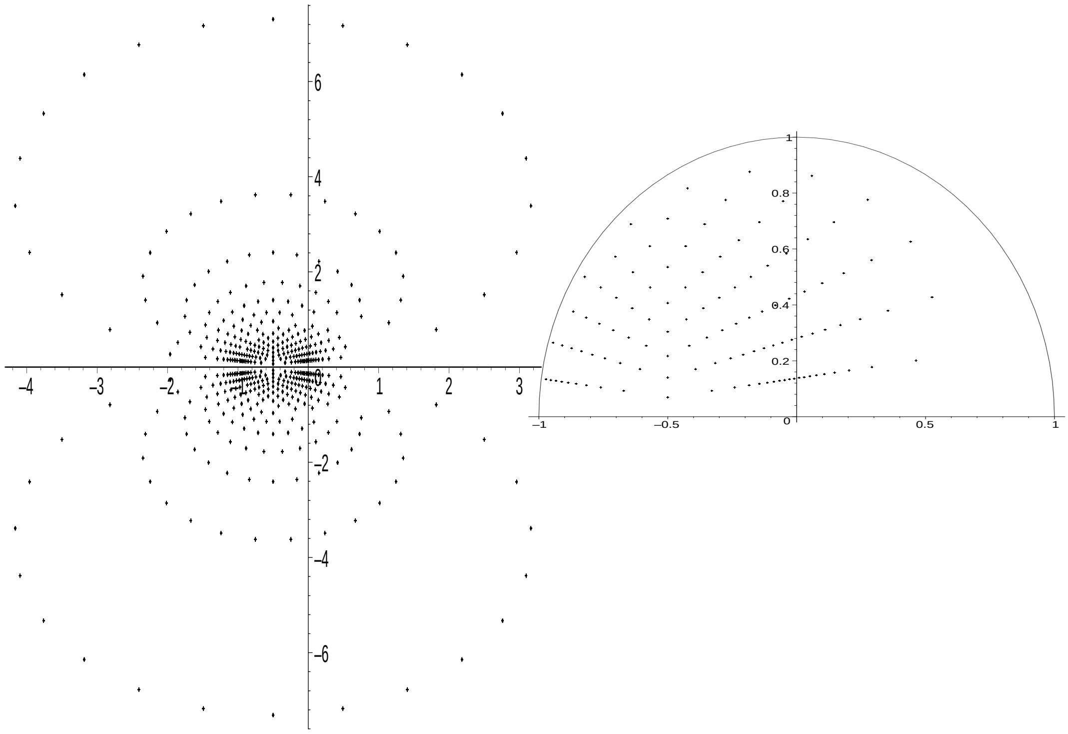

Figure 6.1: The special zeros of $\mathcal{G}$-functions of integrable equations with order 23, in the complex plane and inside the upper half unit disc form a nice pattern.

There are basically two kinds of zeros, those on the unit circle and those of the unit circle. The following theorem asserts the existence of a certain finite set of integrable equations at any order $n > 3$.

Theorem 6.19. *Let $n > 3$. To any point r in one of the sets*

1. $r \in \mathfrak{P}(\Phi_{2n}, \Phi_{2n})$ *such that* $|r| \neq 1$,
2. $r \in \Phi_{n-1}$,
3. $r \in \Phi_{2n}$ *such that* $r^n = -1$

corresponds an integrable n-th order $\mathcal{B}$-equation, which is not in a hierarchy of order smaller than 4.

Proof. 1. For $r \in \mathfrak{P}(\Phi_{2n}, \Phi_{2n})$ the proof consists of showing that $U_m(r, \bar{r})$ has infinitely many solutions m including $m = n$. By substitution of $r = a\psi = b\phi - 1$, with $\psi, \phi \in \Phi_{2n}$, in $U_m(r, \bar{r})$ we get

$$U_m(\psi, \phi) = (b\bar{\phi})^m + (ab\psi\bar{\phi})^m - (b\phi)^m - (ab\bar{\psi}\phi)^m.$$

This vanishes when $m \equiv 0 \bmod n$. Note that when n is odd and $\eta_1 + \eta_2$ does not divide the quadratic part of the equation, no symmetries appear at even order. When r is real or on the unit circle the set $\{r, 1/r, \bar{r}, 1/\bar{r}\}$ does not contain 4 elements.

2. The $(n-1)$-th roots of unity are all double zeros of $\mathcal{G}_n$. They appear in conjugated pairs and are double zeros at order $m \equiv 1 \bmod n-1$ as well, cf. Lemma 6.1. A real zero and its conjugate do not form a pair.

3. All odd powers of a primitive $(2n)$-th root of unity are mapped to zero for all $m \equiv n \bmod 2n$.

□

The following theorem asserts that there are no 'new' integrable $\mathcal{B}$-equations, i.e., equations that do not commute with an integrable $\mathcal{B}$-equation we have proven to exist in Theorem 6.19.

Theorem 6.20. *Any integrable $\mathcal{B}$-symmetry is a symmetry of*

- *a $\mathcal{B}$-equation described in Theorem 6.19, or*
- *a 1-st, 2-nd or 3-rd order $\mathcal{B}$-equation, or*
- *a 5-th or 7-th order $\mathcal{B}$-equation with equal eigenvalues, or*

Proof. Suppose the eigenvalues are nonzero. Let H be a divisor of infinitely many $\mathcal{G}_m$. Any set of zeros Z of H is a subset of a set of the form (6.9). If $0 \in Z$, the eigenvalues of the equation are equal. It follows from Theorem 5.8 that the equation is a symmetry of a $\mathcal{B}$-equation of order $2, 3, 5$ or 7. If

$$Z \subset \{-1, r, \frac{1}{r}\}$$

and the multiplicity of r is 1, the equation is a symmetry of a $\mathcal{B}$-equation of order $1, 2$ or 3, cf. Theorem 6.15. If

$$Z \subset \{-1, r, \frac{1}{r}\}$$

and the multiplicity of r is 2, r is a root of unity. If

$$Z \subset \{-1, r, \frac{1}{r}, \bar{r}, \frac{1}{\bar{r}}\}$$

the biunit coordinates of r are roots of unity because otherwise none of the pairs

$$\frac{r}{\bar{r}}, \frac{1+r}{1+\bar{r}} \text{ or } r\bar{r}, \frac{1+r}{1+\frac{1}{\bar{r}}}$$

are roots of unity. □

The following theorem asserts that any integrable $\mathcal{B}$-equation has no other $\mathcal{B}$-symmetries than the symmetries we have proven to exist.

Theorem 6.21. *If the integrable $\mathcal{B}$-equation*

$$\begin{cases} u_t = & (1+r^n)u_n + K \\ v_t = & (1+r)^n v_n \end{cases}$$

is not in a lower hierarchy it has no $\mathcal{B}$-symmetries other than the symmetries on order m, with:

1. $m \equiv 0 \mod n$ *if* $r \in \mathfrak{P}(\Phi_{2n}, \Phi_{2n})$, *and* $2|n$ *or* $2 \not| n,\ \eta_1 + \eta_2 | \widehat{K}$,
2. $m \equiv n \mod 2n$ *if* $r \in \mathfrak{P}(\Phi_{2n}, \Phi_{2n})$, *and* $2 \not| n,\ \eta_1 + \eta_2 \not| \widehat{K}$,
3. $m \equiv 1 \mod n-1$ *if* $r \in \Phi_{n-1}$,
4. $m \equiv n \mod 2n$ *if* $r \in \Phi_{2n}$.

Proof. From the proof of Theorem 6.19, we know that symmetries exist at these orders. We now prove that the equations do not have any other symmetries.

1. We write $U_m(r, \bar{r})$ in terms of ψ and ϕ using equations 4.7 and 4.11. Furthermore, we perform the transformations

   $$\psi^2 \to \mu\nu,\ \phi^2 \to \nu.$$

   Thus, we obtain the Diophantine equation

   $$\left(\frac{1-\mu}{1-\nu}\right)^m = \frac{1-\mu^m}{1-\nu^m} \tag{6.10}$$

   for roots of unity μ, ν. By Theorem E.1, under the conditions

   $$\mu, \nu \neq \pm 1,\ \mu \neq \nu, \bar{\nu},\ \mu^m, \nu^m \neq 1,$$

   the equation (6.10) has no solution unless $m = 1$. We check the conditions. When $\mu = -1$ we find that $\phi = \pm i\psi$, we have

   $$|r + \frac{1}{2}| = \frac{1}{2}.$$

   In this case equation (6.10) reduces to

   $$\begin{aligned} \nu^m &= 1, && \text{when } m \text{ even}, \\ (1-\nu)^m &= 2^{m-1}(1-\nu^m), && \text{when } m \text{ odd}, \end{aligned}$$

   with $\nu \neq \pm 1$ a root of unity. The same equation, in μ instead of ν is obtained when $\nu = -1$, i.e., when $\phi = \pm i$ or

   $$r + \bar{r} = -2.$$

By Proposition E.3 the equation for odd order has no solutions $m > 1$. For the even solutions, note that we are in the case where n is even. The equation is not in a lower hierarchy if ψ is a primitive $(2n)$-th root of unity. This implies that ν is a primitive n-th root of unity and the even solutions are given by $m \equiv 0 \bmod n$.

2. When n is odd and $\eta_1 + \eta_2$ does not divide $\widehat{K}$ there is no symmetry at any odd order since $\eta_1 + \eta_2$ does not divide $\mathcal{G}_{2m+1}$.

3. When $m \not\equiv 1 \bmod n-1$ the point $r \in \Phi_{n-1}$ is not a double zero of $\mathcal{G}_m$.

4. Two $(2n)$-th roots of unity $r = \psi, s = \phi$ are both zeros of $\mathcal{G}_m$ if $U_m(\psi, \phi) = 0$. By applying the transformation

$$\psi \to -\mu,\ \phi \to -\nu$$

we obtain

$$\left(\frac{1-\mu}{1-\nu}\right)^m = \frac{1+(-\mu)^m}{1+(-\nu)^m}$$

for $(2n)$-th roots of unity μ, ν. Suppose that

$$\mu, \nu \neq -1,\ \mu \neq \nu, \bar{\nu}.$$

Then, by Theorem E.1, the equation has no odd solutions $m > 1$ such that $\mu^m, \nu^n \neq 1$. For even m we use Theorem E.4, which states that

$$\left(\frac{1-\mu}{1-\nu}\right)^m = \frac{1+\mu^m}{1+\nu^m}$$

has no solutions $m > 1$ such that $\mu^m, \nu^n \neq -1$.

□

6.4.1 Quadratic part of the integrable $\mathcal{B}$-equations

Abstract. We describe the quadratic part of the integrable $\mathcal{B}$-equations and we show that the equations are real (up to a complex scaling).

If $a_1 = 0$ then K can be anything because the $\mathcal{G}$-function of the equation divides the $\mathcal{G}$-functions of all the symmetries. Take $a_1 \neq 0$. Let Q be the greatest common divisor of

$$\mathcal{G}_n[1 + r^n, (1+r)^n]$$

and

$$\eta_1\eta_2(\eta_1 + \eta_2)(\eta_1 - r\eta_2)(r\eta_1 - \eta_2)(\eta_1 - \bar{r}\eta_2)(\bar{r}\eta_1 - \eta_2).$$

Q is the common factor of all $\mathcal{G}$-functions of the symmetries. The quadratic part of the equation can be written as:

$$\widehat{K} = \frac{\mathcal{G}_n}{Q} P,$$

with P an arbitrary symmetric polynomial that does not have Q as a divisor.

Although in the analysis complex roots of unity play an important role, in the end the equations turn out to be real.

Proposition 6.22. *All integrable $\mathcal{B}$-equations with nonzero eigenvalues are real (up to a complex scaling).*

Proof. Since

$$\frac{a_1}{a_2} = \frac{1+r^n}{(1+r)^n} = \frac{1+\bar{r}^n}{(1+\bar{r})^n} = \frac{\bar{a}_1}{\bar{a}_2}$$

all ratios of eigenvalues of integrable $\mathcal{B}$-equations are real valued. What about the quadratic part? We have

$$\begin{aligned}(\eta_1 - r\eta_2)(r\eta_1 - \eta_2)(\eta_1 - \bar{r}\eta_2)(\bar{r}\eta_1 - \eta_2) \quad &= \quad r\bar{r}(\eta_1^4 + \eta_2^4) \\ &\quad +(r\bar{r}+1)(r+\bar{r})\eta_1\eta_2(\eta_1^2+\eta_2^2).\end{aligned}$$

Hence Q is real. Therefore the quadratic part is real if the eigenvalues and P are chosen to be real. □

We demonstrate our method by calculating an integrable equation together with its first higher order symmetry.

Example 6.23. *Take $n = 6$. The line*

$$\mathbb{R}e^{\frac{1}{3}\pi i} - 1$$

intersects the imaginary axis in the point

$$r = \sqrt{3}i.$$

This is a zero of the $\mathcal{G}$-function $\mathcal{G}_6[-13, 32]$, since

$$\frac{1+r^6}{(1+r)^6} = -\frac{13}{32}.$$

The polynomial dividing all $\mathcal{G}$-functions of the symmetries is

$$Q = (3\eta_1^2 + \eta_2^2)(\eta_1^2 + 3\eta_1^2).$$

The quadratic part of the equation is, with $P = \frac{1}{2}$,

$$K = \frac{\mathcal{G}_6[-13, 32]}{2Q} = \frac{15}{2}(\eta_1^2 + \eta_2^2) + 13\eta_1\eta_2.$$

Therefore the equation

$$\left\{\begin{array}{rl} u_t = & -13u_6 + 15vv_2 + 13v_1^2 \\ v_t = & 32v_6 \end{array}\right.$$

is integrable. Let us calculate the symmetry $\mathcal{B}_{12}[b_1, b_2](S)$. To obtain the eigenvalues we compute

$$\frac{b_1}{b_2} = \frac{1 + r^{12}}{(1 + r)^{12}} = \frac{365}{2048}.$$

The symbolic quadratic part is given by

$$\widehat{S} = \frac{\mathcal{G}_m[365, 2048]}{\mathcal{G}_6[-13, 32]} \widehat{K}.$$

By applying the inverse Gel'fand and Dikiĭ transformation we obtain the symmetry at order 12:

$$\left[\begin{array}{c} 365u_{12} + 561vv_8 - 1460v_1v_7 - 9900v_2v_6 - 21900v_3v_5 - 13893v_4^2 \\ 2048v_{12} \end{array} \right].$$

6.5 $\mathcal{B}$-equations in a lower hierarchy

Abstract. We describe all $\mathcal{B}$-equations that belong to a lower hierarchy. This solves the recognition problem.

Let K be the quadratic part of an integrable $\mathcal{B}$-equation. Let k be the degree of

$$Q = \frac{\mathcal{G}_n}{\gcd(\mathcal{G}_n, \widehat{K})}.$$

If the equation is nondegenerate we have $0 < k < 8$. If $k < 4$ or $k = 7$ the equation is in a k-th order hierarchy. It never happens that $k = 6$ because whenever $\mathcal{G}_6$ has zeros 0 and r it does not have $\bar{r}$ as zero. The remaining cases are enumerated as in Theorem 6.19.

1. Let $r \neq -1$ be a zero of the polynomial Q. It has biunit coordinates (ζ^a, ζ^b) where ζ is a primitive $(2n)$-th root of unity. The equation is in a hierarchy of order d, with d (> 3) a divisor of n, if a/n and b/n are integer multiples of $1/d$.

2. The equation is in a hierarchy of order $d + 1$, with d (> 2) a divisor of $n - 1$, if the double zero r is a d-th root of unity.

3. When $n = lm$ with l odd and $m > 3$ the equation can be in a m-th order hierarchy. This is the case if $\mathcal{G}_n/\mathcal{G}_m$ divides $\widehat{K}$.

6.6 The number of integrable $\mathcal{B}$-equations

Abstract. We present formulas for the number of n-th order integrable equations and for the number of n-th order integrable equations that are not in a lower order hierarchy of order higher than 3.

1. The number of points $r \in \mathfrak{P}(\Phi_{2n}, \Phi_{2n})$ leading to different eigenvalues of integrable equations is

$$f(n) = \begin{cases} \frac{(n-2)^2}{4} & \text{if } n \text{ is even} \\ \frac{(n-1)(n-3)}{4} & \text{if } n \text{ is odd} \end{cases}$$

We count the number of points in the upper half plane (because conjugation leaves the set invariant) excluding the points on the unit circle where $\bar{r} = r^{-1}$. Put $\zeta = e^{\frac{1}{n}\pi i}$. The imaginary part of $\mathfrak{P}(\zeta^a, \zeta^b)$ is positive only when $0 < b < a$. There are exactly

$$\sum_{a=1}^{n-2} a$$

such points. A point is on the unit circle when the angle of the line through 0 is twice the angle of the line through -1. The set Φ_{n-1} contains

$$\lfloor \frac{n-1}{2} \rfloor$$

points on the upper half unit circle. Subtracting these two numbers and dividing by 2 (because inversion leaves the set invariant) gives the desired number.

If $g(n)$ is the number of these integrable equations not in the hierarchy of an other equation we have

$$f(n) = \sum_{d|n} g(d)$$

and by Möbius' inversion

$$g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d}),$$

with the Möbius function defined as follows: (p_i are prime)

$$\mu(\prod_{i=1}^{j} p_i^{\alpha_i}) = \begin{cases} 1 & \text{if } \alpha_i = 0 \text{ for all } i, \\ 0 & \text{if } \alpha_i > 1 \text{ for some } i, \\ (-1)^j & \text{if } \alpha_i = 1 \text{ for all } i. \end{cases}$$

2. The number of complex $(n-1)$-th roots of unity giving different eigenvalues at fixed n is

$$f(n) = \begin{cases} \frac{n-2}{2} & \text{if } n \text{ is even} \\ \frac{n-3}{2} & \text{if } n \text{ is odd} \end{cases}$$

We counted the zeros that are above the real line.

If $g(n)$ is the number of these integrable equations not in the hierarchy of an other we have

$$f(n) = \sum_{d|n-1} g(d+1)$$

and by Möbius' inversion

$$g(n) = \sum_{d|n-1} \mu(d) f(\frac{n-1}{d} + 1).$$

3. The last case is concerned with a vanishing first eigenvalue. When n is prime, twice a prime or a power of 2 , the equation can not be in a lower hierarchy.

At order 5 there is the extra equation with eigenvalue 1. Its $\mathcal{G}$-function has the set of zeros

$$\{0, -1, \zeta_3, \zeta_3^2\}.$$

Thus, there are exactly

$$\begin{array}{rll} n(n-2)/4 & \text{when} & n \text{ even}, \\ (n+1)(n-3)/4 & \text{when} & n \text{ odd}, \\ 4 & \text{when} & n = 5 \end{array}$$

nondegenerate n-th order integrable $\mathcal{B}$-equations.

Finally, the number of n-th order integrable equations that are not in a lower hierarchy with $3 < n < 24$ is given in Table 6.1.

n	4	5	6	7	8	9	10	11	12	13
#	3	5	7	8	12	15	18	23	26	33
n	14	15	16	17	18	19	20	21	22	23
#	37	44	45	61	57	76	74	89	87	116

Table 6.1: The number of integrable equations not in a lower hierarchy with orders between 3 and 24.

Chapter 7

On the spectrum of integrable equations

The classification of integrable $\mathcal{B}$-equations is based on the knowledge of all divisors H of $\mathcal{G}_{1,n}^{-1,2}$ that are divisors of infinitely many $\mathcal{G}_{1,m}^{-1,2}$. For this particular kind of equation, a hierarchy of integrable equations corresponds to any such divisor. The results also have an immediate implication for the general N-component equation. If part of a diagonal equation is a non-integrable $\mathcal{B}$-equation, the equation is not integrable. In other words, by the classification of $\mathcal{B}$-equations we have obtained a condition on the spectrum of more general equations and on the order of their symmetries. Other conditions on the spectrum or on the order of the symmetries can be obtained by requiring terms with other gradings to be nonvanishing.

7.1 Nonvanishing terms linear in both u_k and v_l

Abstract. We give all divisors H of

$$\mathcal{G}_{1;n}^{0,1}[a_1, a_2] = a_1 \left(\xi_1^n - (\xi_1 + \eta_1)^n\right) + a_2 \eta_1^n$$

such that there are infinitely many $m \in \mathbb{N}$ and $b_1, b_2 \in \mathbb{C}$ for which H divides $\mathcal{G}_{1;m}^{0,1}[b_1, b_2]$. We assume that $a_1, a_2 \in \mathbb{C}$ are not both zero.

If a $\mathcal{G}$-function of a given equation does not have a common divisor with the symbolic expression for the corresponding nonlinear part, then it should have a common divisor with the $\mathcal{G}$-function of any symmetry of the equation.

This observation leads to the following problems: How can we determine all common divisors of $\mathcal{G}_{k,n}^{i,j-i}$ with infinitely many $\mathcal{G}_{k,m}^{i,j-i}$ for fixed i, j and k. Note that we may take $k = 1$ since the results for $k = 2$ follow from relation (4.5). Moreover, when $k = 1,\ i = j$ the problem is basically the same as a problem that was solved for the classification of integrable scalar equations. Here no conditions on the spectrum are obtained. Thus for $j = 1$ the only case to be considered is $k = 1,\ i = 0$, which we will do in this section.

We start by assuming that one of the eigenvalues a_1 or a_2 is zero.

Proposition 7.1 ($a_1 = 0$)**.** *For every $m \geq n$ and $b_2 \in \mathbb{C}$ the $\mathcal{G}$-function $\mathcal{G}^{0,1}_{1;n}[0, a_2]$ divides $\mathcal{G}^{0,1}_{1;m}[0, b_2]$.*

Proof. We have

$$\frac{\mathcal{G}^{0,1}_{1;m}[0, b_2]}{\mathcal{G}^{0,1}_{1;n}[0, a_2]} = \frac{b_2}{a_2}\eta_1^{m-n}.$$

□

Proposition 7.2 ($a_2 = 0$)**.** *For every $m \equiv 0$ mod n and $b_1 \in \mathbb{C}$ the $\mathcal{G}$-function $\mathcal{G}^{0,1}_{1;n}[a_1, 0]$ divides $\mathcal{G}^{0,1}_{1;m}[b_1, 0]$.*

Proof. We have

$$\left(\xi_1^{kn} - (\xi_1 + \eta_1)^{kn}\right) = \left(\xi_1^n - (\xi_1 + \eta_1)^n\right)\left(\sum_{i=0}^{k-1} \xi_1^{ni}\eta_1^{n(k-i-1)}\right).$$

□

When n is odd, the problem of finding all the divisors that appear infinitely many times is solved by relating it to the $\mathcal{G}$-function we treated in the preceding chapter. For odd n we have

$$\mathcal{G}^{0,1}_{1;n}[a_1, a_2](r, 1) = -(1 + r)^n \mathcal{G}^{-1,2}_{1;n}[a_2, a_1](1, f_4(r)), \tag{7.1}$$

where f_4 is the anharmonic transformation, cf. 4.9,

$$f_4 : r \to -\frac{r}{1 + r}.$$

To see what is going on we will treat the cases $n < 4$ without using the relation 7.1. We will find that our $\mathcal{G}$-function divides infinitely many higher $\mathcal{G}$-functions for any value of a_1, a_2 but in the case $n = 3$ only at odd order. Note that this resembles the situation in Section 6.2.

Proposition 7.3 ($n = 1$)**.** *Let $n = 1$ and $a_1 \neq a_2$. Then for every $m \in \mathbb{N}$ and $b_1, b_2 \in \mathbb{C}$ the $\mathcal{G}$-function $\mathcal{G}^{0,1}_{1;n}[a_1, a_2]$ divides $\mathcal{G}^{0,1}_{1;m}[b_1, b_2]$.*

Proof. We have

$$\mathcal{G}^{0,1}_{1;n}[a_1, a_2] = (a_2 - a_1)\eta_1$$

and

$$\mathcal{G}^{0,1}_{1;m}[b_1, b_2](\xi_1, 0) = 0.$$

□

In the sequel we will assume that both the eigenvalues are nonzero and that the degree of the common divisor is higher than 1.

Proposition 7.4 ($n = 2$)**.** *For every $m > 1$ there are $b_1, b_2 \in \mathbb{C}$ such that $\mathcal{G}^{0,1}_{1;2}[a_1, a_2]$ divides $\mathcal{G}^{0,1}_{1;m}[b_1, b_2]$.*

Proof. The point $(r, 1)$ is a solution to $\mathcal{G}^{0,1}_{1;2}[a_1, a_2] = 0$ if

$$r = \frac{a_2 - a_1}{2a_1}.$$

It is a zero of $\mathcal{G}^{0,1}_{1;m}[b_1, b_2]$ when

$$\frac{b_1}{b_2} = \frac{1}{(1 + r)^m - r^m}.$$

□

Proposition 7.5 ($n = 3$)**.** *For every odd m, there are b_1 and b_2 such that $\mathcal{G}^{0,1}_{1;3}[a_1, a_2]$ is a divisor of $\mathcal{G}^{0,1}_{1;m}[b_1, b_2]$.*

Proof. The expression for the ratio b_1/b_2 is invariant under

$$f_3 : r \to -1 - r$$

if m is odd. Therefore in this case the point $(-1 - r, 1)$ is a zero of $\mathcal{G}^{0,1}_{1;m}[b_1, b_2]$ if $(r, 1)$ is. □

Thus, we are in the following situation: any common divisor H of $\mathcal{G}^{0,1}_{1;n}[a_1, a_2]$ and $\mathcal{G}^{0,1}_{1;m}[b_1, b_2]$ of degree 2 divides a $\mathcal{G}$-function of order 2. If H has degree 3 and is of the form

$$\eta_1(\xi_1 - r\eta_1)(\xi_1 + (1 + r)\eta_1)$$

it divides a $\mathcal{G}$-function of order 3. We will describe all common divisors H of $\mathcal{G}^{0,1}_{1;n}[a_1, a_2]$ and infinitely many $\mathcal{G}^{0,1}_{1;m}[b_1, b_2]$ which have degree higher than 2 and are not of the form above. We start with a description of the zeros with higher multiplicity.

