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ON THE FOURIER TRANSFORM OF THE GREATEST COMMON DIVISOR

Peter H. van der Kamp
Dept. of Mathematics and Statistics, La Trobe University, Victoria, Australia
P.vanderkamp@latrobe.edu.au

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Abstract

The discrete Fourier transform of the greatest common divisor

$$\widehat{\mathrm{id}}[a](m) = \sum_{k=1}^{m} \gcd(k,m)\alpha_m^{ka},$$

with α_m a primitive m-th root of unity, is a multiplicative function that generalizes both the gcd-sum function and Euler's totient function. On the one hand it is the Dirichlet convolution of the identity with Ramanujan's sum, $\widehat{\mathrm{id}}[a] = \mathrm{id} * c_\bullet(a)$, and on the other hand it can be written as a generalized convolution product, $\widehat{\mathrm{id}}[a] = \mathrm{id} *_a \phi$. We show that $\widehat{\mathrm{id}}[a](m)$ counts the number of elements in the set of ordered pairs (i,j) such that $i \cdot j \equiv a