Proposition 7.6. *Suppose that $\mathcal{G}^{0,1}_{1;n}[a_1, a_2]$ has a multiple zero $(r, 1)$. Then we have*

$$r = \frac{\zeta}{\zeta - 1},$$

where $\zeta \in \Phi_{n-1}$. If n is odd we also have $r = -1/2$. The ratio of eigenvalues is

$$\frac{a_1}{a_2} = (1 + r)^{1-n}.$$

The multiple zero has multiplicity two. If n is even it is the only multiple zero and if n is odd there is one other double zero $(-1 - r, 1)$.

Proof. We solve the simultaneous equations

$$\mathcal{G}^{0,1}_{1;n}[a_1, a_2](r, 1) = \partial_r \mathcal{G}^{0,1}_{1;n}[a_1, a_2](r, 1) = 0$$

in r. Explicitly:

$$a_2 = a_1 \left((1 + r)^n - r^n\right) \text{ and } r^{n-1} = (1 + r)^{n-1}.$$

From the last equation we get $r = \zeta(r + 1)$ where $\zeta^{n-1} = 1$. By taking $\zeta = -1$ we have $r = -\frac{1}{2}$, which is a double root only if n is odd. When $\zeta \neq 1$ we have

$$r = \frac{\zeta}{1 - \zeta}.$$

In particular we have $|r| = |r + 1|$, which implies that $\Re(r) = -\frac{1}{2}$. By substituting the second equation into the first equation we get

$$\frac{a_1}{a_2} = (1 + r)^{1-n}.$$

Consider the second derivative

$$(\partial_r)^2 \mathcal{G}^{0,1}_{1;n}[a_1, a_2](r, 1) = a_1 n(n - 1) \left((r + 1)^{n-2} - r^{n-2}\right).$$

It has no simultaneous zero with the first derivative. Therefore $(r, 1)$ is a double zero. Suppose that we have a second double zero $(s, 1)$. Then we have

$$\frac{a_1}{a_2} = (1 + r)^{1-n} = (1 + s)^{1-n},$$

and hence $|1 + r| = |1 + s|$. Together with $\Re(r), \Re(s) = -\frac{1}{2}$ this implies that $s = r$ or $s = \bar{r}$. Note that if $\Re(r) = -\frac{1}{2}$ we have $\bar{r} = -1 - r$. □

Next we assume that our common divisor H has (at least) two zeros $(r, 1)$ and $(s, 1)$ where $s \neq r, -1 - r$. When H divides $\mathcal{G}^{0,1}_{1;m}[a_1, a_2]$ we have

$$(1 + r)^m - r^m - (1 + s)^m + s^m = 0. \tag{7.2}$$

Since H should divide infinitely many other $\mathcal{G}$-functions, equation (7.2) should have infinitely many integer solutions m. By Corollary D.4 at least one of the pairs

$$\frac{1 + r}{1 + s}, \frac{r}{s} \text{ or } \frac{1 + r}{s}, \frac{r}{1 + s}$$

consists of roots of unity. If

$$\frac{1 + r}{1 + s}, \frac{r}{s}$$

are roots of unity we have in particular that $|r| = |s|$ and $|r + 1| = |s + 1|$. This implies, see Figure 7.1, that $s = \bar{r}$ since we assumed $s \neq r$. If

$$\frac{1 + r}{s}, \frac{r}{1 + s}$$

are roots of unity we have in particular that $|r + 1| = |s|$ and $|r| = |s + 1|$. This implies, see Figure 7.1, that $s = -\bar{r} - 1$ since we assumed $r \neq -s - 1$.

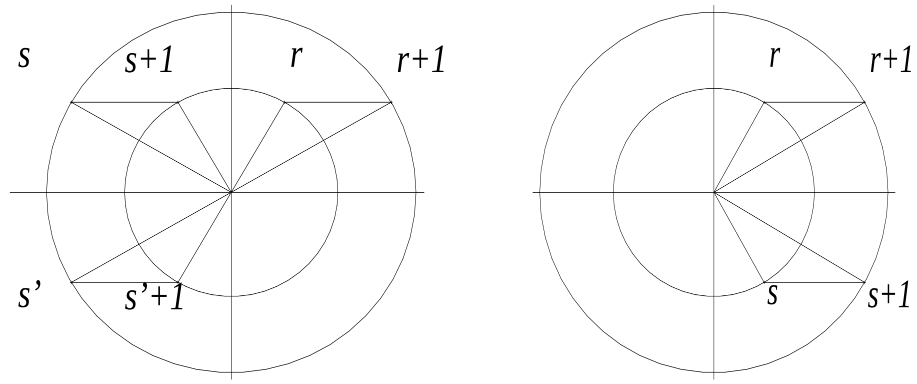

Figure 7.1: Points in the complex plane, we see that $s = \bar{r}$ or $s = -1 - \bar{r}$.

We first treat the cases in which the order is odd. Recall that if n is odd, $(-\bar{r} - 1, 1)$ is a zero of the $\mathcal{G}$-function if $(\bar{r}, 1)$ is.

Theorem 7.7. *Let n be odd. Equation (7.2) with*

$$s \in \{\bar{r}, -1 - \bar{r}\}, \ s \notin \{r, -1 - r\}$$

has infinitely many solutions $m \equiv 0$ mod n when

$$r \in \mathfrak{P}(\Phi_{2n}, \Phi_{2n}), \ r + \bar{r} \neq -1.$$

There are no other solutions $m \in \mathbb{N}$.

Proof. This follows from relation (7.1) and Theorems 6.19, 6.21. The set $\mathfrak{P}(\Phi_m, \Phi_m)$ is invariant under $\mathfrak{A}$ since it is invariant under the generators f_2, f_3 which is clear from the biunit coordinate descriptions (4.10). The image of the unit circle, the exceptional case in Theorem 6.19, is the line $r + \bar{r} = -1$. □

We will find a bigger set of points at even orders. When $s = \bar{r}$ the same set is obtained and $s = -1 - \bar{r}$ gives an extra set.

Theorem 7.8. *Let n be even. Equation (7.2) with*

$$s = \bar{r}, \ s \neq r, -1 - r$$

has infinitely many solutions $m \equiv 0$ mod n when

$$r \in \mathfrak{P}(\Phi_{2n}, \Phi_{2n}), \ r + \bar{r} \neq -1.$$

There are no integer solutions other than $m \equiv 0$ mod n.

Proof. We use biunit coordinates (ψ, ϕ) to describe r. For some $\alpha, \beta \in \mathbb{R}$ we have

$$r = \alpha\psi = \beta\phi - 1 \text{ and } s = \bar{r} = \alpha\frac{1}{\psi} = \beta\frac{1}{\phi} - 1.$$

The left hand side of equation (7.2) becomes

$$(\beta\phi)^m - (\alpha\psi)^m - (\beta\phi^{-1})^m + (\alpha\psi^{-1})^m.$$

This vanishes when $m \equiv 0 \bmod n$ and $\psi^{2n} = \phi^{2n} = 1$. When $\psi, \phi = \pm 1$ we have $s = r$. Hence we have obtained

$$r \in \mathfrak{P}(\Phi_{2n}, \Phi_{2n}).$$

When $r + \bar{r} = -1$ we have $s = -1 - r$ which we excluded in the theorem. This proves the first part of our theorem.

By Corollary D.4 the ratios

$$\frac{r}{s}, \ \frac{1+r}{1+s}$$

are roots of unity. Therefore ψ, ϕ are roots of unity. By using the formula (4.7) and performing the transformation

$$\phi^2 = \nu, \ \psi^2 = \mu,$$

equation (7.2) becomes:

$$\left(\frac{1-\mu}{1-\nu}\right)^m = \frac{1-\mu^m}{1-\nu^m}.$$

By Theorem E.1 this equation has no solution unless $m = 1$ if

$$\mu, \nu \neq \pm 1, \ \mu \neq \nu, \nu^{-1}, \ \mu^m, \nu^m \neq 1.$$

We check these conditions.

- To $\mu, \nu = 1$ corresponds $\psi, \phi = \pm 1$. The point r is real, which is excluded in the theorem.
- To $\mu = \nu$ corresponds $\psi = \pm\phi$. The lines $\alpha\psi, \beta\phi - 1$ do not intersect.
- To $\mu = \nu^{-1}$ corresponds $\psi = \pm\phi^{-1}$. The point r has real part $-\frac{1}{2}$ which is excluded in the theorem.
- To $\mu, \nu = -1$ corresponds $\psi, \phi = \pm i$ where $i^2 = -1$. The point r satisfies $r + \bar{r} = 0$ or $r + \bar{r} = -2$.The equation reduces to

  $$(1-\mu)^n = 2^{n-1}(1-\mu^n)$$

  and, in both cases, Proposition E.3 is used.

□

Proposition 7.9. *If n is even the points $(r, 1)$, with*

$$r \in \mathfrak{P}(\Phi_{2n}, \Phi_{2n}),\ r + \bar{r} = -1$$

are zeros of $\mathcal{G}^{0,1}_{1;n}[a_1, 0]$.

Proof. For such a point we have $\phi = \pm\psi^{-1}$, hence r is given by

$$r(\psi, \pm\psi^{-1}) = -\frac{\psi^2}{1 + \psi^2},$$

where $\psi \in \Phi_{2n}$. With $\zeta = -\psi^2$ we obtain

$$r = \zeta(1 + r).$$

If n is even we have $\zeta \in \Phi_n$. By taking the n-th power we obtain

$$r^n = (1 + r)^n.$$

□

Theorem 7.10. *Let n be even. Equation (7.2) with*

$$s = -1 - \bar{r},\ s \neq r$$

has infinitely many solutions $m \equiv n$ mod $2n$ when

$$r \in \mathfrak{P}(\Phi_{4n} \setminus \Phi_{2n}, \Phi_{4n} \setminus \Phi_{2n}),\ r + \bar{r} \neq -1$$

There are no other solutions $m > 1$.

Proof. We use biunit coordinates (ψ, ϕ) to describe r. For some $\alpha, \beta \in \mathbb{R}$ we have

$$r = \alpha\psi,\ 1 + r = \beta\phi,\ s = -\beta\frac{1}{\phi},\ 1 + s = -\alpha\frac{1}{\psi}.$$

The left hand side of equation (7.2) becomes

$$(\beta\phi)^m - (\alpha\psi)^m - (-\alpha\psi^{-1})^m + (-\beta\phi^{-1})^m.$$

This vanishes when $m \equiv n$ mod $2n$ and $\psi^{2n} = \phi^{2n} = -1$. Thus we have obtained

$$r \in \mathfrak{P}(\Phi_{4n} \setminus \Phi_{2n}, \Phi_{4n} \setminus \Phi_{2n}).$$

When $r + \bar{r} = -1$ we have $s = r$ which we excluded in the theorem.

The ratios

$$\frac{1 + r}{s}, \frac{r}{1 + s}$$

are roots of unity. Therefore ψ, ϕ are roots of unity. By using the formula (4.7) and performing the transformations

$$\psi^2 = \mu,\ \phi^2 = \nu,$$

equation (7.2) becomes
$$\left(\frac{1-\mu}{1-\nu}\right)^m = \frac{1+(-\mu)^m}{1+(-\nu)^m}. \tag{7.3}$$
We have
$$\mu, \nu \neq \pm 1, \mu \neq \nu, \nu^{-1}.$$
This is seen as follows:

- to $\mu, \nu = 1$ corresponds $\psi, \phi = \pm 1$. Then $\psi, \phi \notin \Phi_{4n} \setminus \Phi_{2n}$.
- to $\mu = \nu$ corresponds $\psi = \pm\phi$. Here the lines $\alpha\psi, \beta\phi - 1$ do not intersect.
- to $\mu = \nu^{-1}$ corresponds $\psi = \pm\phi^{-1}$. The point r has real part $-\frac{1}{2}$ which is excluded in Theorem 7.10.
- to $\mu, \nu = -1$ corresponds $\psi, \phi = \pm i$ where $i^2 = -1$. Then $\psi, \phi \notin \Phi_{4n} \setminus \Phi_{2n}$.

By Theorem E.1 the odd solutions $m > 1$ to equation (7.3) satisfy
$$\mu^m = 1, \ \nu^m = 1$$
which does not happen when $\psi, \phi \in \Phi_{4n} \setminus \Phi_{2n}$. By Theorem E.4 the even solutions $m > 1$ to equation (7.3) satisfy
$$\mu^m = -1, \ \nu^m = -1$$
yielding no other solutions than $m \equiv n \bmod 2n$. Hence, the last statement in the theorem is proved. □

To see where we are and where we are going, we will summerise and give a short outlook. All greatest common divisors of $\mathcal{G}^{i,1-i}_{1,n}$ with infinitely many $\mathcal{G}^{i,1-i}_{1,m}$ were described:

- in Theorem 5.8 for $i = 1$,
- in Theorem 6.19 for $i = -1, n > 3$,
- in Theorem 7.7 for $i = 0, n > 3, n$ odd,
- in Theorems 7.8 and 7.10 for $i = 0, n > 3, n$ even.

We immediately obtain similar results on the common divisors of $\mathcal{G}^{i,1-i}_{2,m}$-functions, since by the relation (4.5), any $\mathcal{G}^{i,1-i}_{2;m}$-function is related to a $\mathcal{G}^{1-i,i}_{1;m}$-function.

What we have solved is the problem of finding the possible eigenvalues for which the equation can be integrable if we require the nonvanishing of the k-th component of u-grading i and v-grading $1 - i$. No restriction on the eigenvalues of 1-st, 2-nd or 3-rd order equation was obtained.

The next step is to require the nonvanishing of the k-th component of u-grading i and v-grading $1 - i$ together with the nonvanishing of the l-th component of u-grading j and v-grading $1 - j$, where of course $i \neq j$ if $k = l$.

7.2 Nonvanishing quadratic terms with different gradings

Abstract. We first consider equations that have nonvanishing terms $K_1^{-1,2}$ and $K_2^{1,0}$. This leads to an equation of Lech–Mahler type with four terms, which will be solved in a similar way as the previous problems. New conditions on the spectrum are obtained when n is even.

The other combinations of nonvanishing quadratic terms of different type yield equations of Lech–Mahler type with five terms. These problems will be solved using the algorithm of Smyth. In particular, a finite number of solutions will be obtained.

7.2.1 Nonvanishing terms $K_1^{-1,2}$ and $K_2^{1,0}$

We give all divisors H_1 of $\mathcal{G}_{1;n}^{-1,2}[a_1, a_2]$ and H_2 of $\mathcal{G}_{2;n}^{1,0}[a_1, a_2]$ such that infinitely many m and b_1, b_2 exist for which H_1 divides $\mathcal{G}_{1;m}^{-1,2}[b_1, b_2]$ and H_2 divides $\mathcal{G}_{2;m}^{1,0}[b_1, b_2]$.

When the order is odd, no new conditions on the spectrum are obtained since the relations (4.5) and (7.1) imply that

$$\mathcal{G}_{2;n}^{1,0}[a_1, a_2](1, r) = -(1 + r)^n \mathcal{G}_{1;n}^{-1,2}[a_1, a_2](1, f_4(r)).$$

Therefore we turn to the even order case. Let $r, s \in \mathbb{C}$ be such that

$$\mathcal{G}_{1;m}^{-1,2}[b_1, b_2](1, r) = \mathcal{G}_{2;m}^{1,0}[b_1, b_2](1, s) = 0.$$

Then we have

$$1 + r^m + (s(1 + r))^m - ((1 + s)(1 + r))^m = 0. \tag{7.4}$$

An integrability condition for the equation at order n is that this equation has infinitely many integer solutions m including n.

Theorem 7.11. *Let n be even. Suppose that $r, s \notin \{0, -1\}$. If equation (7.4) has infinitely many solutions $m > 1$, including $m = n$, these are exhaustively given by $m \equiv n \bmod 2n$ and we have*

$$r \in \mathfrak{P}(\Phi_{4n} \setminus \Phi_{2n}, \Phi_{4n} \setminus \Phi_{2n}), \;\; s = -\frac{1}{1 + \bar{r}}$$

or

$$r \in \mathfrak{P}(\Phi_{4n} \setminus \Phi_{2n}, \Phi_{2n}), \;\; s = -\frac{\bar{r}}{1 + \bar{r}}.$$

Proof. By Corollary D.4 one of the pairs

$$\frac{1}{s(r+1)}, \frac{r}{(1+s)(1+r)} \quad \text{or} \quad \frac{1}{(1+s)(1+r)}, \frac{r}{s(1+r)}$$

should be roots of unity. The first pair consisting of roots of unity implies that

$$|\frac{1}{s}| = |1 + r| \;\; \text{and} \;\; |r| = |1 + \frac{1}{s}|.$$

The second pair consisting of roots of unity implies that

$$|\frac{1}{s}| = |1 + \frac{1}{r}| \text{ and } |\frac{1}{r}| = |1 + \frac{1}{s}|.$$

Since f_3 leaves invariant the zeros of $\mathcal{G}_{1;n}^{-1,2}$, we may choose the first pair being roots of unity. This implies that

$$s = -\frac{1}{1+r} \text{ or } s = -\frac{1}{1+\bar{r}},$$

see Figure 7.1. When $s = -1/(1+r)$ equation (7.4) with $r = \alpha\psi = \beta\phi - 1$ becomes

$$1 + (\alpha\psi)^m + (-1)^m - (\alpha\psi)^m = 0,$$

which is not true for $m = n$ since n is even. When $s = -1/(1+\bar{r})$ the left hand side of equation (7.4) becomes

$$1 + (\alpha\psi)^m + (-\phi^2)^m - (\alpha\bar{\psi}\phi^2)^m.$$

This vanishes if $\psi^{2n} = \phi^{2n} = -1$ and $m \equiv n \bmod 2n$. Therefore we have

$$r \in \mathfrak{P}(\Phi_{4n} \setminus \Phi_{2n}, \Phi_{4n} \setminus \Phi_{2n}).$$

If (r, s) solves equation (7.4), then so does $(\frac{1}{r}, s)$. By Lemma 7.12 this gives the other points

$$r \in \mathfrak{P}(\Phi_{4n} \setminus \Phi_{2n}, \Phi_{2n}).$$

We translate the last part of the theorem to a statement about the solutions of a diophantine equation. By using the formula (4.7) and performing the transformation

$$\psi^2 = \mu\nu, \ \phi^2 = \nu$$

equation (7.4) with $s = -1/(1+\bar{r})$ becomes

$$\left(\frac{1-\mu}{1-\nu}\right)^n = \frac{1-(-\mu)^n}{1+\nu^n}$$

When $\nu = -1$ and m even we apply Proposition E.6, for odd m we obtain $\mu^m = -1$. In the other cases we can apply Theorem E.1, since

- when $\nu = 1$ or $\mu = 1/\nu$ we have $r \in \mathbb{R}$,
- when $\mu = 1$ we have $r \notin \mathbb{C}$,
- when $\mu = -1$ the equation becomes $\nu^n = -1$,
- when $\mu = \nu$ we have $\psi = \pm\phi^2$, but $\pm\phi^2 \notin \Phi_{4n} \setminus \Phi_{2n}$ if $\phi \in \Phi_{4n}$.

and the theorem is proved. □

Lemma 7.12. *The image of* $\mathfrak{P}(\Phi_{2n} \setminus \Phi_n, \Phi_{2n} \setminus \Phi_n)$ *under the map* $f_3 : r \to \frac{1}{r}$ *is* $\mathfrak{P}(\Phi_{2n} \setminus \Phi_n, \Phi_n)$.

Proof. The biunit coordinates of

$$r \in \mathfrak{P}(\Phi_{2n} \setminus \Phi_n, \Phi_{2n} \setminus \Phi_n)$$

are (ψ, ϕ) with $\psi^{2n} = \psi^{2n} = -1$. By equation (4.10) the biunit coordinates of the point $f_3(r)$ are $(\psi^{-1}, \phi\psi^{-1})$. We have $(\psi^{-1})^{2n} = -1$ and $(\phi\psi^{-1})^{2n} = 1$. Since $f_3 \circ f_3 = 1$ the map is onto. □

Notice that by the relation (4.5) the results obtained in this section immediately transfer to equations with nonvanishing terms $K_1^{0,1}$ and $K_2^{2,-1}$ such as equation (4.6). Apparently we have obtained a condition on the spectrum of such equations at even order.

However, the integrable equation (4.6) has a continuous spectrum! How to explain these seemingly contradictory observations? The requirement in this section is the existence of divisors of infinitely many $\mathcal{G}$-functions. However, this is not a necessary condition for integrability. It might occur that a divisor of the $\mathcal{G}$-function is a divisor of the quadratic part of the equation. Thus, instead of a condition on the spectrum this determines the form of the quadratic part. We have

$$G_{1,n}^{0,1}[1, \alpha](\xi, \eta) = \eta\left((\alpha - 1)\eta - 2\xi\right).$$

Therefore $(\alpha - 1)\eta - 2\xi$ should be a divisor of $\widehat{K_1^{0,1}}$ which is true for equation (4.6).

7.2.2 Nonvanishing terms $K_1^{-1,2}$ and $K_1^{0,1}$

The nonvanishing of both $K_1^{-1,2}$ and $K_1^{0,1}$ as well as the nonvanishing of both $K_2^{2,-1}$ and $K_2^{1,0}$ yields the equation

$$(1 + r^n)\left((1 + s)^n - s^n\right) - (1 + r)^n = 0. \tag{7.5}$$

Lemma 7.13. *If equation (7.5) has infinitely many solutions, then*

$$r, \frac{1 + r}{s}, \frac{1 + s}{s}$$

are roots of unity.

Proof. The equation is of the form

$$aA^k + bB^k + cC^k + dD^k + eE^k = 0.$$

By using the Lech-Mahler theorem we have proven that if such an equation has infinitely many integers k as solutions, then three of the numbers A, B, C, D, E have a root of unity as a ratio and the same holds for the other two, cf. Corollary D.5.

Thus we have to consider ten cases. Take

$$A = 1 + s, B = r(1 + s), C = s, D = rs, E = 1 + r$$

Suppose that $A/D, B/C, B/E, C/E$ are roots of unity. This means that

$$\frac{1+r}{s}, \frac{1+s}{rs}, \frac{1+r}{r(1+s)}$$

are roots of unity. By multiplying the last two and dividing by the first one sees that r is a root of unity and the result follows. Suppose that $A/B, C/D, C/E, D/E$ are roots of unity. This means that

$$r, \frac{1+r}{s}$$

are roots of unity. We need a little more subtle argument here. Write equation (7.5) as

$$(1 + r^n)\left(\left(\frac{1+s}{s}\right)^n - 1\right) - \left(\frac{1+r}{s}\right)^n = 0.$$

and suppose it has infinitely many solution $n \in \mathbb{N}$. Since r and $(1+r)/s$ are roots of unity, their powers yield a finite number of values. Moreover, for the infinite number of solutions we have $1 + r^n \neq 0$. Hence for these infinite number of solutions

$$\left(\frac{1+s}{s}\right)^n - 1$$

has only finitely many values. This happens only when $(1 + s)/s$ is a root of unity.

All other eight cases are treated similarly to one of the above and the statement is proved. □

Notation 7.14. *The set of all* **primitive n-th roots of unity** *is denoted* Φ'_n.

Proposition 7.15. *If equation (7.5) has infinitely many solutions and* $r, s \notin \{0, -1\}$ *then one of the following cases applies:*

- $r, s \in \Phi'_3$
  with solutions odd n such that 3 is not a divisor of n.
- $\zeta \in \Phi'_{12},\ r \in \{\zeta, \zeta(1 - \zeta^2)\}, s = -\zeta^2(1 + \zeta),$
  with solutions odd n such that 3 is not a divisor of n.
- $\zeta \in \Phi'_{10},\ r \in \{\zeta^2, -\zeta^3\}, s = -\zeta(1 + \zeta^2),$
  with solutions odd n such that 5 is not a divisor of n.

Proof. By Lemma 7.13 the ratios

$$r, \frac{1+r}{s}, \frac{1+s}{s}$$

are roots of unity. Put $y = (1+s)/s$. The algorithm of Smyth is used to solve

$$((1+r)(y-1)(1+1/r)(1/y-1)-1)ry = 0$$

for roots of unity r, y and s is found from $s = 1/(y-1)$. By substitution of r, s in the equation the values $n \in \mathbb{N}$ are obtained. □

Using Proposition 7.15 we obtain the following ratios of eigenvalues:

- $a_1/a_2 = 1$ at order 5.
- $a_1/a_2 = 26 - 30\zeta + 15\zeta^2$ with $\zeta \in \Phi'_{12}$ at order 5.
- $a_1/a_2 = -2 + 3\zeta^2 - 3\zeta^3$ with $\zeta \in \Phi'_{10}$ at order 3.

7.2.3 Nonvanishing terms $K_1^{-1,2}$ and $K_2^{2,-1}$

The nonvanishing of both $K_1^{-1,2}$ and $K_2^{2,-1}$ yields the equation

$$(1+r^n)(1+s^n) - ((1+r)(1+s))^n = 0. \tag{7.6}$$

Lemma 7.16. *If equation (7.6) has infinitely many solutions, then*

$$r, s, (1+r)(1+s)$$

are roots of unity.

Proof. As in the proof of Lemma 7.13 we use Corollary D.5 and have to consider ten cases. Take

$$A = 1,\ B = r,\ C = s,\ D = rs,\ E = (1+r)(1+s)$$

Suppose that $A/D, B/C, B/E, C/E$ are roots of unity. This means that $rs, r/s$ and $(1+r)(1+s)/s$ are roots of unity. By multiplying the first two we see that r^2 is a root of unity and hence r, s and $(1+r)(1+s)$ as well. Suppose that $A/B, C/D, C/E, D/E$ are roots of unity. This means that r and $(1+r)(1+s)/s$ are roots of unity. We write equation (7.6) as

$$(1+r^n)\left(\left(\frac{1}{s}\right)^n - 1\right) - \left(\frac{(1+r)(1+s)}{s}\right)^n = 0.$$

and suppose that it has infinitely many solution $n \in \mathbb{N}$. Since r and $(1+r)(1+s)/s$ are roots of unity, their powers yield a finite number of values. Moreover, for the infinite number of solutions we have $1 + r^n \neq 0$. Hence for these infinite number of solutions $\left(\frac{1}{s}\right)^n - 1$ has only finitely many values. This happens only when s is a root of unity and the statement is verified in this case.

All other eight cases are treated similarly to one of the above. □

Proposition 7.17. *If equation (7.6) has infinitely many solutions and* $r, s \notin \{0, -1\}$ *then one of the following cases applies:*

- $r, s \in \Phi_3'$,
  with solutions $n \in \mathbb{N}$ *such that* 3 *is not a divisor of* n.
- $s \in \Phi_{12}'$, $r \in \{-s, -s - s^3\}$,
  with solutions odd n *such that* 3 *is not a divisor of* n *and solutions even* n *such that* 4 *divides* n.
- $s \in \Phi_5'$, $r \in \{s^2, s^3\}$,
  with solutions odd n *such that* 5 *is not a divisor of* n.

Proof. By Lemma 7.16 the points $r, s, (1+r)(1+s)$ are roots of unity. The algorithm of Smyth is used to solve

$$((1 + r)(1 + s)(1 + 1/r)(1 + 1/s) - 1)rs = 0$$

for roots of unity r, s. By substitution of r, s in the equation the values n are obtained. □

Using Proposition 7.17 we obtain the following ratios of eigenvalues:

- $a_1/a_2 = 1$ at order 5.
- $a_1/a_2 = 7 + 8\zeta - 4\zeta^3$ with $\zeta \in \Phi_{12}'$ at order 4.
- $a_1/a_2 = -2 + 3\zeta^2 + 3\zeta^3$ with $\zeta \in \Phi_5'$ at order 3.

7.2.4 Nonvanishing terms $K_1^{0,1}$ and $K_2^{1,0}$

The nonvanishing of both $K_1^{0,1}$ and $K_2^{1,0}$ yields the equation

$$((1 + r)^n - r^n)((1 + s)^n - s^n) = 1. \tag{7.7}$$

Lemma 7.18. *If equation (7.7) has infinitely many solutions, then*

$$\frac{1 + r}{r}, \frac{1 + s}{s}, rs$$

are roots of unity.

Proof. As in the proof of Lemma 7.13 we use Corollary D.5 and have to consider ten cases. Take

$$A = (1 + r)(1 + s), B = r(1 + s), C = s(1 + r), D = rs, E = 1$$

Suppose that $A/D, B/C, B/E, C/E$ are roots of unity. Then $r(1 + s), s(1 + r)$ and $(1 + r)(1 + s)/(rs)$ are roots of unity. By dividing the last by the first two, we see that rs is a root of unity and hence the statement follows in this case. Suppose that

$A/B, C/D, C/E, D/E$ are roots of unity. This means that rs and $(1+r)/r$ are roots of unity. We write equation (7.7) as

$$\left(\left(\frac{1+r}{r}\right)^n - 1\right)((r(1+s))^n - (rs)^n) = 1.$$

and suppose that it has infinitely many solutions $n \in \mathbb{N}$. Since rs and $(1+r)/r$ are roots of unity, their powers yield a finite number of values. Moreover, for the infinite number of solutions we have $\left(\frac{1+r}{r}\right)^n - 1 \neq 0$. Hence for these infinite number of solutions $(r(1+s))^n$ has only finitely many values. This happens only when $r(1+s)$ is a root of unity. Hence $(1+s)/s$ is a root of unity as well.

All other eight cases are treated similarly to one of the above. □

Proposition 7.19. *If equation (7.7) has infinitely many solutions and* $r, s \notin \{0, -1\}$ *then* $r, s \in \Phi'_3$*, with solutions* $n \in \mathbb{N}$ *such that* 3 *is not a divisor of* n*.*

Proof. By Lemma 7.18 the ratios $(1+r)/r, (1+s)/s$ and rs are roots of unity. The algorithm of Smyth is used to solve

$$rs(-1-s)(-1-r) - 1 = 0$$

for roots of unity r, s. By substitution of r, s in the equation the values n are obtained. □

Using Proposition 7.19 we obtain the ratio of eigenvalues $a_1/a_2 = 1$ at order 5.

The results obtained in this chapter, so far, are directly applicable in the classification of 2-component integrable evolution equations of any order with a diagonalisable homogeneous linear part and nonvanishing quadratic terms:

$$\begin{cases} u_t = & a_1 u_n + K_1^{-1,2} + K_1^{0,1} + K_1^{1,0} + \cdots \\ v_t = & a_2 v_n + K_2^{2,-1} + K_2^{1,0} + K_2^{0,1} + \cdots \end{cases}$$

7.3 Vanishing quadratic terms

Abstract. For the classification of equations without quadratic terms the following problems have to be solved. Determine all common divisors of $\mathcal{G}_{1,n}^{k,2-k}$ with infinitely many $\mathcal{G}_{1,m}^{k,2-k}$ for fixed k. We treat the case $k = -1$ which is used to classify the cubic version of the class of $\mathcal{B}$-equations.

We give all divisors H of $\mathcal{G}_{1;n}^{-1,3}[a_1, a_2]$ such that there are infinitely many m and b_1, b_2 for which H divides $\mathcal{G}_{1;m}^{-1,3}[b_1, b_2]$. In this way we classify equations of the form

$$\mathcal{K}_n[a_1, a_2](K): \begin{cases} u_t = & a_1 u_n + K(v_0, v_1, \ldots) \\ v_t = & a_2 v_n \end{cases},$$

where $a_1, a_2 \in \mathbb{C}$ and K a cubic polynomial in derivatives of v. We assume that $\mathcal{G}_{1;m}^{-1,3}[a_1, a_2]$ does not divide K.

We have

$$\begin{bmatrix} b_1 u_m + S \\ b_2 v_m \end{bmatrix},$$

with $S \in \mathcal{A}$ cubic in derivatives of v, as a symmetry of $\mathcal{K}_n[a_1, a_2](K)$ when

$$\widehat{S} = \frac{\mathcal{G}_{1;m}^{-1,3}[b_1, b_2]}{\mathcal{G}_{1;n}^{-1,3}[a_1, a_2]} \widehat{K}$$

is polynomial. Therefore $\mathcal{G}_{1;n}^{-1,3}[a_1, a_2]$ and $\mathcal{G}_{1;m}^{-1,3}[b_1, b_2]$ should have a common divisor for there being a symmetry. Suppose we can find $a_1, a_2, b_1, b_2 \in \mathbb{C}$ such that $(F, L, T \in \mathbb{C}[\eta_1, \eta_2, \eta_3])$

$$\begin{aligned} \mathcal{G}_{1;n}^{-1,3}[a_1, a_2] &= FL, \\ \mathcal{G}_{1;m}^{-1,3}[b_1, b_2] &= FT. \end{aligned}$$

Then the Lie derivative of $\mathcal{K}_m[b_1, b_2](S)$ with respect to $\mathcal{K}_n[a_1, a_2](K)$ vanishes if we take

$$\widehat{K} = LMv^3$$

and

$$\widehat{S} = MTv^3,$$

where $M \in \mathbb{C}[\eta_1, \eta_2, \eta_3]$ can be chosen freely.

Theorem 7.20. *The cubic $\mathcal{K}$-equation $\mathcal{K}_n[a_1, a_2](K)$ has no symmetries unless $a_1 = a_2$ or $a_2 = 0$. When $a_1 = a_2$ any symmetry is in a hierarchy of order 3.*

Proof. The case $a_2 = 0$ is easily proven. The equation is integrable since

$$\mathcal{G}_{1;m}^{-1,3}[a_1, 0] = -a_1(\eta_1 + \eta_2 + \eta_3)^n$$

divides $\mathcal{G}_{1;m}^{-1,3}[b_1, b_2]$ for arbitrary $m > n$ if $b_2 = 0$. The case $a_2 \neq 0$ is proven by F. Beukers with Proposition 7.21. The eigenvalues a_2, b_2 are scaled to 1. ☐

Proposition 7.21 (Beukers)**.** *The function*

$$\mathcal{G}_{1;m}^{-1,3}[a, 1] = \eta_1^n + \eta_2^n + \eta_3^n - a(\eta_1 + \eta_2 + \eta_3)^n$$

is irreducible unless $a = 1$, n odd or $a = 1/3, n = 2$, where

$$\mathcal{G}_{1;2}^{-1,3}[\frac{1}{3}, 1] = \frac{2}{3}(\eta_1 + \zeta\eta_2 + \zeta^2\eta_3)(\eta_1 + \zeta^2\eta_2 + \zeta\eta_3) \tag{7.8}$$

with $\zeta^2 + \zeta + 1 = 0$.

Proof. The function

$$\mathcal{G}_{1;n}^{-1,3}[1, 1] = \mathcal{G}_n^2$$

has been treated in Theorem 5.6.

The idea of the proof is as follows. Suppose that $\mathcal{G}_{1;n}^{-1,3}[a,1]$ is reducible with factors of degree k and $n-k$, say

$$\mathcal{G}_{1;n}^{-1,3}[a,1] = A \cdot B.$$

The points of intersection of the curves $A=0$ and $B=0$ are singular points of

$$\mathcal{G}_{1;n}^{-1,3}[a,1] = 0. \tag{7.9}$$

In Lemma 7.22 it is shown that the singular points of the curve (7.9) are double points. So the curves $A=0, B=0$ have only simple intersection and by Bézout's theorem the number of intersection points is precisely $k(n-k)$. Therefore the number of singular points of the curve (7.9) is at least $k(n-k)$.

Assume that $a \neq 1$. Suppose that $\mathcal{G}_{1;n}^{-1,3}[a,1]$ is divisible by a linear form L. Suppose that L is not symmetric under all permutations of η_1, η_2, η_3. Then $\mathcal{G}_{1;n}^{-1,3}[a,1]$ is divisible by another linear form L' as well and we have the divisor LL'. So we can assume $k \geq 2$. If L is completely symmetric we necessarily have

$$L = \eta_1 + \eta_2 + \eta_3.$$

Note that when $\eta_1+\eta_2+\eta_3$ divides $\mathcal{G}_{1;n}^{-1,3}[a,1]$, we have that $\eta_1+\eta_2+\eta_3$ divides $\eta_1^n+\eta_2^n+\eta_3^n$. Since

$$\eta_1^n + \eta_2^n + \eta_3^n = 0$$

is a nonsingular curve and $n \geq 2$ we arrive at a contradiction.

From Lemma 7.22 it follows that the number of singular points is at most twelve. Hence, using $k \geq 2$ we get

$$k(n-k) \geq 2(n-2) \leq 12$$

and thus $n \leq 8$. When $n \leq 8$ we have from Lemma 7.22 that the number of singular points is at most six. So, when $n \leq 8$ we get

$$2(n-2) \leq 6.$$

This implies $n \leq 5$. But in those cases Lemma 7.22 tells us that the number of singularities is at most three. So

$$2(n-2) \leq 3,$$

hence $n < 4$. When $n = 3$ we must have a linear factor L. By Lemma 7.22 the singular points are either the single point $(1:1:1)$ or the point $(1:1:-1)$ and its permutations. When $(1:1:1)$ is the only singular point, we get

$$2(n-2) \leq 1,$$

which is impossible when $n=3$. When $(1:1:-1)$ is a singular point, we have $a=1$, which we had excluded in this case. When $n=2$ the only singular point is $(1:1:1)$ where $a=1/3$. Here the function is divisible by the two linear factors in equation (7.8). $\square$

Lemma 7.22. *Let $\lambda \in \mathbb{C}$ be nonzero and n an integer ≥ 2. Consider the projective algebraic curve $C_{\lambda,n}$ given by*

$$\lambda(x^n + y^n + z^n) - (x + y + z)^n = 0.$$

Let $\mu \in \mathbb{C}$ be such that $\mu^{n-1} = \lambda$. Then the singular points of $C_{\lambda,n}$ are given by all triples $x_0, y_0, z_0 \in \mathbb{C}$ such that

$$x_0 + y_0 + z_0 = \mu,\ x_0^{n-1} = y_0^{n-1} = z_0^{n-1} = 1.$$

Moreover,

(a) Each singularity is an ordinary double point.

In the following results we assume either $\lambda \neq 1$ or n even. Then,

(b) There are at most twelve singularities.

(c) When $n \leq 8$, there are at most six singularities.

(d) When $n \leq 5$, there are at most three singularities.

Proof. The singular points of a projective algebraic curve $F(x, y, z) = 0$ can be solved by simultaneous solution of

$$\partial_x F(x, y, z) = \partial_y F(x, y, z) = \partial_z F(x, y, z) = 0.$$

In our case this yields

$$(x + y + z)^{n-1} = \lambda x^{n-1} = \lambda y^{n-1} = \lambda z^{n-1}.$$

As a result we see that the ratios $x^{n-1} = y^{n-1} = z^{n-1}$ in the singular points are of the form $x = x_0 l$, $y = y_0 l$, $z = z_0 l$ where x_0, y_0, z_0 are $(n-1)$-th roots of unity and $l \in \mathbb{C}$ nonzero. Hence we get $(x_0 + y_0 + z_0)^{n-1} = \lambda$. By multiplication of x_0, y_0, z_0 with a common $(n-1)$-th root of unity, if necessary, we can see that

$$x_0 + y_0 + z_0 = \mu.$$

This proves the first part of our lemma.

To prove part (a) we must write our equation $F_{\lambda,n} = 0$ locally around a singular point. Without loss of generality we can take $(x_0 : y_0 : 1)$ with $x_0^{n-1} = y_0^{n-1} = 1$ for such a singularity. We let $\mu = 1 + x_0 + y_0$. Put

$$x = x_0 + \xi, y = y_0 + \eta.$$

Then the local equation up to terms of order two in ξ, η reads

$$\binom{n}{2} \left(\mu^{n-2}(\xi + \eta)^2 - \mu^{n-1}(x_0^{n-2}\xi^2) - \mu^{n-1}(y_0^{n-2}\eta^2)\right) = 0.$$

Up to a factor, and using $x_0^{n-1} = y_0^{n-1} = 1$ this quadratic form reads

$$(\xi + \eta)^2 - \frac{\mu}{x_0}\xi^2 - \frac{\mu}{y_0}\eta^2.$$

Its discriminant equals

$$4 - 4(1 - \mu/x_0)(1 - \mu/y_0) = -4\mu/(x_0 y_0),$$

which is nonzero. Hence every singularity is a double point.

Part (b) follows directly from Lemma 7.23(a).

Part (c) and (d) are proved as follows: if $n = 6, 8$ then $n-1$ is prime and Lemma 7.23(b) applies. We see that $\mu = x_0 + y_0 + z_0$ has at most six triples of $(n-1)$-th roots of unity as solution.

When $n = 7$ we have by assumption $\lambda \neq 1$. The number of singular points is equal to the number of ordered pairs of 6-th roots of unity x_0, y_0 such that

$$(x_0 + y_0 + 1)^6 = \lambda.$$

The 36 values are easily computed. It is straightforward to verify that values of $\lambda \neq 0, 1$ are assumed at most six times.

The cases $n = 4,\ 5$ are treated similarly to the case $n = 7$, only we work with 3-rd and 4-th roots of unity respectively. □

Lemma 7.23. *Let $\nu \in \mathbb{C}$ be nonzero and not a root of unity. Consider the equation $\nu = \zeta_1 + \zeta_2 + \zeta_3$. Then the following two statements hold.*

(a) The equation has at most twelve solutions in roots of unity.

(b) Let p be a prime. Then the number of solutions in p-th roots of unity is at most six.

Proof. To prove (a) we suppose that $(\zeta_1, \zeta_2, \zeta_3)$ and $(\zeta_1', \zeta_2', \zeta_3')$ are solutions. Then

$$\zeta_1 + \zeta_2 + \zeta_3 - \zeta_1' - \zeta_2' - \zeta_3' = 0 \tag{7.10}$$

is a vanishing sum of roots of unity. Let us suppose that $\zeta_1', \zeta_2', \zeta_3'$ is not a permutation of $\zeta_1, \zeta_2, \zeta_3$. According to [Rin01] vanishing sums of six roots of unity are of the following form.

1. The roots of unity cancel pairwise.

2. The sum is, up to permutation, of the form

$$\zeta(1 + \omega + \omega^2) + \zeta'(1 + \omega + \omega^2) = 0,$$

where $\eta = e^{\pm 2\pi i/3}$.

3. The sum has, up to permutation, the form

$$\zeta(-\omega - \omega^2 + \eta + \eta^2 + \eta^3 + \eta^4) = 0,$$

   where $\eta = e^{2\pi i/5}$.

Suppose we are in case (1). If any two of ζ_i cancel, then ν is root of unity, contrary to our assumptions. So the ζ_i cancel the $-\zeta'_j$ and $\zeta'_1, \zeta'_2, \zeta'_3$ is a permutation of $\zeta_1, \zeta_2, \zeta_3$.

Suppose we are in case (2). The triple ζ_i, $i = 1, 2, 3$ cannot be of the form $\zeta, \zeta\omega, \zeta\omega^2$, since the sum would be zero. But then the values of ζ_i, $i = 1, 2, 3$ and the relation uniquely determine the set of ζ'_i. We argue similarly in case (3). From any of the two possibilities we can take all permutations, hence the total number of solutions is at most twelve.

To prove (b) suppose that

$$a_0 + a_1\zeta_p + \cdots + a_{p-1}\zeta_p^{p-1} = b_0 + b_1\zeta_p + \cdots + a_{p-1}\zeta_p^{p-1}$$

where $\zeta_p = e^{2\pi i/p}$ and $a_i, b_j \in \{0, 1, 2, 3\}$ such that

$$\sum_i a_i = \sum_i b_i = 3. \tag{7.11}$$

Since ζ_p has $1+x+x^2+\cdots+x^{p-1}$ as its minimal polynomial over $\mathbb{Q}$, we conclude that all numbers $a_i - b_i$ have the same value β. Because of condition (7.11) the number β must be zero, and we conclude that $a_i = b_i$ for all i. Hence, in any relation (7.10) in p-th roots of unity, the ζ'_i must be a permutation of the ζ_i. We thus conclude that the number of representations of ν as sum of three p-th roots of unity is at most six. □

Chapter 8

Almost integrable evolution equations and p-adic numbers

This chapter is devoted to almost integrable $\mathcal{B}$-equations of finite depth, i.e., equations of the form (6.8) with a finite number of symmetries. We give a short introduction to p-adic numbers and treat the method of Skolem, which allows us to conclude that only a finite number of symmetries exist for a given equation. We present a method by which all n-th order $\mathcal{B}$-equations that possess a m-th order symmetry are obtained. We show the existence of a 2-component evolution equation with 2 generalised symmetries and prove, with the method of Skolem, that it is a counterexample to Fokas' conjecture. In the end we formulate a new conjecture.

8.1 The conjecture of Fokas

A.S. Fokas formulated the following conjecture in 1987, cf. [Fok87].

Conjecture 8.1 (Conjecture of Fokas). *If a scalar equation possesses at least one time-independent non-Lie point symmetry, then it possesses infinitely many. Similarly for N-component equations one needs N symmetries.*

Note that for $N = 1$ the conjecture of Fokas is proven to be true for the class of equations (1.1) by the classification result of Sanders and Wang [SW98], which is reviewed in Chapter 5, cf. Theorem 5.11.

A candidate counterexample that possesses at least one higher order symmetry was given by I.M. Bakirov in [Bak91]. This is the 2 component equation

$$\left\{ \begin{array}{rl} u_t = & 5u_4 + v^2 \\ v_t = & v_4 \end{array} \right. .$$

As can easily be checked this equation possesses the sixth order symmetry

$$\left[\begin{array}{c} 11u_6 + 5vv_2 + 4v_1^2 \\ v_6 \end{array}\right].$$

Bakirov showed, by performing computer calculations, that the equation does not possess other symmetries of order $n \leq 53$.

The use of p-adic methods in integrability theory was initiated in [BSW98]. Beukers, Sanders and Wang proved that the Bakirov equation does not possess generalised symmetries other than the symmetry at order 6. Their results were based on a p-adic method of Skolem which provides a way of calculating all orders at which a symmetry can appear. A remark was made that this method also works for other cases, i.e., the fourth order equations with ratio of eigenvalues $29, 11, \frac{17}{3}$ are almost integrable of depth 1 with symmetries at order $10, 28, 16$ respectively[1].

8.2 p-adic numbers

For an elementary introduction to p-adic number theory, see [Gou97]. We just recall some of the basic notions.

Definition 8.2. *Let K be a field. A mapping $|\cdot| : K \to \mathbb{R}^+$ is an* **absolute value** *on K if*

$$\begin{array}{rcl} |x| = 0 & \Leftrightarrow & x = 0, \\ |xy| & = & |x||y|, \\ |x+y| & \leq & |x| + |y|. \end{array}$$

It is called **non-archimedian** *if*

$$|x+y| \leq max(|x|, |y|).$$

The p-**adic valuation**, *$v_p : \mathbb{Z} \to \mathbb{N}$ is defined as follows: write $n \in \mathbb{Z}$ as $n = p^\alpha n'$, where p is not a divisor of n'. Then, by definition*

$$v_p(n) = \alpha, \;\; v_p(0) = \infty.$$

The p-**adic valuation** *on $\mathbb{Q}$ is $v_p : \mathbb{Q} \to \mathbb{Z}$ with*

$$v_p(a/b) = v_p(a) - v_p(b).$$

Notice that this is only well defined if p is a prime number. The properties of this valuation are

$$\begin{array}{rcl} v_p(ab) & = & v_p(a) + v_p(b), \\ v_p(a+b) & \geq & min(v_p(a), v_p(b)). \end{array}$$

[1]A misprint was made in [BSW98]: instead of $\frac{17}{3}$ it was written $\frac{1}{3}$.

Therefore the mapping $|\cdot|_p : Q \to \mathbb{R}^+$ *defined by*

$$|x|_p = p^{-v_p(x)}$$

is a non-archimedian absolute value. The p**-adic field** $\mathbb{Q}_p$ *is the completion of* $\mathbb{Q}$ *with respect to* $|\cdot|_p$*. In canonical representation* $x \in \mathbb{Q}_p$ *is written*

$$x = \sum_{n \geq n_0} x_n p^n, \; 0 \leq x_n < p.$$

Any element in the **valuation ring** $\mathbb{Z}_p = \{x \in \mathbb{Q}_p | v_p(x) \geq 0\}$ *can be written as*

$$x = \sum_{n \geq 0} x_n p^n, \; 0 \leq x_n < p.$$

In the field of p**-adic units** $\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p | v_p(x) = 0\}$ *we have* $x_0 \neq 0$*. These are the invertible elements.*

Example 8.3. *In* $\mathbb{Z}_5$ *we write*

$$\begin{aligned} 57 &= 2 \cdot 5^0 + 1 \cdot 5^1 + 2 \cdot 5^2, \\ \frac{3}{4} &= 1 + \frac{1}{1-5} = 2 \cdot 5^0 + \sum_{i=1}^{\infty} 1 \cdot 5^i. \end{aligned}$$

Note that the higher the power of the prime involved, the smaller the number.

8.2.1 Hensel's lemma

Abstract. We include a version of Hensel's lemma, which provides a method to check whether a polynomial has a zero in $\mathbb{Z}_p^\times$. The proof is included and by means of an example it is shown to be constructive.

The procedure given in the following lemma is called Hensel lifting.

Lemma 8.4 (Hensel)**.** *A polynomial*

$$f(r) = \sum_{i=0}^{m} c_i r^i \text{ with } c_i \in \mathbb{Z}_p$$

has a zero in $\mathbb{Z}_p^\times$ *if there is an* $\alpha_1 \in \mathbb{Z}/p$ *such that*

- $f(\alpha_1) \equiv 0 \text{ mod } p$,
- $\frac{df}{dr}(\alpha_1) \not\equiv 0 \text{ mod } p$.

Proof. It is possible to construct a sequence $\{\alpha_n\}$ with (induction hypothesis)

$$\begin{aligned}\alpha_n &\equiv \alpha_{n-1} \bmod p^{n-1},\\ f(\alpha_n) &\equiv 0 \bmod p^n.\end{aligned}$$

Calculate $\beta \in \mathbb{Z}/p$ such that

$$0 = f(\alpha_{n+1}) = f(\alpha_n + \beta p^n) \equiv f(\alpha_n) + \frac{df}{dr}(\alpha_1)\beta p^n \bmod p^{n+1}.$$

By the induction hypothesis $\gamma \in \mathbb{Z}/p$ exists such that

$$f(\alpha_n) \equiv \gamma p^n \bmod p^{n+1}.$$

Substituting this and dividing by p^n gives an equation that can be solved in $\mathbb{Z}/p$:

$$\beta \equiv -\gamma(\frac{df}{dr}(\alpha_1))^{-1} \bmod p.$$

By completeness the sequence converges and, since f is continue for the p-adic topology, its limit is a zero of f. $\square$

Example 8.5. *The square roots of 2 are in $\mathbb{Z}_7$. Take*

$$f(r) = r^2 - 2.$$

Then we have

$$\begin{array}{ll} f(3) \equiv 0 \; mod\; 7, & f(4) \equiv 0 \; mod\; 7, \\ \frac{df}{dr}(3) \equiv 6 \; mod\; 7, & \frac{df}{dr}(4) \equiv 1 \; mod\; 7. \end{array}$$

Therefore Hensel's lemma can be applied. The number 3 is lifted as follows. We have

$$f(3) = 1 \cdot 7,$$

so $\gamma = 1$. The inverse of 6 in $\mathbb{Z}_7$ is 6. Then

$$\beta \equiv -1 \cdot 6 \equiv 1.$$

Indeed

$$f(3 + 1 \cdot 7) = 2 \cdot 7^2.$$

One step further gives

$$f(3 + 1 \cdot 7 + 2 \cdot 7^2) = 6 \cdot 7^3 + 4 \cdot 7^4.$$

This example illustrates that the method of Hensel is constructive.

8.2.2 The method of Skolem

Abstract. Skolem's method allows us to conclude that a given equation has only a finite number of symmetries. The method is based on the fact that if some equation does not have a solution in some p-adic field, it can not have a solution in $\mathbb{C}$. The method reduces the number of orders that need to be checked to a finite number.

If $x_i \in \mathbb{Z}_p^\times$ then, by Fermat's little theorem, there exists $y_i \in \mathbb{Z}_p$ such that

$$x_i^{p-1} = 1 + y_i p.$$

Choose $j \in \mathbb{N}$. Let

$$u_n^m = \sum_{i=1}^{j} c_i y_i^m x_i^n, \ \ m, n \in \mathbb{N}.$$

For instance, $U_n(r,s)$ has the form u_n^0, with $j = 4, c_i = (-1)^i$ and

$$x_1 = 1 + s, \ x_2 = 1 + r, \ x_3 = r(1 + s), \ x_4 = s(1 + r),$$

cf. equation 6.4 and Lemma 6.18.

Lemma 8.6 (Skolem)**.** *If*

$$u_k^0 \not\equiv 0 \ mod \ p$$

then for all r we have

$$u_{k+r(p-1)}^0 \neq 0.$$

Proof. We have

$$u_{k+r(p-1)}^0 = \sum_{i=1}^{j} c_i x_i^k (1 + y_i p)^r \equiv u_k^0 \bmod p \not\equiv 0.$$

Therefore $u_{k+r(p-1)}^0 \neq 0$. □

Lemma 8.7 (Skolem)**.** *If*

$$u_k^0 = 0, \ \ u_k^1 \not\equiv 0 \ mod \ p$$

then for all $r > 0$ we have

$$u_{k+r(p-1)}^0 \neq 0.$$

Proof. Assume $u_{k+r(p-1)}^0 = 0$. Then we have

$$\sum_{i=1}^{j} c_i x_i^k (1 + y_i p)^r = \sum_{t=1}^{r} \begin{pmatrix} r \\ t \end{pmatrix} p^t u_k^t = 0.$$

By using

$$\frac{1}{r} \begin{pmatrix} r \\ t \end{pmatrix} = \frac{1}{t} \begin{pmatrix} r-1 \\ t-1 \end{pmatrix}$$

and dividing by pr this is written

$$u_k^1 + \sum_{t=2}^{r} \left(\begin{array}{c} r-1 \\ t-1 \end{array} \right) \frac{p^{t-1}}{t} u_k^t = 0.$$

This contradicts the second assumption since $(p^{t-1})/t$ always contains a divisor p. To see this, write $t = p^\alpha s$ with $p \not| s$. Then s is invertible and

$$\frac{p^{t-1}}{t} = \frac{1}{s} p^{p^\alpha s - \alpha - 1}.$$

The power of p is bigger than 1, for when $\alpha = 0$ we know $s \geq 2$ and when $\alpha > 0$ we have $s \geq 1$ and $p^\alpha \geq \alpha + 2$ (because $p > 2$). Hence we conclude $u^0_{k+r(p-1)} \neq 0$. $\square$

How to prove that $u_n^0 = 0$ has finitely many integer solutions n? Use Hensel's method to look for a prime number p such that the x_i are in $\mathbb{Z}_p^\times$ and check the conditions in the lemmas of Skolem. Note that only finitely many orders, have to be checked and that the computations to be done are all modulo p or p^2.

8.3 Almost integrable $\mathcal{B}$-equations

Abstract. We describe a method by which all n-th order $\mathcal{B}$-equation can be found that have a symmetry at order m.

To find a symmetry of $\mathcal{B}_n[a_1, a_2](K)$ one has to find m, b_1, b_2 such that $\mathcal{G}_1[a_1, a_2]$ has a common divisor with $\mathcal{G}_m[b_1, b_2]$. From Theorem 6.15 we know that if this common divisor has a degree smaller than 4 the corresponding equations are always in a hierarchy of order $1, 2$ or 3. Therefore we consider divisors of degree at least 4. According to Lemma 6.18 the function

$$\mathcal{G}_n[1 + r^n, (1 + r)^n](\xi_1, \xi_2)$$

has a fourth degree divisor whenever there exists an s such that $U_n(r, s) = 0$. In the following we disregard the trivial divisors of $U_n(r, s)$ which are $(r - s)(rs - 1)$ for all m and $(r + 1)(s + 1)$ when m is odd. The following method was introduced in [vdKS01].

Lemma 8.8. *Take $n > 3$. All ratios of eigenvalues of n-th order $\mathcal{B}$-equations with a symmetry on order m can be obtained by calculating the resultant of $U_n(r, s)$ and $U_m(r, s)$ with respect to s. To any of its zeros r corresponds the ratio*

$$\frac{a_1}{a_2} = \frac{1 + r^n}{(1 + r)^n}.$$

of an integrable $\mathcal{B}$-equation.

Proof. A necessary condition for two polynomials to have a nontrivial common divisor is the vanishing of their resultant. If the resultant of $U_n(r,s)$ and $U_m(r,s)$ with respect to s vanishes for some $r \in \mathbb{C}$, by Lemma 6.18 the $\mathcal{G}$-functions

$$\mathcal{G}_n[1+r^n,(1+r)^n](\xi_1,\xi_2)$$

and

$$\mathcal{G}_m[1+r^m,(1+r)^m](\xi_1,\xi_2)$$

have a common 4-th degree divisor. This implies that the n-th order $\mathcal{B}$-equation with eigenvalues

$$a_1 = 1+r^n,\ a_2 = (1+r)^n$$

and quadratic part $\mathcal{G}_n[a_1,a_2]$ divided by this 4-th degree divisor has a symmetry on order m. □

Example 8.9 (Bakirov)**.** *The resultant of $U_4(r,s)$ and $U_6(r,s)$ with respect to s contains the divisor*

$$f = 2r^4 + 10r^3 + 15r^2 + 10r + 2.$$

We have

$$1+r^4 \equiv -\frac{5}{2}r(2r^2+2+3r) \ mod\ f$$

and

$$(1+r)^4 \equiv -\frac{1}{2}r(2r^2+2+3r) \ mod\ f.$$

Their ratio is 5, the ratio of the eigenvalues of the Bakirov equation. As expected $\mathcal{G}_4[5,1](1,r)$ is proportional to f.

8.3.1 One symmetry does not imply integrability

Abstract. We demonstrate the use of the method of Skolem while treating the Bakirov equation.

In [BSW98] it was proven that the Bakirov equation provides a counterexample to the old believe that one symmetry implies integrability. We let p increase and, using Hensel lifting, look in $\mathbb{Z}_p^\times$ for zeros of the resultant calculated in Example 8.9:

$$f(r) = 2r^4 + 10r^3 + 15r^2 + 10r + 2.$$

The lowest 'good' prime is 181. In $\mathbb{Z}/181$ we find

$$f(66) = f(139) = 0.$$

These numbers can be lifted to elements of $\mathbb{Z}_{181}^\times$. Modulo p^2 they are

$$r \equiv 66 + 13p,\ s \equiv 139 + 29p.$$

The function $U_m(r,s)$ has the form u_m^0 with $c_i = (-1)^i$, $j = 4$ and

$$\begin{aligned} x_1 &= 1 + s \equiv 140 + 29p \bmod p^2, \\ x_2 &= 1 + r \equiv 67 + 13p \bmod p^2, \\ x_3 &= r(1+s) \equiv 9 + 165p \bmod p^2, \\ x_4 &= s(1+r) \equiv 82 \bmod p^2. \end{aligned}$$

For $0 \le m < 180$ we have $u_m^0(r,s) \equiv 0 \bmod p$ only when $m \in \{0,1,4,6\}$. Applying

$$x_i \to \frac{x_i^{p-1} - 1}{p}$$

gives

$$\begin{aligned} y_1 &\equiv 40 \bmod p, \\ y_2 &\equiv 33 \bmod p, \\ y_3 &\equiv 140 \bmod p, \\ y_4 &\equiv 46 \bmod p. \end{aligned}$$

For $m \in \{0,1,4,6\}$ the function

$$u_m^1 = 33 \cdot 67^m + 46 \cdot 82^m - 40 \cdot 140^m - 140 \cdot 9^m$$

is nonzero modulo p. Both Skolem's lemmas can be applied and it is shown that there is no nontrivial symmetry except at order 6.

8.3.2 The counterexample to Fokas' conjecture

Abstract. We present a counterexample to Fokas' conjecture.

The counterexample was first presented in [vdKS02].

Theorem 8.10. *The 2-component equation*

$$\left\{ \begin{array}{ll} u_t = & 2r^2u_7 + 7(2r^2+4r+3)(v_3v_0 + (3-r)v_2v_1) \\ v_t = & (16r^2+28r+21)v_7 \end{array} \right. , \tag{8.1}$$

with

$$r^3 + r^2 - 1 = 0,$$

possesses exactly two nontrivial generalised symmetries.

Proof. The resultant of U_7 and U_{11} has the following divisor in common with the resultant of U_7 and U_{29}:

$$(r^3 - r - 1)(r^3 + r^2 - 1)\left((r^2+r+1)^3 - (1+r)^2r^2\right). \tag{8.2}$$

Therefore, by Lemma 8.8 there are three 7-th order equations possessing symmetries at order 11 and 29. We explicitly compute the three equations and their symmetries.

Each zero of r^3+r^2-1 is mapped to a different eigenvalue. We take $\mathbb{C}[r]/(r^3+r^2-1)$ as our coefficient ring. As eigenvalues of the equations we take

$$1+r^7 = 2r^2 \text{ and } (1+r)^7 = 16r^2+28r+21.$$

The quadratic parts are

$$\mathcal{G}_7[2r^2, 16r^2+28r+21](\xi_1, \xi_2)$$

divided by

$$2(\xi_1 - r\xi_2)(\xi_1 - (r+r^2)\xi_2)(\xi_1^2 + (1-r-r^2)\xi_1\xi_2 + \xi_2^2),$$

i.e.,

$$\frac{7}{2}(2r^2+4r+3)(\xi_1+\xi_2)(\xi_1^2+(2-r)\xi_1\xi_2+\xi_2^2).$$

By applying the inverse Gel'fand–Dikiĭ transformation we obtain the quadratic part of the equations. The symmetries can be calculated in the same way, leading to the 11-th order symmetry $S = (S_1, S_2)$ with

$$\begin{aligned} S_1 &= (-3r^2+r+2)u_{11} + 11\big((14r^2+24r+18)v_7v_0 + (35r^2+57r+42)v_6v_1 \\ &\quad + (48r^2+70r+49)v_5v_2 + (51r^2+65r+42)v_4v_3\big) \end{aligned}$$

and

$$S_2 = (151r^2+265r+200)v_{11},$$

and the 29-th order symmetry $S = (S_1, S_2)$ with

$$\begin{aligned} S_1 = \; & (-40r^2+9r+17)u_{29} \\ & +29\,\big(30(1081r^2+1897r+1432)v_{25}v_0 \\ & +(311920r^2+547311r+413143)v_{24}v_1 \\ & +(706832r+533441+403277r^2)v_{23}v_2 \\ & +(449543r^2+782050r+589257)v_{22}v_3 \\ & +(537572r+402545+317304r^2)/2v_{21}v_4 \\ & +(1026233r^2+1635821r+1205570)v_{20}v_5 \\ & +(1101516r+779787+787277r^2)/2v_{19}v_6 \\ & +(2656229r+1710194+2393075r^2)v_{18}v_7 \\ & +(3831912r^2+3208669r+1731205)v_{17}v_8 \\ & +(6105788r^2+4007995r+1678107)v_{16}v_9 \\ & +(4807604r+1421555+8899703r^2)v_{15}v_{10} \\ & +(5263833r+11440843r^2+915604)v_{14}v_{11} \\ & +3(1793035r+155000+4312473r^2)v_{13}v_{12}\big) \end{aligned}$$

and

$$S_2 = (3761840r^2 + 6601569r + 4983377)v_{29}.$$

In $\mathbb{Z}/101$ the first divisor of (8.2) has zero 20 and the third zero 52. They can be lifted and both Skolem's lemmas can be applied. We take $p = 101$. Modulo p^2 the zeros are

$$r \equiv 20 + 76p, \;\; s \equiv 52 + 76p.$$

The function $U_m(r, s)$ has the form u_m^0 with $c_i = (-1)^i$, $j = 4$ and

$$\begin{aligned} x_1 &= 1 + s \equiv 53 + 76p \bmod p^2, \\ x_2 &= 1 + r \equiv 21 + 76p \bmod p^2, \\ x_3 &= r(1 + s) \equiv 50 + 3p \bmod p^2, \\ x_4 &= s(1 + r) \equiv 82 + 3p \bmod p^2. \end{aligned}$$

For $0 \le m < 99$ we have $u_m^0(r, s) \equiv 0 \bmod p$ only when $m \in \{0, 1, 7, 11, 29\}$. The numbers y_i for which $x_1^{p-1} = 1 + y_i p$ are modulo p:

$$\begin{aligned} y_1 &\equiv 99 \bmod p, \\ y_2 &\equiv 54 \bmod p, \\ y_3 &\equiv 16 \bmod p, \\ y_4 &\equiv 97 \bmod p. \end{aligned}$$

For $m \in \{0, 1, 7, 11, 29\}$ the function u_m^1:

$$54 \cdot 21^m + 97 \cdot 82^m - 99 \cdot 53^m - 16 \cdot 50^m$$

is nonzero modulo p. Both Skolem's lemmas can be applied now. In this way it is proven that

$$\{r, \frac{1}{r}, s, \frac{1}{s}\} \equiv \{20, 96, 52, 68\} \bmod 101 \tag{8.3}$$

is not a set of zeros of a $\mathcal{G}_m$-function when $m \notin \{0, 1, 7, 11, 29\}$. If one of the other sets corresponding to the modulo 101 sets

$$\{40, 48, 42, 89\} \text{ or } \{32, 60, 63, 93\}$$

is set of zeros of a function $\mathcal{G}_m$ for some m, their minimal polynomials divides the resultant of $U_7(r, s)$ with $U_m(r, s)$. That means that the set (8.3) consist of zeros of this $\mathcal{G}_m$ as well, and hence m equals $0, 1, 7, 11$ or 29. It is shown that there are no nontrivial symmetries except at order 11 and at order 29. □

8.3.3 On the depth of non-integrable $\mathcal{B}$-equations

Abstract. We calculated 46300 almost integrable $\mathcal{B}$-equations of depth at least 1. Some refinements of the method of Skolem are presented. These made it possible to prove that the depth of the calculated $\mathcal{B}$-equations, with the exception of the counterexamples (8.1), is exactly 1. A new conjecture is formulated.

We calculated the resultant of $U_n(r,s)$ and $U_m(r,s)$ with respect to s for $4 \leq n \leq 10$ and $n+1 \leq m \leq n+150$. To obtain the equations with finitely many symmetries only, we filtered out the integrable equations using the results of Chapter 6.

We give an indication of the size of the expressions. The resultant of U_{10} and U_{160} has degree 556. The coefficients of r^n with $244 < n < 312$ have 207 decimal digits. The number of n-th order equations we calculated is equal to the sum of the degree's of the resultants divided by 4. This number is shown in table 8.1.

n	4	5	6	7	8	9	10	4-10
#	2745	2701	5679	5644	8740	8839	11952	46300

Table 8.1: The number of n-th order non-integrable $\mathcal{B}$-equations with a symmetry at order $n < m < n+151$.

In Figure 8.1 and 8.2 the positions of the zeros of these resultants in the complex plane are plotted. As a fundamental domain the upper half unit circle is chosen. The full pictures are invariant under $r \to \frac{1}{r}r$ and $r \to \bar{r}$.

Refinements of the method of Skolem

All these equations are almost integrable of depth at least 1. To calculate the depth of each equation we implemented the method of Skolem together with the following two refinements, made in [vdKS01].

1. Most of the resultants we have calculated are irreducible. By the argument in the proof of Theorem 8.10 it suffices to prove the statement for one particular set of zeros.

2. Sometimes it is much more efficient to use two pairs of zeros. The argument goes as follows: the s-resultant of $U_5(r,s)$ and $U_{19}(r,s)$ contains the divisor

$$\begin{aligned} f(r) = \; & r^{12} + 4r^{11} + 10r^{10} + 19r^9 + 28r^8 + 34r^7 \\ & +37r^6 + 34r^5 + 28r^4 + 19r^3 + 10r^2 + 4r + 1 \end{aligned}$$

which is irreducible over $\mathbb{Q}$ and splits into linear factors over $\mathbb{Z}_{509}^{\times}$. The numbers $(264, 407)$ form a solution for $U_m(r,s)$ when

$$m \in \{0, 1, 5, 19, 256, 414\}.$$

The numbers $(267, 300)$ form a solution for $U_m(r,s)$ when

$$m \in \{0, 1, 5, 19, 162, 254\}.$$

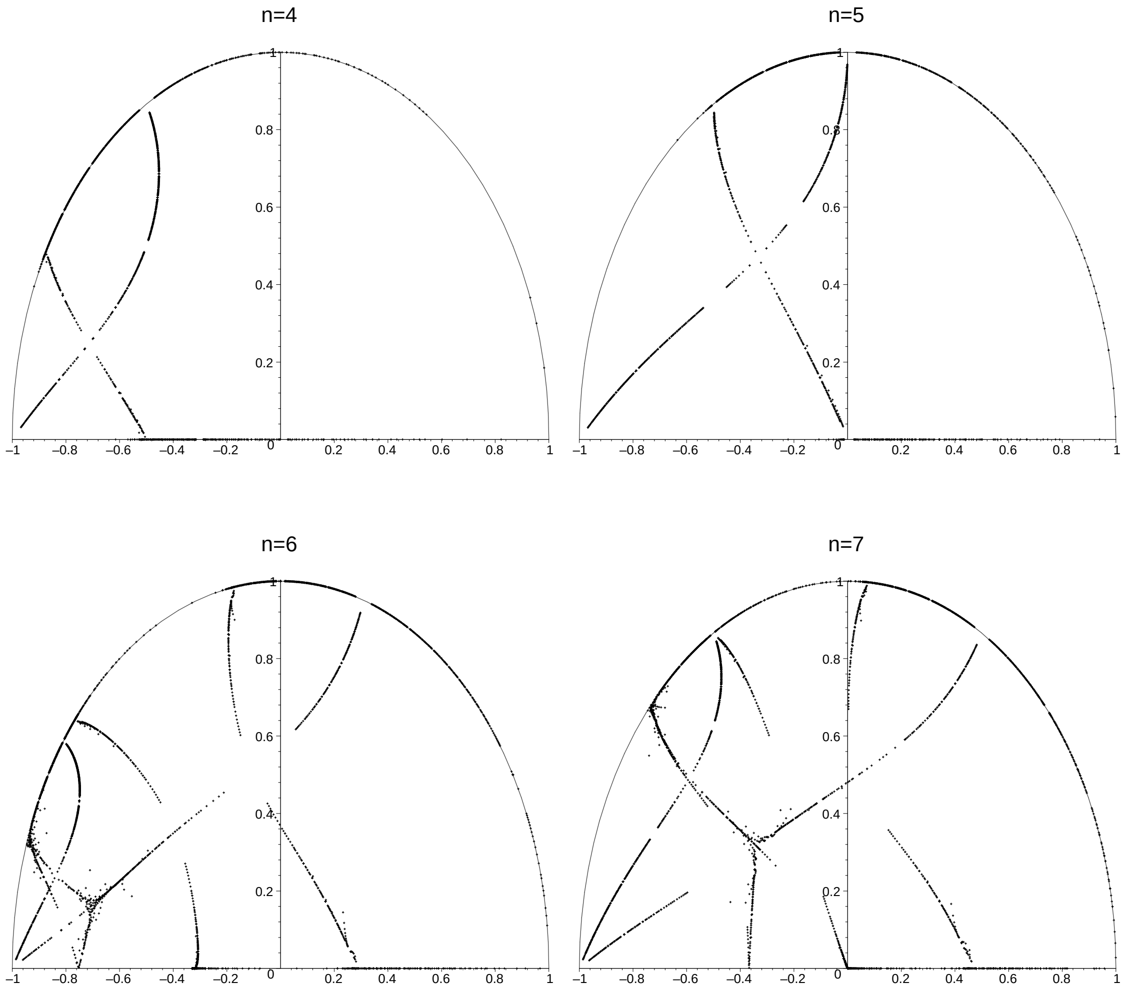

Figure 8.1: Zeros of the $\mathcal{G}$-functions corresponding to almost integrable equations of order $4, 5, 6$ and 7.

> By using both pairs we can apply Lemma 8.6 for all $0 \leq m < 508$ but $\{0, 1, 5, 19\}$, for which we can use Lemma 8.7. It is hard task, even for a computer, to find a prime for which the normal procedure works. The calculations that were performed showed that such a prime is bigger than 8146.

With these improvements we have been able to prove that all equations we calculated have exactly one nontrivial symmetry, with the exception of the seventh order equations with two symmetries at order 11 and 29.

Theorem 8.11. *Take* $3 < n < 11, n < m < n + 151$ *and* $m \neq 11, 29$ *when* $n = 7$. *Then, the n-th order non-integrable $\mathcal{B}$-equations with a symmetry of order m is almost integrable of depth 1.*

The following MAPLE output can be used to verify the above statement for $n = 7, 29 \leq m \leq 37$.

```
prf29:=[101, {20, 52}],[97, {4, 32}]:
prf30:=[2531, {75, 871}]:
prf31:=[1021, {16, 42}]:
prf32:=[877, {226, 214}]:
prf33:=[601, {23, 409}]:
prf34:=[2857, {2457, 716}, {742, 391}]:
prf35:=[661, {401, 330}, {122, 245}]:
prf36:=[179, {17, 76}]:
prf37:=[233, {30, 56}, {20, 84}]:
```

The sequence `prf.m` consists of a prime number p and one or two sets of modulo p solutions of $\mathrm{res}_s(U_7(r,s), U_m(r,s))$ such that all conditions in the lemmas of Skolem are satisfied.

The exceptions, where the resultant has two divisors, are

$$(n, m) = (4, 24), (4, 28), (6, 42), (7, 8), (7, 49), (8, 56), (10, 70)$$

Three divisors appear at $n = 7, m = 11$ and four at $n = 7, m = 29$.

The following could be inferred from Theorem 8.11.

Conjecture 8.12. *The only integer $N > 2$ such that*

- ⋆ *there exists $r, s \in \mathbb{C}$ such that the Diophantine equation*

$$(1 + r^m)(1 + s)^m = (1 + s^m)(1 + r)^m \tag{8.4}$$

has exactly N solutions $m > 1$.

is $N = 3$. Moreover, when $N = 3$ the solutions are given by $m = 7, 11, 29$.

Note that there are finitely many points $r, s \in \mathbb{C}$ such that $m = 7, 11, 29$ are all the solutions $m > 1$ to equation (8.4). These points are obtained from

$$1 + r = r^3 = \frac{(1 + s)^2}{s}.$$

and the transformations $r \to 1/r$ and $r \leftrightarrow s$.

When we describe the zeros of

$$(r^3 - r - 1)(r^3 + r^2 - 1)$$

in biunit coordinates, something quite peculiar is found. Using the biunit coordinate description (4.10) of the anharmonic ratios (4.9) we obtain equations for the real and complex parts separately. Suppose that (ψ, ϕ) are the biunit coordinates of r. Then

$$1 + r = r^3, \; 1 + r = r^{-2}$$

imply

$$\phi = \psi^3, \; \phi = \psi^{-2}$$

respectively. By substituting these expression for ϕ in the equations for the real parts, in both cases we obtain:

$$(\psi^4 + \psi^2 + 1)^3 = \psi^4(1 + \psi^2)^2.$$

Thus ψ^2 is a zero of the third divisor of the polynomial (8.2). Finally we note that

$$\frac{(r^2 + r + 1)^3}{r^2(1 + r)^2}$$

is an **absolute invariant** of $\mathfrak{A}$, cf. [MM97, Section 4.6].

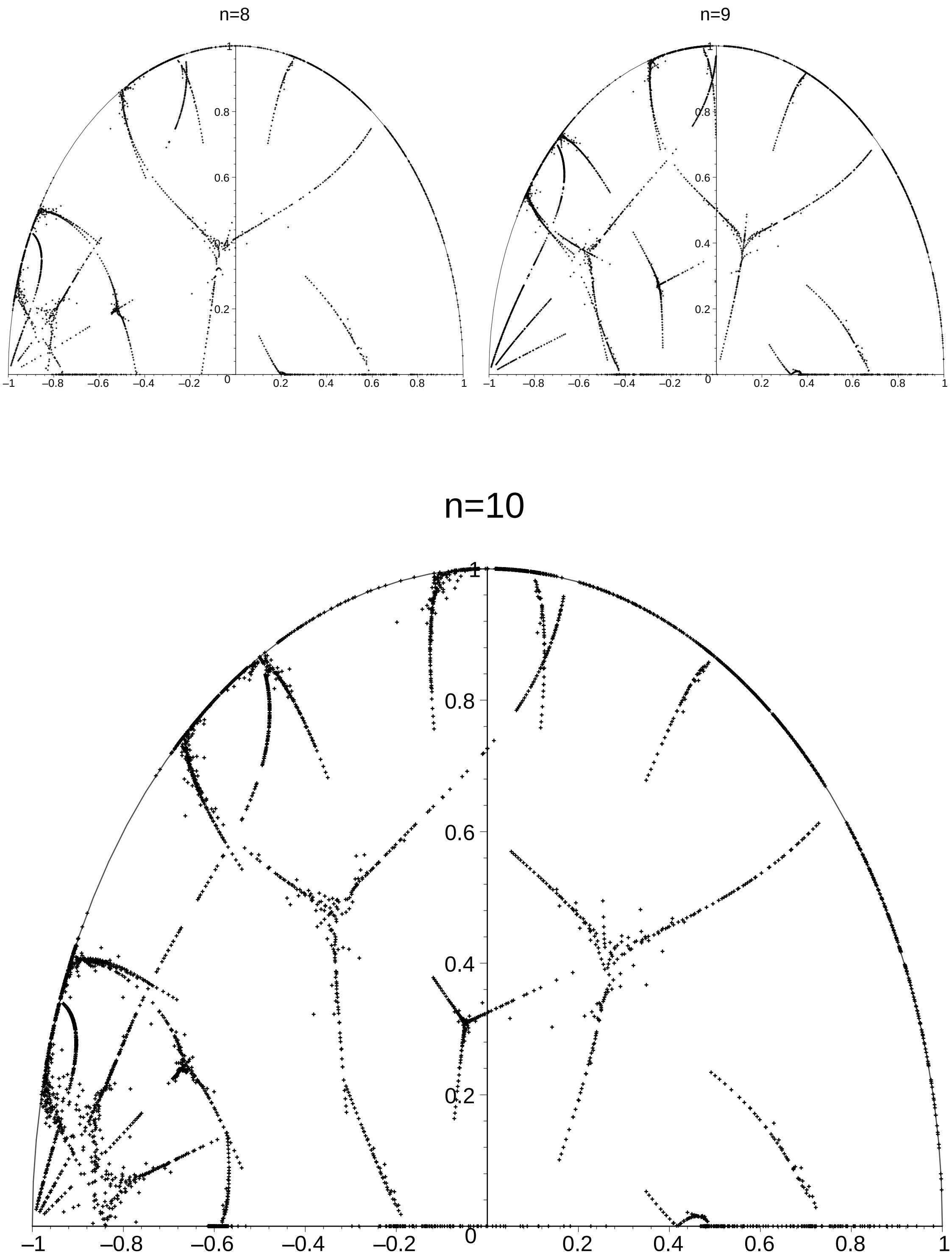

Figure 8.2: Zeros of the $\mathcal{G}$-functions corresponding to almost integrable equations of order $8, 9$ and 10.

Chapter 9

Nonpolynomial symmetries

We will prove that the KDV equation coupled to a purely nonlinear equation

$$\begin{cases} u_t = & u_3 + 3uu_1 \\ v_t = & \alpha u_1 v + uv_1 \end{cases}$$

has polynomial symmetries of even weight only if α is a negative and rational number. Moreover we prove that, allowing multiplying with v^c where $c \in \mathbb{C}$, this equation possesses several mutually noncommuting hierarchies of nonpolynomial symmetries for any value of $\alpha \in \mathbb{C}$.

9.1 Foursovs conjecture, generalised KDV

In [Fou00] a classification of third order symmetrically coupled KDV-like equations with respect to the existence of two symmetries is presented. One equation in the list is quite special;

$$\begin{cases} u_t = & \frac{1}{2}u_3 + \frac{1}{2}v_3 + (2-\alpha)uu_1 + (6-\alpha)vu_1 + \alpha uv_1 + (4-\alpha)vv_1 \\ v_t = & \frac{1}{2}v_3 + \frac{1}{2}u_3 + (2-\alpha)vv_1 + (6-\alpha)uv_1 + \alpha vu_1 + (4-\alpha)uu_1 \end{cases} . \tag{9.1}$$

For all values of α odd order symmetries were found. At even order symmetries were found as well, but only for some particular values of α. Foursov calculated all weight 2,4,6,8 and 10 symmetries with the help of computer algebra and formulated the following conjecture.

Conjecture 9.1 (Foursov, [Fou00]). *The equation 9.1 has symmetries of order $2k$ and weight $2k + 2n$ when*

$$\alpha = 2(1 - \frac{k}{n})$$

for any nonnegative integer k and any positive integer n.

A particularly easy case is $\alpha = 2$: symmetries of zero order and weight $2n$ are

$$\begin{bmatrix} (u-v)^n \\ -(u-v)^n \end{bmatrix}.$$

No extra odd weight symmetries were found because it was assumed that the symmetries were polynomial. The crucial observation one has to make is that the weight can be any number, i.e., the above equation is a symmetry when $\alpha = 2$ for all $n \in \mathbb{C}$.

We put the equation 9.1 in Jordan form by the invertible linear transformation

$$u \to \frac{1}{2}(u+v), v \to \frac{1}{2}(u-v).$$

Moreover, we apply a scale transformation $u \to u/2$ and the parameter translation $\alpha \to \alpha + 2$ to obtain the equation we denote $K(\alpha)$:

$$\left\{ \begin{array}{ll} u_t = & u_3 + 3uu_1 \\ v_t = & \alpha u_1 v + u v_1 \end{array} \right. \tag{9.2}$$

a generalisation of the famous KDV equation. The Foursov conjecture states that for all negative $\alpha \in \mathbb{Q}$ the equation has a hierarchy of even order polynomial symmetries. This is the case as we will show that all conditions of the implicit function theorem are satisfied. Since we allow the symmetries to be nonpolynomial, we find symmetries at any order for any $\alpha \neq 0$. The results described in this chapter were published in [vdK02b].

9.2 Generalisations of the KDV symmetries

Abstract. The first condition in the implicit function theorem, cf. Theorem 3.8, is finding one symmetry S. Instead of explicitly giving S, we show that for all α the equation has infinitely many odd order symmetries.

Lemma 9.2. *Let Z_n be the (odd) n-th order symmetry of the KDV equation*

$$u_t = u_3 + 3uu_1$$

Then for all n we have

$$S_n(\alpha) = \begin{pmatrix} Z_n \\ (\alpha v + v_1 D_x^{-1}) Z_{n-2} \end{pmatrix}$$

as a symmetry of $K(\alpha)$, i.e., equation (9.2).

Proof. The Lie derivative

$$\mathcal{L}(K)S(\alpha) = D_{S_n}[K](\alpha) - D_K[S_n](\alpha)$$

has as first component $D^u_{Z_n}[Z_3]-D^u_{Z_3}[Z_n]$, which vanishes because Z_n is a symmetry of KDV (Z_3). The second component is expanded in powers of α. The zeroth power has coefficient

$$\begin{aligned}
& v_1 D_x^{-1} D^u_{Z_{n-2}}[Z_3] + D_x^{-1} Z_{n-2}(uv_2 + u_1 v_1) \\
& -v_1 Z_n - uv_2 D_x^{-1} Z_{n-2} - uv_1 Z_{n-2} \\
= \; & v_1 \left((u_1 D_x^{-1} + D_x^{-1}(D_x^3 + 3uD_x + 3u_1) - u) Z_{n-2} - Z_n \right) \\
= \; & v_1 \left((D_x^2 + 2u + u_1 D_x^{-1}) Z_{n-2} - Z_n \right),
\end{aligned}$$

which vanishes because of the recursion relation for KDV symmetries; apply the scaling $u \to 3u$ to the operator (10.3). The coefficient of α,

$$v \left((D_x^3 + 2uD_x + 3u_1 + u_2 D_x^{-1}) Z_{n-2} - D_x Z_n \right),$$

vanishes for the same reason, since

$$u_1 D_x^{-1} - D_x^{-1} u_1 = D_x^{-1} u_2 D_x^{-1}.$$

Finally α^2 has coefficient

$$u_1 v Z_{n-2} - u_1 v Z_{n-2} = 0.$$

Therefore the $S_n(\alpha)$ with n odd form a hierarchy of the equation $K(\alpha)$ for all α. □

9.3 Nonlinear injectiveness, relative 2-primeness

Abstract. We prove that $K^0(\alpha)$ is nonlinear injective and relatively 2-prime with respect to any of its symmetries $S_n(\alpha)$.

The standard choice for a grading on the Lie algebra is the total degree. However, we take the degree in u as our grading, which is more convenient here.

Lemma 9.3. *The linear part of $K(\alpha)$ is nonlinear injective.*

Proof. Suppose that Q has u-grading i. We will prove that $\mathcal{L}(K^0)Q = 0$ implies $i = 0$. The first symmetry condition reads:

$$\begin{aligned}
0 &= \mathcal{L}(K^0)Q \\
&= \begin{bmatrix} D^u_{Q_1} & D^v_{Q_1} \\ D^u_{Q_2} & D^v_{Q_2} \end{bmatrix} \begin{bmatrix} u_3 \\ 0 \end{bmatrix} - \begin{bmatrix} D_x^3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\
&= D^u_{Q_1} u_3 - \begin{bmatrix} D_x^3 Q_1 \\ -D^u_{Q_2} u_3 \end{bmatrix}.
\end{aligned}$$

This implies first of all that Q_1 does not contain a part that depends on v because it would be changed by the operation D_x^3 and left unchanged by $D^u_{Q_1}$. That Q_1 has

u-grading 0 is easily seen by using the symbolic method. When Q_1 is nonzero we need

$$\xi_1^3 + \xi_2^3 + \cdots + \xi_{i+1}^3 - (\xi_1 + \xi_2 + \cdots + \xi_{i+1})^3 = 0,$$

which implies $i = 0$, as in the scalar case, cf. Lemma 4.6. Secondly, $\mathcal{L}(K^0)Q = 0$ implies that Q_2 does not depend on u or its derivatives, i.e $Q_2 \in \mathcal{L}^0$. □

In other words, this lemma states that $K(\alpha)$ is nonlinear injective.

Lemma 9.4. *$K(\alpha)$ is relatively 2-prime with respect to $S_n(\alpha)$.*

Proof. The symmetries we consider have the form $(0, Q)$. Suppose now that Q has u-grading i. The actions of $\mathcal{L}(K^0)$ and $\mathcal{L}(S_n^0)$ are symbolically given by multiplication with the $\mathcal{G}$-functions

$$\mathcal{G}_n^i = \xi_1^n + \xi_2^n + \cdots + \xi_{i+1}^n.$$

In the symbolic language

$$\mathcal{L}(S_n)Q \in Im(\mathcal{L}(K))$$

implies

$$Q \in Im(\mathcal{L}(K))$$

whenever $\mathcal{G}_3^i$ and $\mathcal{G}_n^i$ are relatively prime, compare with lemma 4.3. All $\mathcal{G}_n^i$ with $i \geq 2$ are irreducible because the projective hypersurfaces given by

$$\mathcal{G}_n^i = 0$$

are nonsingular, compare with theorem 7.21. This shows that $K(\alpha)$ is relatively 2-prime with respect to $S_n(\alpha)$. □

9.4 Solving the symmetry conditions

Abstract. We solve the first nontrivial symmetry condition for several different forms of the term of lowest grading Q^0. We also give a recursive formula for the higher order terms in a special case.

We look for symmetries of the form $(0, Q_k)$. Automatically the first equation

$$\mathcal{L}(K^0)Q_k^0 = 0$$

is satisfied. The next, and already the last, symmetry condition to solve reads

$$\begin{aligned} 0 \;=\; & \begin{bmatrix} 0 & 0 \\ D^u_{Q_k^1} & D^v_{Q_k^1} \end{bmatrix} \begin{bmatrix} u_3 \\ 0 \end{bmatrix} - \begin{bmatrix} D_x^3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ Q_k^1 \end{bmatrix} \\ & + \begin{bmatrix} 0 & 0 \\ D^u_{Q_k^0} & D^v_{Q_k^0} \end{bmatrix} \begin{bmatrix} 3uu_1 \\ \alpha u_1 v + uv_1 \end{bmatrix} - \begin{bmatrix} 3(u_1 + uD_x) & 0 \\ \alpha v D_x + v_1 & \alpha u_1 + uD_x \end{bmatrix} \begin{bmatrix} 0 \\ Q_k^0 \end{bmatrix}. \end{aligned}$$

which is equivalent to the equation

$$D^u_{Q^1_k}[u_3] = uD_xQ^0_k + \alpha u_1 Q^0_k - D^v_{Q^0_k}[\alpha u_1 v + uv_1].$$

This can be solved if the coefficients of u, u_1 and u_2 vanish. Expanding the terms in the right hand side gives

$$\begin{aligned} uD_xQ^0_k + \alpha u_1 Q^0_k - D^v_{Q^0_k}[\alpha u_1 v + uv_1] \;&=\; u(D_xQ^0_k - v_{i+1}\partial_{v_i}Q^0_k) \\ &\quad +u_1(\alpha Q^0_k - (\alpha+i)v_i\partial_{v_i}Q^0_k) \\ &\quad +u_2(-\alpha i - \frac{i(i-1)}{2})v_{i-1}\partial_{v_i}Q^0_k \\ &\quad +\cdots \end{aligned}$$

where the sum over i is taken. Since total differentiation is done by the operator $D_x = v_{i+1}\partial_{v_i}$ (summation is assumed) the coefficient of u vanishes identically.

Let $\alpha \neq 0$. We make the following Ansatz.

Ansatz 9.5. *The term of lowest grading has the form*

$$Q^0_k \equiv \sum_{j=0}^{2k} c_j v_j v_{2k-j} v^{w/2-k-1},$$

of order $2k$ and weight w. Here k is a positive integer and w can be any number.

The operator $iv_i\partial_{v_i}$ counts the order, it multiplies Q^0_k with $2k$. The operator $v_i\partial_{v_i}$ counts the degree in v, it multiplies Q^0_k with $w/2-k+1$. Therefore the coefficient of u_1 vanishes when

$$w = 2k\frac{\alpha-2}{\alpha}.$$

When we put $w = 2k+2n$ we get

$$\alpha + 2 = 2(1-\frac{k}{n})$$

as predicted by Foursov in his conjecture. Only if $n \in \mathbb{N}$ the symmetries are polynomial, however, this would be an peculiar choice for n.

Straightforward calculation shows that the vanishing of the u_2-coefficient implies

$$c_j = c_{j-1}\frac{(j-1-2k)(2\alpha+2k-j)}{j(2\alpha+j-1)}.$$

This recursion relation can be solved as long as

$$\alpha \neq 0, -\frac{1}{2}, \cdots, \frac{1}{2}-k.$$

The result does not vanish since $c_{k+i} = c_{k-i}$ when $k \in \mathbb{N}$, which can easily be proven by induction on i. It is possible to do the computations for higher order. We use the symbolic calculus. With Ansatz 9.5 the symmetries of $K(\alpha)$ are symbolically given by

$$Q(\alpha, k) = \left[\begin{array}{c} 0 \\ \sum_{n=0}^{k} Q^n \end{array} \right],$$

where the term of lowest grading

$$Q^0 = F[k, \alpha](\eta_1, \eta_2) v^{-\frac{2k+a}{a}}$$

and F satisfies the linear differential equation

$$\alpha(\partial_{\eta_1} + \partial_{\eta_2})F + \frac{1}{2}(\eta_1 \partial_{\eta_1}^2 + \eta_2 \partial_{\eta_2}^2)F = 0.$$

The higher order Q^i satisfy the recurrence relation

$$\begin{aligned} \left(n \sum_{i=1}^{n} \xi_i^3\right) Q^n \quad = \quad & \sum_{j=1}^{n} \Bigg(((\sum_{i=1, i \neq j}^{n} \xi_i) + 2(\alpha + k)\xi_j + \eta_1 + \eta_2) Q^{n-1}(\xi_{n/j}, \eta_1, \eta_2) \\ & - \sum_{i=1}^{2} (\alpha \xi_j + \eta_i) Q^{n-1}(\xi_{n/j}, \eta_{3-i}, \xi_j + \eta_i) \\ & -3 \sum_{i>j}^{n} (\xi_j + \xi_i) Q^{n-1}(\xi_{n/j/i}, \xi_j + \xi_i, \eta_1, \eta_2) \Bigg), \end{aligned}$$

where

$$\xi_{n/i} = \xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n.$$

The implicit function theorem guarantees that this relation generates polynomials, which can be transformed into differential polynomials.

Example 9.6. *The equation $K(-\frac{4}{3})$ has the $k = 2$ symmetry*

$$R = \left[\begin{array}{c} 0 \\ (v_4 v + \frac{1}{2} v_1 v_3 - \frac{3}{20} v_2^2 + \frac{3}{2} u v_1^2 + \frac{4}{3} u_2 v^2 + 5 u_1 v_1 v + \frac{18}{5} u v_2 v + \frac{16}{15} u^2 v^2) v^2 \end{array} \right],$$

of weight 10 *and order* 4.

The procedure also works for complex α.

Example 9.7. *Let α be a primitive third root of unity. For $k = 1$ the procedure gives the symmetry*

$$Q = \left[\begin{array}{c} 0 \\ \frac{1}{6}(\alpha - 1)(3v_1^2 - 4vv_2 + 2\alpha uv^2 - 2\alpha vv_2 - 2v^2 u) v^{1+2\alpha} \end{array} \right].$$

One can also look for odd order solutions by taking k half integer. In this case we have

$$c_{k+1/2+i} = -c_{k-1/2-i}$$

which implies $Q_k^0 = 0$. However, when

$$-2\alpha \in \mathbb{N},\ 0 < 2k + 2\alpha \leq k$$

we have $c_j = 0$ for all

$$j \geq 2k + 2\alpha.$$

This means that when $-\alpha$ is integer or half integer there are respectively $-\alpha$ or $-2(\alpha+1)$ additional odd order solutions.

Example 9.8. *The only additional odd order symmetry with this form of $K(-\frac{3}{2})$ is $(0, P)$, where*

$$\begin{aligned} P &= vv_5 + \frac{5}{3}v_1v_4 + \frac{25}{3}u_1v_1^2 + \frac{25}{3}uv_1v_2 + 10u_1vv_2 \\ &\quad +5uvv_3 + 9u_2vv_1 + \frac{3}{2}u_3v^2 + \frac{9}{2}uu_1v^2 + 6u^2vv_1. \end{aligned}$$

To cover the higher values of k for integer or half integer negative α we start counting coefficients from the other side of the polynomial. The assumption we must make here is that $k \leq -\alpha$ or $k > -2\alpha$ whenever $-2\alpha \in \mathbb{N}$.

Ansatz 9.9. *Let*

$$Q_k^0 \equiv \sum_{i=0}^{k} b_i v_{k+i} v_{k-i} v^{w/2-k-1}.$$

Then the recurrence relation for the coefficients becomes

$$\begin{aligned} b_1 &= 2b_0 \frac{k(1-k-2\alpha)}{(k+1)(2\alpha+k)}, \\ b_i &= b_{i-1}\frac{(k+1-i)(i-k-2\alpha)}{(k+i)(k+i-1+2\alpha)}. \end{aligned}$$

Note that when

$$-2\alpha \in \mathbb{N},\ k = -2\alpha + 1 + i, i \in \mathbb{N},$$

all coefficients b_j with $j > i$ vanish.

There is more symmetry. We make another Ansatz.

Ansatz 9.10. *Let*

$$Q_k^0 \equiv \sum_{j=0}^{k} a_j v_{k-j} v_1^j v^{w/2-k/2-j},$$

of order k and weight w, again k is a positive integer and $w \in \mathbb{C}$.

The coefficient of u_1 vanishes if

$$w = k\frac{\alpha - 2}{\alpha}.$$

The coefficient of u_2 vanishes if

$$a_{j+1} = \frac{a_j(k-j)(j+1-2\alpha-k)}{2\alpha(j+1)}.$$

This procedure works for all integer $k > 1$ and all $w \in \mathbb{C}$. We have $Q_k^0 = 0$ when $k = 1$. For $k = 2$ one obtains the same symmetries as taking $k = 1$ in Ansatz 9.5 or 9.9. Note that when α is a negative integer or half integer we have $a_j = 0$ for all $j > k - 1 + 2\alpha$.

Example 9.11. *The equation $K(-\frac{4}{3})$ has the extra symmetry $T = (0, P)$ with*

$$\begin{aligned} P &= v^3v_4 + \frac{1}{2}v^2v_3v_1 - \frac{3}{16}vv_2v_1^2 + \frac{15}{256}v_1^4 \\ &\quad + \frac{4}{3}u_2v^4 + 5v^3u_1v_1 + 4uv^3v_2 + \frac{5}{4}uv^2v_1^2 + \frac{4}{3}u^2v^4 \end{aligned}$$

The weight and the order is the same as in Example 9.6.

Lemma 9.12. *The approximate symmetries Q commute with the symmetries S^n in lowest grading.*

Proof. The first component of S_n^0 does not depend on v and its second component vanishes. Moreover the first component of Q_k^0 vanishes and its second component does not depend on u. These properties assure that $\mathcal{L}(S_n^0)Q_k^0 = 0$. □

9.5 Noncommuting symmetries

We have shown that the KDV equation coupled to a purely nonlinear equation:

$$K(\alpha) = \begin{cases} u_t = & u_3 + 3uu_1 \\ v_t = & \alpha u_1 v + uv_1 \end{cases}$$

has infinitely many odd order symmetries $S_n(\alpha)$, that its linear part is nonlinear injective and that the linear part of any odd order symmetry $S_n(\alpha)$ is relatively 2-prime with $K(\alpha)$, cf. Lemmas 9.2, 9.3 and 9.4.

We solved the first two symmetry conditions for infinitely many Q (twice) for all α and showed that $\mathcal{L}(S_n^0)Q^0 = 0$. By the implicit function theorem, Theorem 3.8, all $Q_k(\alpha)$ commute with $K(\alpha)$ and with all $S_n(\alpha)$.

There is a linear map that transforms every symmetry of $K(\alpha)$ into a symmetry of the equation (9.1) found by Foursov. His conjecture, Conjecture 9.1, turns out to be true inside the class of polynomial symmetries. However, the symmetry structure of the equation is bigger than that.

The several symmetries $Q_k(\alpha)$ are mutually noncommuting, i.e., if R and T are two such symmetries of $K(\alpha)$ we have $\mathcal{L}(R)T \neq (0, 0)$. Since the Lie derivative is a representation, it follows that $\mathcal{L}(R)T$ is a symmetry of $K(\alpha)$.

Example 9.13. *Let R be the symmetry of example 9.6 and T be the symmetry of example 9.11. We have that $\mathcal{L}(R)T = (0, Q)$, with*

$$\begin{aligned}
Q \;=\;& (23400v^2v_1^4u_2 + 12288v^6u_2^2 + 6912v^4v_4^2 + 1296v^2v_2^4 + 270v_1^6u \\
&+90v_1^5v_3 - 27v_1^4v_2^2 + 63360v^2u^2v_1^4 + 107520v^5u_1^2v_2 + 79872v^4u^3v_1^2 \\
&-9216v^4u_2v_2^2 + 34560v^4v_3^2u + 23130vv_1^5u_1 + 2880v^3v_3^2v_2 + 12288v^5v_3u_3 \\
&+5310vv_1^4v_4 - 324vv_2^3v_1^2 + 2880v^2v_1^3v_5 - 20736v^3uv_2^3 + 28672v^6u_1^2u \\
&+5688v^2v_1^2v_3^2 + 18432v^5u_2v_4 + 16384v^6u_1u_3 + 456192v^4u_1v_1uv_2 \\
&+93312v^3v_1v_3uv_2 + 116736v^5u_1v_1u^2 + 49152v^5u_2uv_2 + 84480v^4v_3u_1v_2 \\
&+204288v^4u_2uv_1^2 + 18432v^4uv_1v_5 - 9216v^4v_2v_1u_3 + 36864v^4v_4uv_2 \\
&+104448v^4v_1v_3u^2 + 92160v^4v_1v_3u_2 + 99840v^4v_4u_1v_1 + 144960v^3v_1^2v_3u_1 \\
&-54144v^3v_2v_1^2u_2 + 29952v^3v_4v_3v_1 - 6912v^3v_2v_1v_5 + 378240v^3u_1v_1^3u \\
&+251136v^3v_2v_1^2u^2 + 74880v^3v_4uv_1^2 - 97920v^3u_1v_1v_2^2 - 22320v^2v_2v_1^3u_1 \\
&-69552v^2v_2^2v_1^2u - 7776v^2v_1v_3v_2^2 + 46080v^2v_1^3v_3u - 11232v^2v_2v_1^2v_4 \\
&+26460vv_2v_1^4u + 1764vv_2v_1^3v_3 - 6912v^3v_4v_2^2 + 67584v^5v_3u_1u + 24576v^5uv_1u_3 \\
&+202752v^5u_2u_1v_1 + 9216v^4v_3v_5 + 41472v^4u^2v_2^2 + 6144v^5v_4u^2 + 312960v^4u_1^2v_1^2 \\
&+12288v^5u_1v_5 + 8192v^6u_2u^2 + 3840v^3v_1^3u_3 + 12288v^5u^3v_2)\frac{v}{7680}
\end{aligned}$$

is a symmetry of $K(-\frac{4}{3})$. The term of u-grading 0 *is neither of the forms given in Ansatz 9.5, 9.9 or 9.10.*

The full symmetry structure of the equation $K(\alpha)$ is related to the sets of polynomial solutions of the linear differential equations

$$(\sum_{i=1}^{n} 2\alpha\partial_{\eta_i} + \eta_i\partial_{\eta_i}^2)F = 0, \quad n = 2, 3, 4, \ldots.$$

We have not studied the relations between these symmetries. We do not know whether it is possible to generate all symmetries by taking Lie derivatives starting from some smaller set of symmetries.

Chapter 10

Complex of variational calculus

We describe the complex of variational calculus. From the representations on the space of vertical vector fields and on the space of densities new representations are constructed. The several corresponding invariants are called cosymmetries, symplectic operators, cosymplectic operators and recursion operators.

10.1 n-Forms

In general, a **complex** is defined as a sequence of vector spaces and linear maps between successive spaces with the property that the composition of any pair of successive maps is identically zero. The framework of formal variational calculus was developed by Gel'fand and Dikiĭ. Apparently Loday, cf. [Lod91][Chapter 10], was the first to notice that the construction of a Lie algebra complex can be lifted from the antisymmetric case to the general case. He speaks of a 'simple, but striking result'. The sequence of vector spaces we consider consist of the following spaces.

Notation 10.1. *Take $n \in \mathbb{N}$. A multilinear map from $\mathfrak{h}^n$ to the space of densities Ω^0 is called a* n**-form**. *The space of n-forms is notated Ω^n.*

We will construct representations $\mathcal{L}^n$ of $\mathfrak{g}$ on the spaces Ω^n. and linear maps $\mathfrak{d}^n \in \operatorname{Hom}(\Omega^n, \Omega^{n+1})$ that satisfy $\mathfrak{d}^{n+1}\mathfrak{d}^n = 0$. We also prove that the $\mathfrak{d}^n$ are $\mathfrak{g}$-module maps, i.e., that the diagram

$$\begin{array}{ccccccc}
\Omega^0 & \xrightarrow{\mathfrak{d}^0} & \Omega^1 & \xrightarrow{\mathfrak{d}^1} & \Omega^2 & \xrightarrow{\mathfrak{d}^2} & \cdots \\
\big\downarrow{\scriptstyle \mathcal{L}^0(w)} & & \big\downarrow{\scriptstyle \mathcal{L}^1(w)} & & \big\downarrow{\scriptstyle \mathcal{L}^2(w)} & & \\
\Omega^0 & \xrightarrow{\mathfrak{d}^0} & \Omega^1 & \xrightarrow{\mathfrak{d}^1} & \Omega^2 & \xrightarrow{\mathfrak{d}^2} & \cdots
\end{array} \tag{10.1}$$

commutes. This is the complex of variational calculus. The coboundary operator is related to the Euler operator, or 'variational derivative'.

10.2 Constructing new representations

Abstract. We give a general rule to construct new representations and use this rule to define the Lie derivative on the spaces Ω^n, $\mathrm{Hom}(\Omega^1, \mathfrak{h})$, $\mathrm{Hom}(\mathfrak{h}, \Omega^1)$, $\mathrm{Hom}(\mathfrak{h}, \mathfrak{h})$ and $\mathrm{Hom}(\Omega^1, \Omega^1)$.

Proposition 10.2. *Let $H = Hom(F, G)$. If F and G are U-modules then so is H.*

Proof. Let the product of U be $P : U \to \mathrm{End}(U)$. By assumption we have representations of U on F and on G. They are denoted

$$Q_1 : U \to \mathrm{End}(F), Q_2 : U \to \mathrm{End}(G).$$

They satisfy

$$Q_i(P(x)y) = Q_i(x)Q_i(y) - Q_i(y)Q_i(x), \ i = 1, 2.$$

We prove the existence of a representation of U on H, i.e., a linear mapping

$$Q_3 : U \to \mathrm{End}(H)$$

satisfying the above equation for $i = 3$.

Let $\iota : F \to \mathrm{Hom}(H, G)$ be defined by

$$\iota(f)h = h(f).$$

Define Q_3 by the **Cartan identity**:

$$\iota(y)Q_3(x) = Q_2(x)\iota(y) - \iota(Q_1(x)y). \tag{10.2}$$

We first calculate

$$\begin{aligned}
&\iota(z)Q_3(x)Q_3(y) \\
&= Q_2(x)\iota(z)Q_3(y) - \iota(Q_1(x)z)Q_3(y) \\
&= Q_2(x)Q_2(y)\iota(z) - Q_2(x)\iota(Q_1(y)z) \\
&\quad - Q_2(y)\iota(Q_1(x)z) + \iota(Q_1(y)Q_1(x)z).
\end{aligned}$$

By interchanging x with y

$$\begin{aligned}
&\iota(z)(Q_3(x)Q_3(y) - Q_3(y)Q_3(x)) \\
&= (Q_2(x)Q_2(y) - Q_2(y)Q_2(x))\iota(z) \\
&\quad + \iota((Q_1(y)Q_1(x) - Q_1(x)Q_1(y))z) \\
&= Q_2(P(x)y)\iota(z) - \iota(Q_1(P(x)y)z) \\
&= \iota(z)Q_3(P(x)y).
\end{aligned}$$

Since $\mathrm{Ker}(\iota(z)) = 0$, i.e., if $h \in H$ maps all elements in $z \in F$ to zero then $h = 0$, it follows that

$$Q_3(P(x)y) = Q_3(x)Q_3(y) - Q_3(y)Q_3(x).$$

□

Corollary 10.3. *Suppose*

$$G = Hom(V^n, F),\ \ H = Hom(V^{n+1}, F).$$

If F and G are U-modules then so is H.

Proof. The proof is similar to the proof of Lemma 10.2, except that

$$\iota : V \to \mathrm{Hom}(H, G)$$

is defined by

$$\iota(v_0)h(v_1, \dots, v_n) = h(v_0, v_1, \dots, v_n).$$

Q_2 is now a representation of U on $\mathrm{Hom}(V^n, F)$ and the existence of a representation Q_3 of U on $\mathrm{Hom}(V^{n+1}, F)$ is proven. □

It is surprising that in the corollary we do not need to assume any structure on V. Often there is a structure on V. In our application V is a Leibniz subalgebra of U. In this way we get a representation of U on V for free, as well as representations of V on F and on G.

Remark 10.4. *A more general construction can be made. Call F a $(U\&V)$*-**module** *if it is a U-module and a V-module and there exist linear mappings*

$$P : U \to End(V),\ \ Q : U \to End(F),\ \ R : V \to End(F)$$

such that

$$R(P(x)y) = Q(x)R(y) - R(y)Q(x).$$

The word U-module can be replaced by $(U\&V)$-module in both Lemma 10.2 and Corollary 10.3.

Since Ω^0 and $\mathfrak{h}$ are $\mathfrak{g}$-modules, by Proposition 10.2 the space $\Omega^1 = \mathrm{Hom}(\mathfrak{h}, \Omega^0)$ is a $\mathfrak{g}$-module. The Lie derivative of $\omega_1 \in \Omega^1$ in the direction of $v \in \mathfrak{g}$ is

$$(\mathcal{L}^1(v)\omega_1)(w) = \mathcal{L}^0(v)\omega_1(w) - \omega_1(\mathcal{L}_1(v)w).$$

Since both Ω^1 and $\mathfrak{h}$ are $\mathfrak{g}$-modules by Proposition 10.2 the spaces

$$\mathrm{Hom}(\Omega^1, \mathfrak{h})\ \ \mathrm{Hom}(\mathfrak{h}, \Omega^1),\ \ \mathrm{Hom}(\mathfrak{h}, \mathfrak{h}),\ \ \mathrm{Hom}(\Omega^1, \Omega^1)$$

are also $\mathfrak{g}$-modules. With Corollary 10.3 we can recursively construct the representations

$$\mathcal{L}^n : \mathfrak{g} \to \mathrm{End}(\Omega^n).$$

10.2.1 Cosymmetries

Definition 10.5. *A* **cosymmetry** *is an element $\omega_1 \in \Omega^1$ that is in the kernel of $\mathcal{L}(D_t)$.*

Notation 10.6. *We write $\bar{\mathcal{H}}$ for the N-dimensional $\mathcal{A}$-module with basis*

$$du^1, du^2, \dots, du^N.$$

And we define a product between $\omega = \sum_{i=1}^N \omega^i du^i \in \bar{\mathcal{H}}$ and $V = \sum_{i=1}^N V_i \partial_{u^i} \in \mathcal{H}$ as follows:

$$\omega \cdot V = \sum_{i=1}^N \omega^i V_i \in \mathcal{A},$$

which resembles the **inner product** *on $\mathcal{A}^N$.*

We have applications in mind where Ω^1 can be identified with $\bar{\mathcal{H}}$, the dual to $\mathcal{H}$. We write ω_1 for the element in Ω^1 that is identified with the covector $\omega \in \bar{\mathcal{H}}$.

Notation 10.7. *The* **pairing** *between $\omega_1 \in \Omega^1$ and $\delta(V) \in \mathfrak{h}$ is*

$$\omega_1(\delta(V)) = \int \omega \cdot V \in \Omega^0.$$

Proposition 10.8. *The pairing is nondegenerate, i.e., $\omega_1 = 0$ implies $\omega = 0$.*

Proof. We prove that

$$\int \omega \cdot V = 0$$

for all $V \in \mathcal{H}$ implies $\omega = 0$. Suppose that the i-th component of ω contains the highest order derivative u_n^α. Take for V the i-th basis vector. We have that $\int \omega \cdot \partial_{u^i} = 0$ implies that $\omega^i \in \mathrm{Im}(D_x)$. This means that ω depends linearly on u_n^α. With V the i-th unit vector multiplied with u_n^α, we get $\int \omega \cdot u_n^\alpha \partial_{u^i} = 0$. But we have

$$\omega \cdot u_n^\alpha \partial_{u^i} \notin \mathrm{Im}(D_x).$$

This contradiction shows that ω does not depend on u^β or its derivatives. Neither can ω depend on x, t nor can it be constant. Therefore $\omega = 0$. □

We like to write the Lie derivative in terms of Fréchet derivatives. The Fréchet derivative $D_f[V]$ of $f \in \mathcal{A}$ in the direction of $V \in \mathcal{H}$ was defined in Definition 2.7. It satisfies the Leibniz rule, which is easily shown by

$$D_{fg}[V] = \delta(V)(fg) = \delta(V)(f)g + f\delta(V)(g) = D_f[V]g + fD_g[V].$$

Both the Fréchet derivatives of $\omega \in \bar{\mathcal{H}}$ and $W \in \mathcal{H}$ are defined in terms of their components. We have $D_\omega \in \mathrm{Hom}(\mathcal{H}, \bar{\mathcal{H}})$, $D_W \in \mathrm{End}(\mathcal{H})$ and

$$D_{\omega \cdot W}[V] = D_\omega[V] \cdot W + \omega \cdot D_W[V].$$

Definition 10.9. *The operator $P^\star \in End(\Omega^1)$ is the* **conjugate operator** *to $P \in End(\mathfrak{h})$ if*

$$P^\star(\omega_1)(v) = \omega_1(P(v)),$$

for all $\omega_1 \in \Omega^1$ and $v \in \mathfrak{h}$ in the domain of P. The operator $P^\dagger \in Hom(\mathfrak{h}, \Omega^1)$ is the **adjoint operator** *to $P \in Hom(\mathfrak{h}, \Omega^1)$ if*

$$P^\dagger(v)(w) = P(w)(v)$$

for all $v, w \in \mathfrak{h}$.

Proposition 10.10. *The* **Lie derivative** *of $\omega_1 \in \Omega^1$ in the direction of D_t is written in terms of the Fréchet derivative as follows*

$$\mathcal{L}(D_t)\omega = \partial_t\omega + D_\omega[K] + D_K^\star[\omega].$$

Proof. With $\omega(V) = \omega_1(\delta(V))$ and equations (2.8), (2.9) it follows that for $\omega \in \bar{\mathcal{H}}$ and any $V \in \mathcal{H}$

$$\begin{aligned}
&(\mathcal{L}(D_t)\omega)(V)\\
&= \mathcal{L}(D_t)\omega(V) - \omega(\mathcal{L}(D_t)V)\\
&= \int (\partial_t(\omega \cdot V) + D_{(\omega\cdot V)}[K] - \omega \cdot (\partial_t V + D_V[K] - D_K[V]))\\
&= \int (\partial_t\omega + D_\omega[K] + D_K^\star[\omega]) \cdot V.
\end{aligned}$$

By the nondegeneracy of the pairing, the formula is obtained. □

Example 10.11 (KDV)**.** *The first three cosymmetries of the KDV equation are*

$$1,\ u,\ 2u_2 + u^2.$$

We show that they are in the kernel of $\mathcal{L}(D_t)$. The conjugate of the Fréchet derivative of $K = u_3 + uu_1$ is

$$D_K^\star = -(D_x^2 + u)D_x.$$

Therefore

$$\begin{aligned}
\mathcal{L}(D_t)1 \quad &= D_1[K] + D_K^\star[1]\\
&= 0.\\
\mathcal{L}(D_t)u \quad &= D_u[K] + D_K^\star[u]\\
&= K - K\\
&= 0.\\
\mathcal{L}(D_t)(2u_2 + u^2) \quad &= D_{2u_2+u^2}[K] + D_K^\star[2u_2 + u^2]\\
&= 2(D_x^2 + u)D_x(u_2 + \frac{1}{2}u^2) - (D_x^2 + u)D_x(2u_2 + u^2)\\
&= 0.
\end{aligned}$$

If $\omega_1 \in \Omega^1$ is homogeneous, the product $\omega \cdot v \in \mathcal{A}$ should be homogeneous for all homogeneous $v \in \mathcal{H}$. This means that the components of ω should satisfy

$$\mathcal{L}(\sigma)(\omega^\beta) = (\lambda - \lambda(u^\beta))\omega^\beta$$

for ω_1 to be homogeneous with weight λ.

Example 10.12 (KDV)**.** *Let $\lambda(x) = -1$ and $\lambda(u) = 2$. The element $2u_2 + u^2 \in \mathcal{A}$ is homogeneous with weight 4. Therefore the cosymmetry it represents is homogeneous with weight 6.*

10.3 Invariant operators

Definition 10.13. *A* **recursion operator** *maps symmetries to symmetries and a* **conjugate recursion operator** *maps cosymmetries to cosymmetries.*

Proposition 10.14. *An element $\mathfrak{R} \in End(\mathfrak{h})$ that is in the kernel of $\mathcal{L}(D_t)$ is a recursion operator. An element $\mathfrak{T} \in End(\Omega^1)$ that is in the kernel of $\mathcal{L}(D_t)$ is a conjugate recursion operator.*

Proof. This follows from the Cartan identity. Let v be a symmetry and ω a cosymmetry. We have

$$\begin{aligned} \mathcal{L}(D_t)\mathfrak{R}(v) &= (\mathcal{L}(D_t)\mathfrak{R})v - \mathfrak{R}(\mathcal{L}(D_t)v) = 0, \\ \mathcal{L}(D_t)\mathfrak{T}(\omega) &= (\mathcal{L}(D_t)\mathfrak{T})\omega - \mathfrak{T}(\mathcal{L}(D_t)\omega) = 0. \end{aligned}$$

□

Proposition 10.15. *Let $\mathfrak{H} \in Hom(\Omega^1, \mathfrak{h})$ and $\mathfrak{J} \in Hom(\mathfrak{h}, \Omega^1)$ be in the kernel of $\mathcal{L}(D_t)$. Then $\mathfrak{H}(\omega)$ is a symmetry of $u_t = K$ if $\omega \in \Omega^1$ is a cosymmetry. Also $\mathfrak{J}(v)$ is a cosymmetry if $v \in \mathfrak{h}$ is a symmetry.*

Proof. This follows from the Cartan identity (10.2). We have

$$\begin{aligned} \mathcal{L}(D_t)\mathfrak{H}(\omega) &= (\mathcal{L}(D_t)\mathfrak{H})\omega - \mathfrak{H}(\mathcal{L}(D_t)\omega) = 0, \\ \mathcal{L}(D_t)\mathfrak{J}(v) &= (\mathcal{L}(D_t)\mathfrak{J})v - \mathfrak{J}(\mathcal{L}(D_t)v) = 0. \end{aligned}$$

□

Proposition 10.16. *Let $\mathfrak{H} \in Hom(\Omega^1, \mathfrak{h})$ and $\mathfrak{J} \in Hom(\mathfrak{h}, \Omega^1)$ be in the kernel of $\mathcal{L}(D_t)$. Then $\mathfrak{R} = \mathfrak{H}\mathfrak{J}$ is a recursion operator and $\mathfrak{T} = \mathfrak{J}\mathfrak{H}$ is a conjugate recursion operator.*

Proof. This follows from the Cartan identity (10.2) and Proposition 10.15. Let v be a symmetry and ω a cosymmetry. Then we have

$$\mathcal{L}(D_t)\mathfrak{H}\mathfrak{J}(v) = (\mathcal{L}(D_t)\mathfrak{H})\mathfrak{J}(v) - \mathfrak{H}\mathcal{L}(D_t)\mathfrak{J}(v) = 0,$$
$$\mathcal{L}(D_t)\mathfrak{J}\mathfrak{H}(\omega) = (\mathcal{L}(D_t)\mathfrak{J})\mathfrak{H}(\omega) - \mathfrak{J}\mathcal{L}(D_t)\mathfrak{H}(\omega) = 0.$$

□

Since elements in $\mathfrak{h}$ and Ω^1 are both connected to elements in $\mathcal{A}^N$, the operators we considered can be connected to an operator that can be written in the form:

$$P = \sum_{i=-\infty}^{n} P_i D_x^i,$$

where P_i are $N \times N$ matrices with entries in $\mathcal{A}$. This is done by using the commutation relations

$$\begin{aligned} D_x f &= f D_x + D_x(f), \\ D_x^{-1} f &= f D_x^{-1} - D_x^{-1} D_x(f) D_x^{-1}. \end{aligned}$$

Furthermore we assume that P_n is an invertible matrix. Under this condition one can prove that the operator P is invertible if it is nonzero, cf. [Olv93a, Theorem 5.38]. Such operators are nondegenerate, i.e., $P(V) = 0$ for all V implies that $P = 0$. Therefore we can write the Lie derivatives in terms of Fréchet derivatives.

Definition 10.17. *The* **Fréchet derivative** *of P in the direction of V is*

$$D_P[V] = \sum_{i=-\infty}^{n} D_{P_i}[V] D_x^i,$$

where $(D_{P_i}[V])^{\alpha\beta} = D_{P_i^{\alpha\beta}}[V]$.

Again we have the Leibniz rule:

$$D_{PW}[V] = D_P[V]W + P D_W[V].$$

Proposition 10.18. *The Lie derivatives of $\mathfrak{H} \in Hom(\Omega^1, \mathfrak{h})$ and $\mathfrak{I} \in Hom(\mathfrak{h}, \Omega^1)$ in the direction of D_t are written in terms of the Fréchet derivative as follows*

$$\begin{aligned} \mathcal{L}(D_t)\mathfrak{H} &= \partial_t \mathfrak{H} + D_{\mathfrak{H}}[K] - D_K \mathfrak{H} - \mathfrak{H} D_K^\star, \\ \mathcal{L}(D_t)\mathfrak{I} &= \partial_t \mathfrak{I} + D_{\mathfrak{I}}[K] + D_K^\star \mathfrak{I} + \mathfrak{I} D_K. \end{aligned}$$

Proof.

$$\begin{aligned} &(\mathcal{L}(D_t)\mathfrak{H})(\omega) \\ &= \mathcal{L}_K \mathfrak{H}(\omega) - \mathfrak{H}(\mathcal{L}_K \omega) \\ &= \partial_t \mathfrak{H}(\omega) + D_{\mathfrak{H}(\omega)}[K] - D_K[\mathfrak{H}(\omega)] - \mathfrak{H}(\partial_t \omega + D_\omega[K] + D_K^\star[\omega]) \\ &= (\partial_t \mathfrak{H} + D_{\mathfrak{H}}[K] - D_K \mathfrak{H} - \mathfrak{H} D_K^\star)(\omega), \end{aligned}$$

$$\begin{aligned} &(\mathcal{L}(D_t)\mathfrak{I})(V) \\ &= \mathcal{L}_K \mathfrak{I}(V) - \mathfrak{I}(\mathcal{L}_K V) \\ &= \partial_t \mathfrak{I}(V) + D_{\mathfrak{I}(V)}[K] + D_K^\star[\mathfrak{I}(V)] - \mathfrak{I}(\partial_t V + D_V[K] - D_K[V]) \\ &= (\partial_t \mathfrak{I} + D_{\mathfrak{I}}[K] + D_K^\star \mathfrak{I} + \mathfrak{I} D_K)(V). \end{aligned}$$

□

Example 10.19 (KDV). *$D_x \in Hom(\Omega^1, \mathfrak{h})$ is an invariant of the KDV equation.*

$$\begin{aligned}\mathcal{L}(D_t)D_x &= D_{D_x}[K] - D_K D_x - D_x D_K^\star \\ &= 0 - (D_x^3 + D_x u) \circ D_x - D_x \circ (-D_x^3 - u D_x) \\ &= 0.\end{aligned}$$

Applying the operator D_x to the cosymmetries gives

$$\begin{aligned} D_x(1) &= 0, \\ D_x(u) &= u_1, \\ D_x(u_2 + \frac{1}{2}u^2) &= u_3 + u u_1. \end{aligned}$$

which are indeed symmetries of the equation. The KDV equation admits another invariant operator in $Hom(\Omega^1, \mathfrak{h})$,

$$\mathfrak{H} = D_x^3 + \frac{2}{3}u D_x + \frac{1}{3}u_1.$$

We calculate

$$\begin{aligned}\mathcal{L}(D_t)(\mathfrak{H}) &= D_{\mathfrak{H}}[K] - D_K\mathfrak{H} - \mathfrak{H}D_K^\star \\ &= \frac{2}{3}(u_3 + uu_1)D_x + \frac{1}{3}D_x(u_3 + uu_1) \\ &\quad -(D_x^3 + D_x u) \circ (D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_1) \\ &\quad +(D_x^3 + \frac{2}{3}D_x u - \frac{1}{3}u_1) \circ (D_x^3 + uD_x) \\ &= \frac{2}{3}u_3 D_x + \frac{1}{3}D_x(u_3) + \frac{1}{3}D_x^3 u D_x \\ &\quad -\frac{1}{3}D_x^3 u_1 - \frac{1}{3}D_x u D_x^3 - \frac{1}{3}u_1 D_x^3 \\ &= \frac{1}{3}(2u_3 D_x + u_4 + u_3 D_x + 3u_2 D_x^2 + 3u_1 D_x^3 + u D_x^4 \\ &\quad -u_4 - 3u_3 D_x - 3u_2 D_x^2 - u_1 D_x^3 - u_1 D_x^3 - u D_x^4 - u_1 D_x^3) \\ &= 0.\end{aligned}$$

Applying the operator D_x to the cosymmetries gives

$$\begin{aligned}\mathfrak{H}(1) &= \frac{1}{3}u_1 \\ \mathfrak{H}(u) &= u_3 + uu_1, \\ \mathfrak{H}(u_2 + \frac{1}{2}u^2) &= u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1,\end{aligned}$$

which are indeed symmetries of the equation.

We can invert the first operator, D_x, and obtain a new invariant

$$\mathfrak{J} = D_x^{-1} \in Hom(\mathfrak{h}, \Omega^1)$$

of the KDV equation. We have

$$\begin{aligned}\mathcal{L}(D_t)D_x^{-1} &= D_{D_x^{-1}}[K] + D_K D_x^{-1} + D_x^{-1} D_K^\star \\ &= 0 + (D_x^3 + D_x u) \circ D_x^{-1} + D_x^{-1} \circ (-D_x^3 - uD_x) \\ &= 0.\end{aligned}$$

By Proposition 10.16 we have

$$D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1} \tag{10.3}$$

as an recursion operator of the KDV-equation $u_t = u_3 + uu_1$.

Remark 10.20. *With the expression D_x^{-1} one has to be very careful. The rule $D_x^{-1}D_x = 1$ does not hold, i.e., $D_x^{-1}(D_x(f)) = 0$ when $f \in Ker(D_x)$, cf. [OSW02].*

Proposition 10.21. *The Lie derivatives of $\mathfrak{R} \in Hom(\mathfrak{h}, \mathfrak{h})$ and $\mathfrak{T} \in Hom(\Omega^1, \Omega^1)$ in the direction of D_t are written in terms of the Fréchet derivative as:*

$$\begin{aligned}\mathcal{L}(D_t)\mathfrak{R} &= \partial_t \mathfrak{R} + D_{\mathfrak{R}}[K] - D_K \mathfrak{R} + \mathfrak{R} D_K, \\ \mathcal{L}(D_t)\mathfrak{T} &= \partial_t \mathfrak{T} + D_{\mathfrak{T}}[K] + D_K^\star \mathfrak{T} - \mathfrak{T} D_K^\star.\end{aligned}$$

Proof.

$$\begin{aligned}&(\mathcal{L}_K \mathfrak{R})(V) \\ &= \mathcal{L}_K \mathfrak{R}(V) - \mathfrak{R}(\mathcal{L}_K V) \\ &= \partial_t \mathfrak{R}(V) + D_{\mathfrak{R}(V)}[K] - D_K[\mathfrak{R}(V)] - \mathfrak{R}(\partial_t V + D_V[K] - D_K[V]) \\ &= (\partial_t \mathfrak{R} + D_{\mathfrak{R}}[K] - D_K \mathfrak{R} + \mathfrak{R} D_K)(V),\end{aligned}$$

$$\begin{aligned}&(\mathcal{L}_K \mathfrak{T})(\omega) \\ &= \mathcal{L}_K \mathfrak{T}(\omega) - \mathfrak{T}(\mathcal{L}_K \omega) \\ &= \partial_t \mathfrak{T}(\omega) + D_{\mathfrak{T}(\omega)}[K] + D_K^\star[\mathfrak{T}(\omega)] - \mathfrak{T}(\partial_t \omega + D_\omega[K] + D_K^\star[\omega]) \\ &= (\partial_t \mathfrak{T} + D_{\mathfrak{T}}[K] + D_K^\star \mathfrak{T} - \mathfrak{T} D_K^\star)(\omega).\end{aligned}$$

□

The Leibniz rule, or chain rule, implies that the components should satisfy

$$\begin{aligned}\sigma(\mathfrak{H}^{\alpha\beta}) &= (\lambda + \lambda(u^\alpha) + \lambda(u^\beta))\mathfrak{H}^{\alpha\beta}, \\ \sigma(\mathfrak{J}^{\alpha\beta}) &= (\lambda - \lambda(u^\alpha) - \lambda(u^\beta))\mathfrak{J}^{\alpha\beta}, \\ \sigma(\mathfrak{R}^{\alpha\beta}) &= (\lambda + \lambda(u^\alpha) - \lambda(u^\beta))\mathfrak{R}^{\alpha\beta}, \\ \sigma(\mathfrak{T}^{\alpha\beta}) &= (\lambda - \lambda(u^\alpha) + \lambda(u^\beta))\mathfrak{T}^{\alpha\beta},\end{aligned}$$

to make $\mathfrak{H} \in \mathrm{Hom}(\Omega^1, \mathfrak{h})$, $\mathfrak{J} \in \mathrm{Hom}(\mathfrak{h}, \Omega^1)$, $\mathfrak{R} \in \mathrm{End}(\mathfrak{h})$ and $\mathfrak{T} \in \mathrm{End}(\Omega^1)$ homogeneous of weight λ.

10.4 The coboundary operator

Definition 10.22. *The* **coboundary operator** $\mathfrak{d}^n : \Omega^n \to \Omega^{n+1}$ *is defined recursively by* $\mathfrak{d}^{-1} = 0$ *and*

$$\iota(v)\mathfrak{d}^n = \mathcal{L}^n(v) - \mathfrak{d}^{n-1}\iota(v),\ \ v \in \mathfrak{h}.$$

Proposition 10.23. $\mathfrak{d}^n$ *is a g-module map.*

Proof. Suppose that $\mathcal{L}^n(v)\mathfrak{d}^{n-1} = \mathfrak{d}^{n-1}\mathcal{L}^{n-1}(v)$. Then we have

$$\begin{aligned}\iota(w)\mathcal{L}^{n+1}(v)\mathfrak{d}^n &= \mathcal{L}^n(v)\iota(w)\mathfrak{d}^n - \iota(\mathcal{L}(v)w)\mathfrak{d}^n \\ &= \mathcal{L}^n(v)(\mathcal{L}^n(w) - \mathfrak{d}^{n-1}\iota(w)) - \mathcal{L}^n(\mathcal{L}(v)w) + \mathfrak{d}^{n-1}\iota(\mathcal{L}(v)w) \\ &= \mathcal{L}^n(w)\mathcal{L}^n(v) - (\mathcal{L}^n(v)\mathfrak{d}^{n-1} - \mathfrak{d}^{n-1}\mathcal{L}^{n-1}(v))\iota(w) - \mathfrak{d}^{n-1}\iota^n(w)\mathcal{L}^n(v) \\ &= \iota(w)\mathfrak{d}^n\mathcal{L}^n(v).\end{aligned}$$

The case $n = 0$ follows from $\mathfrak{d}^{-1} = 0$. By induction the diagram (10.1) commutes. □

Proposition 10.24. *The coboundary operator satisfies* $\mathfrak{d}^n\mathfrak{d}^{n-1} = 0$.

Proof. Assume it holds for $n-1$. Then, using the previous proposition,

$$\begin{aligned}\iota^{n+2}(x)\mathfrak{d}^{n+1}\mathfrak{d}^n &= \mathcal{L}^{n+1}(x)\mathfrak{d}^n - \mathfrak{d}^n\iota^{n+1}(x)\mathfrak{d}^n \\ &= \mathfrak{d}^n\mathcal{L}^n(x) - \mathfrak{d}^n(\mathcal{L}^n(x) - \mathfrak{d}^{n-1}\iota^n(x)) \\ &= \mathfrak{d}^n\mathfrak{d}^{n-1}\iota^n(x).\end{aligned}$$

The case $n = 0$ follows from $\mathfrak{d}^{-1} = 0$. By induction the statement is proven. □

Due to $\mathfrak{d}^n\mathfrak{d}^{n-1} = 0$, cohomology can be defined as usual:

Definition 10.25. *The space of* **closed n-forms** *is Ker*$(\mathfrak{d}^n)$ *and the space of* **exact n-forms** *is Im*$(\mathfrak{d}^{n-1})$. *The* **n-th cohomology module** *is Ker*$(\mathfrak{d}^n)/$*Im*$(\mathfrak{d}^{n-1})$.

10.5 The Euler operator

Abstract. We define the Euler operator and show that it equals $\mathfrak{d}^0$.

Definition 10.26. *The* **Euler operator** $E \in Hom(\mathcal{A}, \mathcal{H})$ *is defined by*

$$E(\rho) = \sum_{\beta=1}^{N} E^\beta(\rho),\ \ E^\beta = \sum_{k=1}^{\infty}(-D_x)^k\partial_{u_k^\beta}.$$

Using partial integration we obtain

$$\mathfrak{d}^0\rho(V) = \int\sum_{\beta=1}^{n}\sum_{k=1}^{\infty} D_x^k(V^\beta)\partial_{u_k^\beta}\rho = \sum_{\beta=1}^{N}\int(\sum_{k=1}^{\infty}(-D_x)^k\partial_{u_k^\beta}\rho)V^\beta.$$

By the nondegeneracy of the pairing we obtain the identification $\mathfrak{d}^0\rho = E(\rho) \in \bar{\mathcal{H}}$.

Example 10.27 (KDV). *We apply the Euler operator to the first three densities of example 2.25. We have*

$$\begin{aligned} E(u) &= 1, \\ E(u^2/2) &= u, \\ E(\tfrac{1}{6}u^3 - \tfrac{1}{2}u_1^2) &= u_2 + \frac{1}{2}u^2. \end{aligned}$$

Observe that the resulting expressions are the cosymmetries of Example 10.11.

10.6 Symplectic forms

Definition 10.28. *An element $\omega_2 \in \Omega^2$ such that $\mathfrak{d}^2\omega_2 = 0$ is a* **symplectic form** *if it is nondegenerate. If*

$$\omega_2(v, w) = \mathfrak{J}(v)w$$

then $\mathfrak{J} \in Hom(\mathfrak{h}, \Omega^1)$ is a **symplectic operator**. *In the other direction, if*

$$\iota(v)\omega_2 = \omega_1 \in \Omega^1$$

we write $v = \mathfrak{H}(\omega_1)$ where $\mathfrak{H} \in Hom(\Omega^1, \mathfrak{h})$ is a **cosymplectic operator**.

Proposition 10.29. *For any $\omega_1 \in \Omega^1$, $\mathfrak{d}^1\omega_1 = 0$ is equivalent to $D_{\omega_1} = D^\dagger_{\omega_1}$.*

Proof. We have

$$\begin{aligned} \mathfrak{d}^1\omega_1(v, w) &= \mathcal{L}^1(v)\omega_1(w) - \mathfrak{d}^0\omega_1(v)w \\ &= \mathcal{L}^0(v)\omega_1(w) - \omega_1(\mathcal{L}_1(v)w) - \mathcal{L}^0(w)\omega_1(v) \\ &= \int D_{\omega\cdot w}[v] - D_{\omega\cdot v}[w] - \omega\cdot(D_w[v] - D_v[w]) \\ &= \int D_\omega[v]\cdot w - D_\omega[w]\cdot v \\ &= \int (D_\omega - D^\dagger_\omega)[v]\cdot w. \end{aligned}$$

The result follows from the nondegeneracy of the pairing. □

The following question arises: if $D_\omega - D^\dagger_\omega = 0$, can we find $\omega_0 \in \Omega^0$ such that $\mathfrak{d}^0\omega_0 = \omega_1$? The answer depends on the choice of the ring. It is yes when $\mathcal{A}$ consists of polynomials, due to the vanishing of the first cohomology space. More details and the procedure of finding the solution is described in [Dor93, page 62-73]. The element ω_0 is called the density of ω_1. Since $\mathfrak{d}^0$ is a $\mathfrak{g}$-module map ω_0 is conserved if ω_1 is a cosymmetry.

On the other hand, suppose that $\mathfrak{J} = D_\omega - D^\dagger_\omega$ is nondegenerate. Then $\mathfrak{J}$ is a symplectic operator. If in this case ω_1 is a cosymmetry, $\mathfrak{J} : \mathrm{Hom}(\mathfrak{h}, \Omega^1)$ is an invariant of the equation, i.e., it maps symmetries to cosymmetries.

Example 10.30. *Consider the evolution equation*

$$\begin{cases} u_t = & -u_2 + v^2 \\ v_t = & v_2 \end{cases}.$$

It has symmetries

$$S_0 = \begin{bmatrix} 2u \\ v \end{bmatrix}, \qquad S_1 = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix},$$
$$S_2 = \begin{bmatrix} -u_2 + v^2 \\ v_2 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 3v_1 v - 2u_3 \\ v_3 \end{bmatrix}.$$

It has cosymmetries

$$C_0 = [v, -2u], \qquad C_1 = [v_1, -u_1],$$
$$C_2 = [v_2, u_2 - v^2], \quad C_3 = [v_3, -3v_1 v + 2u_3].$$

The conserved densities from C_1, C_2 are

$$B_1 = \tfrac{1}{2} v_1 u - \tfrac{1}{2} u_1 v, \quad B_2 = \frac{1}{2} v_2 u + \frac{1}{2} v u_2 - \frac{1}{3} v^3.$$

The symplectic operators from C_0, C_3 are

$$\mathfrak{J}_0 = 3 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathfrak{J}_3 = 3 \begin{bmatrix} 0 & D_x^3 \\ D_x^3 & -v_1 - 2vD_x \end{bmatrix}.$$

They satisfy the relation $\mathfrak{J}_3 = \mathfrak{T}_3 \mathfrak{J}_0$ where

$$\mathfrak{T}_3 = \begin{bmatrix} D_x^3 & 0 \\ -v_1 - 2vD_x & -D_x^3 \end{bmatrix}$$

is a conjugate recursion operator. Its conjugate

$$\mathfrak{R}_3 = \begin{bmatrix} -D_x^3 & v_1 + 2vD_x \\ 0 & D_x^3 \end{bmatrix}$$

maps symmetries to symmetries.

Appendix A

Homogeneity

We will show that the problem of finding homogeneous equations with homogeneous invariants is part of the problem of finding nonhomogeneous equations with nonhomogeneous invariants. Also, we show it suffices to find all homogeneous invariants of a homogeneous equation.

However, homogeneity need not be imposed from the start; much of the analysis can be done without! In the classification of $\mathcal{B}$-equations, cf. Chapter 6, and the determination of the spectrum of eigenvalues, cf. Chapter 7, homogeneity was not imposed at all. One may want to work with homogeneous equations when writing down candidate equations possessing candidate symmetries, cf. Chapter 5.

Lemma A.1. *Suppose that $v \in \mathfrak{g}$ and q in some $\mathfrak{g}$-module are homogeneous. Then*

$$\lambda(\mathcal{L}(v)q) = \lambda(v) + \lambda(q).$$

Proof. This is just a reformulation of Lemma 2.31. $\square$

Lemma A.2. *Let p, q be elements in some $\mathfrak{g}$-module of which p is homogeneous. Suppose that, if q is homogeneous, $\lambda(q) \neq \lambda(p)$. Then, if $p + q = 0$ then $p = 0$.*

Proof. Suppose that $p \neq 0$. We have $p + q = 0$ if $q = -p$. But then

$$\lambda(q) = \lambda(-p) = \lambda(p)$$

which contradicts our assumption. Hence $p = 0$. $\square$

In the considerations we assume that we can (formally) expand the right hand side of our equation.

- ⋆ Suppose that the nonhomogeneous equation $u_t = K$ has a nonhomogeneous time-independent invariant Q, i.e., we have $\mathcal{L}(K)Q = 0$. For any choice of $\lambda(t), \lambda(x), \lambda(u)$ we can write

$$K = K^1 + K^2 + \cdots ,$$
$$Q = Q^1 + Q^2 + \cdots ,$$

where $\lambda(K^i) < \lambda(K^j)$ and $\lambda(Q^i) < \lambda(Q^j)$ if $i < j$. The equation $\mathcal{L}(K)Q = 0$ can be written as

$$\mathcal{L}(K^1)Q^1 + R = 0.$$

By Lemma A.1 we have

$$\lambda(\mathcal{L}(K^1)Q^1) = \lambda(K^1) + \lambda(Q^1) \text{ and } \lambda(R) \neq \lambda(K^1) + \lambda(Q^1).$$

Therefore by Lemma A.2 the homogeneous equation $u_t = K^1$ has an invariant Q^1. Different choices of $\lambda(t), \lambda(x), \lambda(u)$ may lead to different homogeneous equations that have invariants.

- ⋆ A similar statement can be made for polynomial equations with time dependent invariants. Suppose that the nonhomogeneous polynomial equation $u_t = K$ has a time dependent nonhomogeneous invariant Q. For any choice of $\lambda(x), \lambda(u)$, we can write

$$K = K_1 + K_2 + \cdots + K_n,$$

where all terms of K with the same weight $a_i\lambda(t) + b_i$ are terms of (K_i). Since K is polynomial all the a_i are positive and the equations $\lambda(\partial_t) = \lambda(K^i)$ can be solved for $\lambda(t)$. We choose $\lambda(t)$ to be the minimum of the set

$$\{-\frac{b_i}{a_i + 1},\ 0 < i \leq n\}.$$

Then, K is written as

$$K = K^1 + K^2 + \cdots ,$$

where $\lambda(K^1) = \lambda(\partial_t)$ and $\lambda(K^i) < \lambda(K^j)$ for all $i < j$. We write

$$Q = Q^1 + Q^2 + \cdots ,$$

with $\lambda(Q^i) < \lambda(Q^j)$ for all $i < j$. The equation $\mathcal{L}(K)Q = 0$ can be written as

$$\mathcal{L}(\partial_t + K^1)Q^1 + R = 0.$$

By Lemma A.1 and our choice of $\lambda(t)$ we have

$$\lambda(\mathcal{L}(\partial_t + K^1)Q^1) = \lambda(K^1) + \lambda(Q^1) \text{ and } \lambda(R) \neq \lambda(K^1) + \lambda(Q^1).$$

Therefore by Lemma A.2 the homogeneous equation $u_t = K^1$ has an invariant Q^1. Different choices of $\lambda(x), \lambda(u)$ may lead to different homogeneous equations that have invariants.

⋆ Suppose that a homogeneous equation $u_t = K$ has a nonhomogeneous invariant Q. Then, for a particular choice of $\lambda(t), \lambda(x), \lambda(u)$ such that the equation is homogeneous with weight w, we write Q as a sum of homogeneous terms

$$Q^1 + Q^2 + \cdots ,$$

where $\lambda(Q^i) \neq \lambda(Q^j)$ if $i \neq j$. For $i = 1, 2, \ldots$, the equation $\mathcal{L}(K)Q = 0$ can be written as

$$\mathcal{L}(K)Q^i + R = 0.$$

By Lemma A.1 we have

$$\lambda(\mathcal{L}(K)Q^i) = w + \lambda(Q^i)$$

and

$$\lambda(R) \neq w + \lambda(Q^i).$$

By repeated application of Lemma A.2, the equation $u_t = K$ has homogeneous invariants $Q_i,\ i = 1, 2, \ldots$. Therefore, it suffices to determine all homogeneous invariants of a homogeneous equation.

Appendix B

An implicit function theorem

We define the concept of filtered algebra. This provides a more general setting than the setting of graded algebra. Also the implicit function theorem of Sanders and Wang is more easily proven. The theorem was formulated and proven in [SW98], where it was used in the classification of scalar equations, cf. Chapter 5.

Definition B.1. *Modules U and V are* **filtered modules** *and $P : U \to End(V)$ is a* **filtered action** *if*

$$U = U^{(0)} \supset U^{(1)} \supset \cdots \quad , \quad \bigcap_{i=0}^{\infty} U^{(i)} = 0,$$
$$V = V^{(0)} \supset V^{(1)} \supset \cdots \quad , \quad \bigcap_{i=0}^{\infty} V^{(i)} = 0.$$

and $P(U^{(i)})V^{(j)} \subset V^{(i+j)}$.

From a graded module $\bar{U}$ we make a filtered module U by

$$U^{(i)} = \sum_{j \geq i}^{\infty} \bar{U}^{(j)}.$$

Now finding a solution to $P(v)q = 0$ consists of solving the set of equations

$$P(v)q \in U^j \text{ for } j = 1, 2, \ldots$$

Under certain conditions all these equations are satisfied provided the first few are.

In the setting of filtered modules v is **nonlinear injective** if $P(v)q \in V^{(i+1)}$ implies $q \in V^{(i+1)}$ for all $q \in V^{(i)}, i > 0$.

Let W be a **filtered $(U\&V)$-module** and P be a **filtered action** of U on V and on W. Then v is **relatively l-prime** with respect to w if $P(w)q \in Im(P(v))$ mod $W^{(i+1)}$ implies $q \in Im(P(v))$ mod $W^{(i+1)}$ for all $q \in W^{(i)}$, $i \geq l$.

Theorem B.2 (Sanders, Wang). *Let W be a filtered $(U\&V)$-module and P be a filtered action of U on V and on W. Suppose that $v \in U$, $w \in V$ and $q \in W$ such that*

* $P(v)w = 0$,
* $P(v)$ *is nonlinear injective,*
* v *is relatively l-prime with respect to w,*
* $P(v)q \in W^{(l)}$,
* $P(w)q \in W^{(1)}$.

Then there exists a unique $\tilde{q} \in W^{(l)}$ such that $\hat{q} = q + \tilde{q}$ is an invariant of both v and w, i.e.

* $P(v)\hat{q} = 0$,
* $P(w)\hat{q} = 0$.

Proof. We know $P(v)P(w)q \equiv P(w)P(v)q \bmod W^{(l)}$. Since $P(w)q \in W^{(1)}$ we can use the nonlinear injectiveness of v to conclude that $P(w)q \in W^{(l)}$. By induction we show that there exists a $\hat{q}$ such that $P(v)\hat{q} \in W^{(p)}$ and $P(w)\hat{q} \in W^{(p)}$ for all $p \geq l$. Suppose $P(v)q \in W^{(p)}$ and $P(w)q \in W^{(p)}$ hold for some $p \geq l$. The case $p = l$ follows from the first argument. We have

$$P(v)P(w)q = P(w)P(v)q$$

and, in particular,

$$P(w)P(v)q \in \operatorname{Im}(P(v)) \bmod W^{(p+1)}.$$

By relative l-primeness of v with respect to w,

$$P(v)q \in \operatorname{Im}(P(v)) \bmod W^{(p+1)}.$$

Therefore, we can uniquely define $\tilde{q} \in W^{(p)}$ by

$$P(v)\tilde{q} = -P(v)q$$

such that $\hat{q} = q + \tilde{q}$ satisfies

$$P(v)\hat{q} \in W^{(p+1)}$$

and, by taking $l = p + 1$ in the first argument,

$$P(w)\hat{q} \in W^{(p+1)}.$$

This implies that q can always be extended such that all homogeneous parts of $P(v)\hat{q}$ and $P(w)\hat{q}$ vanish. Uniqueness follows from $\cap_{i=0}^{\infty} W^{(i)} = 0$. □

Appendix C

Resultants

When using the symbolic method to study generalised symmetries, the problem of determining common divisors of two polynomials quickly arise. Such problems can be solved by computing resultants, a useful algebraic tool.

This section contains material from [GCL92], proofs and algorithms can be found there. Let R be a unique factorisation domain and let $A(x), B(x) \in R[x]$ be nonzero polynomials with

$$A(x) = \sum_{i=0}^{m} a_i x^i \text{ and } B(x) = \sum_{i=0}^{n} b_i x^i.$$

Definition C.1. *The* **Sylvester matrix** *of A and B is the $m \times n$ by $m \times n$ matrix*

$$\begin{bmatrix} a_m & a_{m-1} & \dots & a_1 & a_0 & & \\ & a_m & a_{m-1} & \dots & a_1 & a_0 & \\ & & \dots & \dots & \dots & \dots & \\ & & & a_m & \dots & \dots & a_1 \\ b_n & b_{n-1} & \dots & b_1 & b_0 & & \\ & b_n & b_{n-1} & \dots & b_1 & b_0 & \\ & & \dots & \dots & \dots & \dots & \\ & & & b_n & \dots & \dots & b_1 \end{bmatrix},$$

where the upper part of the matrix consists of n rows of coefficients of $A(x)$, the lower part consists of m rows of coefficients of $B(x)$. The entries not shown are zero.

Definition C.2. *The* **resultant** *of $A(x)$ and $B(x)$ (written $res_x(A, B)$) is the determinant of the Sylvester matrix of A, B. We also define $res_x(0, B) = 0$ and $res_x(a, b) = 1$ for nonzero $a, b \in R$.*

Sylvester's criterion states that two polynomials, $A(x)$ and $B(x)$, have a non-trivial common factor if and only if $\text{res}_x(A, B) = 0$.

The coefficient domain can be another polynomial domain. This makes it possible to use resultants for nonlinear elimination and solve systems of algebraic equations.

Theorem C.3 (Fundamental theorem of resultants). *Let F be an algebraically closed field, and let*

$$f = \sum_{i=0}^{m} a_i(x_2, \ldots, x_r)x_1^i \text{ and } g = \sum_{i=0}^{n} b_i(x_2, \ldots, x_r)x_1^i$$

be elements of $F[x_1, \ldots, x_r]$ of positive degrees in x_1. Then if $(\alpha_1, \ldots, \alpha_r)$ is a common zero of f and g, their resultant with respect to x_1 satisfies

$$res_{x_1}(f, g)(\alpha_2, \ldots, \alpha_r) = 0.$$

Conversely, if the above resultant vanishes at $(\alpha_2, \ldots, \alpha_r)$, then at least one of the following holds:

1. $a_m(\alpha_2, \ldots, \alpha_r) = \cdots = a_0(\alpha_2, \ldots, \alpha_r) = 0$,
2. $b_n(\alpha_2, \ldots, \alpha_r) = \cdots = b_0(\alpha_2, \ldots, \alpha_r) = 0$,
3. $a_m(\alpha_2, \ldots, \alpha_r) = b_n(\alpha_2, \ldots, \alpha_r) = 0$,
4. *$\alpha_1 \in F$ exists such that $(\alpha_1, \ldots, \alpha_r)$ is a common zero of f and g.*

Appendix D

Corollaries of the Lech-Mahler theorem

The following theorem is formulated and proven in [Lec53]. The Lech–Mahler theorem was firstly used in connection with symmetries of evolution equations in [BSW98], where also corollary D.4 was proven.

Theorem D.1 (Lech-Mahler)**.** *Let $a_1, a_2, \ldots, a_n, A_1, A_2, \ldots, A_n$ be nonzero complex numbers. Suppose that none of the ratios A_i/A_j with $i \neq j$ is a root of unity. Then the equation*

$$a_1 A_1^k + a_2 A_2^k + \ldots + a_n A_n^k = 0$$

in the unknown integer k has finitely many solutions.

Corollary D.2. *Let a, b, c, A, B, C be nonzero complex numbers. Suppose that the equation*

$$aA^k + bB^k + cC^k = 0$$

has infinitely many integers k as solution. Then the ratios $A/B, A/C, B/C$ are roots of unity.

Proof. According to Theorem D.1, at least one of the ratios $A/B, A/C, B/C$ must be a root of unity. Without loss of generality we can assume A/B is a root of unity.

- ⋆ Suppose A/B is an n^{th} root of unity. Then, if we replace k by $i + kn$ for $i = 0, 1, 2, \ldots, n-1$ our problem falls into a finite number of problems of the form

  $$(aA^i + bB^i)(A^n)^k + cC^i(C^n)^k.$$

  At least one of them has infinitely many solutions. Hence, according to Theorem D.1, A/C is a root of unity. Together with A/B being a root of unity this implies that B/C is a root of unity.

Interchanging A, B, C in all possible ways proves the statement. $\square$

Corollary D.3. *Let a, b, c, d, A, B, C, D be nonzero complex numbers. Suppose that the equation*

$$aA^k + bB^k + cC^k + dD^k = 0$$

has infinitely many integers k as solution. Then at least one of the pairs $A/B, C/D$ or $A/C, B/D$ or $A/D, B/C$ consists of roots of unity.

Proof. According to Theorem D.1 at least one of the ratios $A/B, A/C, A/D, B/C, B/D, C/D$ must be a root of unity. Without loss of generality we can assume A/B is a root of unity.

- ⋆ Suppose A/B is an n^{th} root of unity. Then, if we replace k by $i + kn$ for $i = 0, 1, 2, \ldots, n-1$ our problem falls into a finite number of problems of the form

$$(aA^i + bB^i)(A^n)^k + cC^i(C^n)^k + dD^i(D^n)^k.$$

  At least one of them has infinitely many solutions. If $aA^i + bB^i = 0$ the ratio C/D must be a root of unity. In case $aA^i + bB^i \neq 0$ by Corollary D.2 the ratios $A/C, A/D, C/D$ are roots of unity. Hence, if A/B is a root of unity at least C/D is a root of unity.

Interchanging A, B, C, D in all possible ways proves the statement. $\square$

Corollary D.4. *Let a, b, c, d, A, B, C, D be nonzero complex numbers. Suppose that $aA^k + bB^k \neq 0$ for all k and that the equation*

$$aA^k + bB^k + cC^k + dD^k = 0$$

has infinitely many integers k as solution. Then at least one of the pairs $A/C, B/D$ or $A/D, B/C$ consists of roots of unity.

Proof. According to Theorem D.1 at least one of the ratios $A/B, A/C, A/D, B/C, B/D, C/D$ must be a root of unity. Without loss of generality we can assume A/B is a root of unity or A/C is a root of unity.

- ⋆ Suppose A/B is an n^{th} root of unity. Then, if we replace k by $i + kn$ for $i = 0, 1, 2, \ldots, n-1$ our problem falls into a finite number of problems of the form

$$(aA^i + bB^i)(A^n)^k + cC^i(C^n)^k + dD^i(D^n)^k = 0.$$

  At least one of them has infinitely many solutions. Since we have $aA^i + bB^i \neq 0$ by Corollary D.2 the ratios $A/C, A/D, C/D$ are roots of unity. Therefore at least A/C or A/D is a root of unity.

- ⋆ Suppose A/C is an n^{th} root of unity. Then, if we replace k by $i + kn$ for $i = 0, 1, 2, \ldots, n-1$ our problem falls into a finite number of problems of the form

$$(aA^i + cC^i)(A^n)^k + bB^i(B^n)^k + dD^i(D^n)^k = 0.$$

At least one of them has infinitely many solutions. If $aA^i + cC^i = 0$ the ratio B/D must be a root of unity. In case $aA^i + cC^i \neq 0$ by Corollary D.2 the ratios $A/B, A/D, B/D$ are roots of unity. Hence, if A/C is a root of unity at least B/D is a root of unity.

Interchanging A, B, C, D in all possible ways proves the statement. $\square$

Corollary D.5. *Let $a, b, c, d, e, A, B, C, D, E$ be nonzero complex numbers. Suppose that the equation*

$$aA^k + bB^k + cC^k + dD^k + eE^k = 0$$

has infinitely many integers k as solution. Then three of the numbers A, B, C, D, E have a root of unity as a ratio and the same is true for the other two.

Proof. According to Theorem D.1 at least one of the ratios $A/B, A/C, A/D, A/E, B/C, B/D, B/E, C/D, C/E, D/E$ must be a root of unity. Without loss of generality we can assume that A/B is a root of unity.

- ⋆ Suppose A/B is an n-th root of unity. Then, if we replace k by $i + kn$ for $i = 0, 1, 2, \ldots, n-1$ our problem falls into a finite number of problems of the form
$$(aA^i + bB^i)(A^n)^k + cC^i(C^n)^k + dD^i(D^n)^k + eE^i(E^n)^k = 0.$$
At least one of them has infinitely many solutions. Suppose that this is the i-th equation. Then $aA^i + bB^i = 0$ or $aA^i + bB^i \neq 0$. When $aA^i + bB^i = 0$ then, by Corollary D.2, $C/D, C/E, D/E$ are roots of unity. When $aA^i + bB^i \neq 0$ then, by Corollary D.3, at least one of the pairs $A/C, D/E$ or $A/D, C/E$ or $A/E, C/D$ consists of roots of unity. Together with A/B being a root of unity this implies that one of the quadruples
$$\begin{array}{ll} A/B, A/C, B/C, D/E & A/B, C/D, C/E, D/E \\ A/B, A/D, B/D, C/E & A/B, A/E, B/E, C/D \end{array}$$
consists of roots of unity.

By interchanging A, B, C, D, E in all possible ways we get that one of the quadruples

$$\begin{array}{ll} A/B, A/C, B/C, D/E & A/B, A/D, B/D, C/E \\ A/B, A/E, B/E, C/D & A/B, C/D, C/E, D/E \\ A/C, A/D, B/E, C/D & A/C, A/E, B/D, C/E \\ A/C, B/D, B/E, D/E & A/D, A/E, B/C, D/E \\ A/D, B/C, B/E, C/E & A/E, B/C, B/D, C/D \end{array}$$

consists of roots of unity. We see that three of the numbers A, B, C, D, E have a root of unity as a ratio and the same is true for the other two. $\square$

Appendix E

Diophantine equations

The material presented here is adapted from work done by F. Beukers.

Theorem E.1 (Beukers)**.** *Let μ, ν be roots of unity. Suppose that $\mu, \nu \neq \pm 1$, $\mu^n, \nu^n \neq 1$ and $\mu \neq \nu, \nu^{-1}$. Then the Diophantine equation*

$$\left(\frac{1-\mu}{1-\nu}\right)^n = \frac{1-\mu^n}{1-\nu^n} \tag{E.1}$$

in the unknown positive integer n has no solution unless $n = 1$.

Proof. The case $n = 2$ is excluded with the following argument. Suppose

$$\left(\frac{1-\mu}{1-\nu}\right)^2 = \frac{1-\mu^2}{1-\nu^2}.$$

Then

$$\frac{1-\mu}{1-\nu} = \frac{1+\mu}{1+\nu}.$$

Hence

$$1 + \nu - \mu - \mu\nu = 1 - \nu + \mu - \mu\nu.$$

So, $\mu = \nu$, a contradiction.

When $n > 2$ we use Lemma E.2. Choose m, a, b positive integers such that

$$\mu = \zeta_m^a, \;\; \nu = \zeta_m^b,$$

where $\zeta_m = e^{2\pi i/m}$ and $\gcd(a, b, m) = 1$. We distinguish two cases.

- $\star$ $\gcd(a, m) = 1$ or $\gcd(b, m) = 1$. Suppose the first case happens. Let a^* be the inverse of a modulo m. Then we see that

  $$\nu = \mu^{a^* b}.$$

  We apply Lemma E.2 with $l \equiv a^* b \pmod{m}$ and conclude that $l = \pm 1$. In other words, $a \equiv \pm b \pmod{m}$ and we see that $\mu = \nu$ or $\mu = \nu^{-1}$.

* $\star$ $\gcd(a, m) > 1$ and $\gcd(b, m) > 1$. In this case the idea is to choose an integer l with $\gcd(l, m) = 1$ such that

$$\nu^l = \nu,\ \ \mu^l \neq 1, \mu, \mu^{-1}.$$

Now replace μ, ν in the original equation by $\mu^l, \nu^l = \nu$. Divide the newly obtained equation by the old one and we obtain an equation of the form (E.2). Now apply Lemma E.2 to conclude that $\mu^l = 1, \mu$ or μ^{-1}. Thus we get a contradiction, i.e., the original equation has no solution once we have found a suitable l.

Now let us choose l. Since $\gcd(a, b, m) = 1$, we can assume that not both $\gcd(a, m)$ and $\gcd(b, m)$ are even. Hence there is an odd prime p which divides one of them, say $\gcd(b, m)$. Because p is odd we can choose an integer

$$l = 1 + \frac{km}{p},$$

with $k = \pm 2$ and $\gcd(l, m) = 1$. Clearly we have $\nu^l = \nu$. Moreover,

$$\mu^l = \zeta_m^{a+akm/p} = \mu\zeta_m^{akm/p} = \mu e^{\pm 4\pi i a/p}.$$

Since a is not divisible by p we see that μ^l/μ is a nontrivial p-th root of unity. Therefore $\mu^l \neq \mu$. Suppose that $\mu^l = \mu^{-1}$. This implies that

$$\mu e^{\pm 4\pi i a/p} = \mu^{-1},$$

i.e., μ is a p-th root of unity. So if μ is not a p-th root of unity, l is found.

Now assume that μ is a p-th root of unity. So p divides m exactly once. Suppose that ν is an N-th root of unity. Since p divides b we get that N is not divisible by p. In particular, $\gcd(p, N) = 1$. Suppose that $p > 3$. Then we choose, using the Chinese remainder theorem, the number l such that

$$l \equiv 1 \bmod N,\ \ l \equiv 2 \bmod p.$$

Note that $\nu^l = l$ and $\mu^l = \mu^2$ which is different from μ, μ^{-1} since $p > 3$. We are left with the case $p = 3$. Now suppose that $N \neq 3, 4, 6$. Then there is an integer c, relatively prime with N such that $c \not\equiv \pm 1 \bmod N$. Choose l such that $l \equiv 1 \bmod 3$ and $l \equiv c \bmod N$. Then

$$\mu^l = \mu,\ \ \nu^l = \nu^c \neq \nu, \nu^{-1}.$$

We apply our argument with ν and μ interchanged to conclude that we get a contradiction once more. Since $N = 3, 6$ are not possible because 3 does not divide N, we are left with the case $p = 3,\ N = 4$. Hence we can assume that $\mu = \omega$, with ω a primitive 3-rd root of unity, and $\nu = i$. Taking absolute values squared on both sides of

$$\left(\frac{1-\omega}{1-i}\right)^n = \frac{1-\omega^n}{1-i^n}$$

yields

$$\left(\frac{3}{2}\right)^n = \frac{3}{2^\epsilon},$$

where $\epsilon = 1$ or 2 depending on whether n is odd or even. This is clearly impossible when $n > 1$.

□

Lemma E.2 (Beukers)**.** *Let μ be a root of unity and l an integer. Suppose that $\mu \neq \pm 1$ and that for some $n > 2$ we have*

$$\left(\frac{1-\mu^l}{1-\mu}\right)^n = \frac{1-\mu^{ln}}{1-\mu^n}. \tag{E.2}$$

Then μ^l is either $1, \mu$ or μ^{-1}.

Proof. Suppose that μ is a primitive m-th root of unity for some $m \geq 3$. By Galois theory equation (E.2) still holds if we replace μ by μ^h for any integer h with $\gcd(h, m) = 1$. So we can assume that

$$\mu = e^{2\pi i/m}.$$

We can also assume that

$$|l| \leq m/2$$

by shifting l over multiples of m if necessary. For any $x \in [-\pi, \pi]$ we have the straightforward inequalities

$$\frac{2}{\pi}|x| \leq |1 - e^{ix}| \leq |x|.$$

From this it follows that

$$\left|\frac{1-\mu^l}{1-\mu}\right| \geq \frac{(2/\pi)(2\pi|l|/m)}{2\pi/m} = \frac{2|l|}{\pi}.$$

On the other hand,

$$\left|\frac{1-\mu^{ln}}{1-\mu^n}\right| = |1 + \mu^n + \mu^{2n} + \cdots + \mu^{(|l|-1)n}| \leq |l|.$$

Hence we find that

$$\left(\frac{2|l|}{\pi}\right)^n \leq |l|.$$

From this it follows that

$$(2|l|/\pi)^{n-1} \leq \pi/2.$$

Using $n > 2$ we get

$$|l| \leq (\pi/2)^{1.5} < 2.$$

Hence $|l| \leq 1$ and we have $\mu^l = 1, \mu$ or μ^{-1}, as asserted. □

Proposition E.3. *Let μ be a root of unity. Suppose that $\mu \neq \pm 1$. Then the diophantine equation*

$$(1-\mu)^n = 2^{n-1}(1-\mu^n)$$

in the unknown positive integer n has no solution unless $n = 1$.

Proof. Division by $1-\mu$ gives

$$(1-\mu)^{n-1} = 2^{n-1}(1+\mu+\cdots\mu^{n-1}).$$

Therefore

$$a = \frac{1-\mu}{2}$$

should be an algebraic integer, i.e., in $\mathbb{Z}[\mu]$. But this can not be true since the norm of any algebraic integer is integer and the absolute value of the norm of a is smaller than 1. This is seen as follows: the norm of a is the product of its conjugates. Each conjugate has the form $(1-\mu^k)/2$ and this has absolute value smaller than 1. □

Theorem E.4 (Beukers)**.** *Let μ, ν be roots of unity. Suppose that*

$$\mu, \nu \neq \pm 1,\ \mu^n, \nu^n \neq -1,\ \mu \neq \nu, \nu^{-1},\ n > 1.$$

Then the Diophantine equation

$$\left(\frac{1-\mu}{1-\nu}\right)^n = \frac{1 \pm \mu^n}{1+\nu^n} \tag{E.3}$$

in the unknown positive integer n has no solutions.

Proof. The proof is similar to the proof of Theorem E.1. The only differences are:

⋆ The case $n = 2$ is excluded with the following argument. Suppose

$$\left(\frac{1-\mu}{1-\nu}\right)^2 = \frac{1+\mu^2}{1+\nu^2}.$$

Then

$$(1-\mu)^2(1+\nu^2) - (1-\nu)^2(1+\mu^2) = 2(\mu-\nu)(\mu\nu-1) = 0.$$

So, $\mu = \nu$ or $\mu = 1/\nu$, a contradiction. Suppose

$$\left(\frac{1-\mu}{1-\nu}\right)^2 = \frac{1-\mu^2}{1+\nu^2}.$$

Then

$$(1-\mu)^2(1+\nu^2) - (1-\nu)^2(1-\mu^2) = 2(\mu-1)(\mu(\nu^2-\nu+1)-\nu) = 0$$

Since $\mu \neq 1$ we have

$$\mu = \nu/(\nu^2-\nu+1).$$

Substituting this into $\mu\bar{\mu} = 1$ and using that $\bar{\nu} = 1/\nu$ we obtain that $\nu^2 = -1$ or $\nu = 1$, contradicting the assumptions.

- ⋆ In the case $\gcd(a,m) = 1$ or $\gcd(b,m) = 1$ we use Lemma E.5 instead of Lemma E.2 to conclude that $\mu^l \in \{\mu, 1, \mu^{-1}\}$.

- ⋆ The absolute value squared of $(1+\omega^n)/(1+i^n)$ yields

$$\begin{array}{rll} 1/4 & \text{when} & n \equiv 4, 8 \bmod 12, \\ 1/2 & \text{when} & n \equiv 1, 5, 7, 11 \bmod 12, \\ 1 & \text{when} & n \equiv 0 \bmod 12, \\ 2 & \text{when} & n \equiv 3 \bmod 6. \end{array}$$

  The absolute value squared of $(1-\omega^n)/(1+i^n)$ yields

$$\begin{array}{rll} 3/4 & \text{when} & n \equiv 4, 8 \bmod 12, \\ 3/2 & \text{when} & n \equiv 1, 5, 7, 11 \bmod 12, \\ 0 & \text{when} & n \equiv 0, 3, 9 \bmod 12. \end{array}$$

□

Lemma E.5. *Let $\mu \neq \pm 1$ be a root of unity and l an integer. Suppose that for some $n \geq 2$ we have*

$$\left(\frac{1-\mu^l}{1-\mu}\right)^n = \frac{1+\alpha\mu^{ln}}{1+\mu^n}, \qquad \alpha = \pm 1,\ \mu^n \neq -1.$$

Then, if $\alpha = 1$ we have $\mu^l = \mu$ or $\mu^l = \mu^{-1}$ and if $\alpha = -1$ we have $\mu^l = 1$.

Proof. Suppose that μ is a primitive m-th root of unity for some $m \geq 3$. By Galois theory (E) still holds if we replace μ by μ^h for any integer h with $\gcd(h,m) = 1$. So we can assume that

$$\mu = e^{2\pi i/m}.$$

We can also assume that

$$|l| \leq \frac{m}{2}$$

by shifting l over multiples of m if necessary.

We have the estimate

$$\left(\frac{2|l|}{\pi}\right)^n \leq \left|\frac{1-\mu^l}{1-\mu}\right|^n.$$

On the other hand, we can give an upper bound for the n-th power of

$$\left|\frac{1-\mu^l}{1-\mu}\right|$$

by using the trivial bound

$$\left|1 \pm \mu^{ln}\right| \leq 2$$

to obtain

$$\left(\frac{2|l|}{\pi}\right)^n \leq \frac{2}{|1+\mu^n|}$$

and hence

$$\left|\cos(\pi\frac{n}{m})\right| \leq \left(\frac{\pi}{2|l|}\right)^n.$$

Then, using the estimate

$$|\cos \pi x| \geq |2x - k|,$$

where k is the nearest odd integer to $2x$, and $|l| \geq 2$ we get

$$\left|\frac{n}{m} - \frac{k}{2}\right| \leq \frac{1}{2}\left(\frac{\pi}{4}\right)^n.$$

From these estimates it follows that $n/m \geq 0.19$. Using this and the trivial lower bound

$$\left|\frac{n}{m} - \frac{k}{2}\right| \geq \frac{1}{2m}$$

we get

$$\frac{1}{m} \leq \left(\frac{\pi}{4}\right)^{0.19m}.$$

Hence $m \leq 100$. But then,

$$\frac{1}{100} \leq \frac{1}{m} \leq \left(\frac{\pi}{2|l|}\right)^n,$$

which in its turn implies that

$$(2|l|/\pi)^n \leq 100.$$

So we are left with a finite number of triples l, m, n. A small computer search yields no solutions with

$$\alpha = \pm 1, \quad 2 \leq |l| \leq m/2.$$

When $\alpha = 1$ there are the solutions $l = \pm 1$ and hence $\mu^l = \mu$ or μ^{-1}. When $\alpha = 0$ we have $\mu^l = 1$. □

Proposition E.6. *Let μ be a root of unity. Suppose that $\mu \neq -1$. Then the diophantine equation*

$$(1-\mu)^n = 2^{n-1}(1+\mu^n)$$

in the unknown positive integer n has no solution unless $n = 1$.

Proof. From the equation 2^{n-1} is a divisor of $(1-\mu)^n$. It follows that 2 is a divisor of $(1-\mu)^2$ and therefore a divisor of $1-\mu^2$. Under the assumption that $\mu^2 \neq 1$ we are in a similar situation as in the proof of Proposition E.3. The remaining case $\mu = 1$ gives no solutions either. □

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Index of mathematical expressions

Index

Integreerbare Evolutievergelijkingen: een Diophantische Aanpak

Samenvatting

Waar gaat dit proefschrift over?

Veel processen in de natuur kunnen worden beschreven met behulp van evolutievergelijkingen. Kenmerkend voor een **evolutievergelijking** is dat de toestand op tijdstip t in principe berekend kan worden wanneer deze gegeven is op tijdstip $t_0 < t$. De Korteweg-De Vries vergelijking is een mooi voorbeeld van zo'n evolutievergelijking:

$$u_t = u_3 + uu_1 \quad (KDV).$$

Hierin is u een functie van x en t, u_t de afgeleide van u naar t en u_i de i-de afgeleide van u naar x. De KDV-vergelijking werd al in de negentiende eeuw afgeleid en beschrijft de beweging van lange golven in smalle en relatief ondiepe kanalen. In zo'n kanaal kan het voorkomen dat een berg water zich over het oppervlak blijft voortbewegen. Hier werd voor het eerst over geschreven door Russell die middels experimenten aantoonde dat dergelijke golven elkaars vorm niet verstoren.

Halverwege de twintigste eeuw bleek de KDV-vergelijking een rol te spelen in diverse andere takken van de natuurkunde. Men herondekte de gelokaliseerde en stabiele oplossing, de zogenaamde 'soliton', en ging op zoek naar een verklaring voor dit interessante fenomeen. Deze kwam in de vorm van behoudswetten. De bekende behoudswetten, die van impuls en energie, bleken er slechts twee van de oneindig vele te zijn. Dit leidde tot de 'inverse scattering' methode, een methode waarmee o.a. de KDV vergelijking exact kon worden opgelost.

Een evolutievergelijking die exact opgelost kan worden, bijvoorbeeld door 'inverse scattering' of door 'linearisatie', wordt **integreerbaar** genoemd. Het blijkt zo te zijn dat iedere integreerbare evolutievergelijking oneindig veel gegeneraliseerde symmetrieën bezit. Voor de KDV vergelijking zijn deze gerelateerd aan haar behoudswetten door een stelling van Noether. Maar ook Burgers' vergelijking, waarvoor slechts één behoudswet geldt, heeft oneindig veel symmetieën. Dit proefschrift gaat over het herkennen en classificeren van integreerbare evolutievergelijkingen met betrekking tot het bestaan van gegeneraliseerde symmetrieën.

Waar stoelt dit proefschrift op?

De volgende ontwikkelingen zijn van groot belang geweest voor het onderzoek dat beschreven wordt in dit proefschrift.

1. Het vermoeden bestond dat wanneer een evolutievergelijking één symmetrie heeft, ze er oneindig veel heeft. Dit werd gepreciseerd door Fokas in 1987.

   > Vermoeden van Fokas:
   > Als een scalaire vergelijking minstens één symmetrie bezit, dan bezit ze er oneindig veel. Evenzo, als een vergelijking met n componenten n symmetrieën heeft, dan heeft ze er oneindig veel.

2. Voor $n = 1$ werd dit vermoeden bevestigd in de klasse van λ-homogene scalaire vergelijkingen (met $\lambda \geq 0$):

   $$u_t = u_n + f(u, u_1, \ldots, u_{n-1}).$$

   De classificatie van deze vergelijkingen met betrekking tot het bestaan van symmetrieën werd uitgevoerd door Sanders en Wang in 1998. Zij was gebaseerd op:

   - de symbolische calculus die geïntroduceerd werd in 1975 door Gel'fand and Dikiĭ
   - een impliciete functiestelling, die laat zien dat onder bepaalde condities het bestaan van één symmetrie integreerbaarheid impliceert.

   Een bepaalde conditie in de impliciete functiestelling werd bewezen door F. Beukers, die gebruik maakte van recente resultaten uit diophantische approximatie theorie.

   Voor $\lambda > 0$ werd een volledige lijst van tien integreerbare vergelijkingen verkregen.

3. De vergelijking die in 1991 gegeven werd door Bakirov:

   $$\begin{cases} u_t = & 5u_4 + v^2 \\ v_t = & v_4 \end{cases}$$

   heeft een symmetrie van orde 6. De computerberekeningen van Bakirov lieten zien dat er geen andere symmetrie bestaat van orde $n < 53$. Met behulp van de p-adische methode van Skolem bewezen Beukers, Sanders en Wang in 1998 dat de vergelijking van Bakirov slechts één gegeneraliseerde symmetrie bezit.

4. Op basis van de stelling van Lech en Mahler werd een vermoeden geformuleerd dat er binnen de klasse van vergelijkingen

   $$\begin{cases} u_t = & a_1 u_n + v^2 \\ v_t = & a_2 v_n \end{cases}$$

een eindig aantal integreerbare vergelijkingen bestaat. Dit bleek inderdaad het geval. Gebruikmakend van een recent algoritme van Smyth, dat polynomiale vergelijkingen $p(x, y) = 0$ oplost voor eenheidswortels x, y, werd de volledige lijst verkregen door Beukers, Sanders en Wang in 2001.

5. De hierboven aangehaalde classificaties onderscheiden zich van andere, doordat de orde van de te classificeren vergelijkingen niet vast gekozen werd.

   Om een recent voorbeeld aan te halen: in 2000 classificeerde Foursov symmetrisch gekoppelde homogene derde orde vergelijkingen met twee componenten u, v van gewicht 2. Interessant vond ik het vermoeden van Foursov dat de vergelijking

$$\begin{cases} u_t = & u_3 + 3uu_1 \\ v_t = & \alpha u_1 v + uv_1 \end{cases}$$

   alleen symmetrieën van een even gewicht heeft als α een negatief rationeel getal is.

Waar draagt dit proefschrift aan bij?

De belangrijkste resultaten die ik behaalde tijdens mijn onderzoek zijn:

1. *De classificatie van integreerbare $\mathcal{B}$-vergelijkingen.* Deze is te vinden in hoofdstuk 6. $\mathcal{B}$-vergelijkingen zijn vergelijkingen van de vorm:

$$\begin{cases} u_t = & a_1 u_n + K(v, v_1, \ldots) \\ v_t = & a_2 v_n \end{cases}$$

   waarbij K quadratisch is. De conditie voor het bestaan van oneindig veel symmetrieën is equivalent met: er is een zekere $r \in \mathbb{C}$ zodanig dat

   (a) a_1, a_2 en r voldoen aan
$$\frac{a_1}{a_2} = \frac{1 + r^n}{(1 + r)^n}.$$

   (b) de diophantische vergelijking
$$(1 + r)^m (1 + \bar{r}^m) = (1 + \bar{r})^m (1 + r^m), \quad r \in \mathbb{C}$$
   heeft oneindig veel oplossingen m inclusief $m = n$.

   De introductie van 'biunit coordinates' maakte het mogelijk te bewijzen dat er precies

$$\begin{array}{rll} n(n-2)/4 & \text{indien} & n \text{ even,} \\ (n+1)(n-3)/4 & \text{indien} & n \text{ oneven,} \\ 4 & \text{indien} & n = 5 \end{array}$$

niet-gedegenereerde n-de orde integreerbare $\mathcal{B}$-vergelijkingen bestaan en dat de ordes van hun symmetrieën gegeven worden door

$$m \equiv 1 \bmod n-1 \quad \text{of} \quad m \equiv 0 \bmod n.$$

2. *De diophantische benadering van integreerbaarheid.* Door de conditie voor integreerbaarheid te schrijven als een diophantische vergelijking voor de nulpunten van $\mathcal{G}$-functies kan het lineaire gedeelte van de vergelijking bepaald worden. Deze benadering is zeer effectief en toepasbaar binnen een grote klasse van vergelijkingen.

   In hoofdstuk 5 kijken we vanuit deze benadering opnieuw naar de classificatie van scalaire vergelijkingen. Het gebruik van de stelling van Lech and Mahler maakt het mogelijk de integreerbare vergelijkingen te classificeren, zonder daarbij gebruik te maken van diophantische approximatie theorie.

   In Hoofdstuk 7 bepalen we het spectrum van integreerbare vergelijkingen in twee componenten u, v met quadratische termen. Ook classificeren we het cubische analogon van de klasse van $\mathcal{B}$-vergelijkingen.

3. *Het vermoeden van Fokas.* In hoofdstuk 8 wordt bewezen dat de vergelijking

$$\begin{cases} u_t = & 2r^2u_7 + 7(2r^2+4r+3)(v_3v_0 + (3-r)v_2v_1) \\ v_t = & (16r^2+28r+21)v_7 \end{cases},$$

   waarin $r^3 + r^2 - 1 = 0$, precies twee symmetrieën heeft. Dit falsificeert het vermoeden van Fokas.

   Ondanks uitvoerige computer-algebraïsche berekeningen zijn er geen andere tegenvoorbeelden gevonden. Een nieuw vermoeden is dat het enige gehele getal $N > 2$ waarvoor geldt:

   > er bestaan $r, s \in \mathbb{C}$ zodanig dat de diophantische vergelijking
   >
   > $$(1+r^m)(1+s)^m = (1+s^m)(1+r)^m$$
   >
   > precies N oplossingen $m > 1$ heeft.

   het getal $N = 3$ is en dat in dit geval de enige oplossingen $m = 7, 11, 29$ zijn.

4. *Het vermoeden van Foursov.* In hoofdstuk 9 zien we dat het vermoeden van Foursov juist is binnen de context van polynomiale symmetrieën. Echter, als we vermenigvuldiging met v^c (waar $c \in \mathbb{C}$) toestaan, dan blijkt het voor elke $\alpha \in \mathbb{C}$ mogelijk om niet-commuterende hierarchieën van symmetrieën te vinden.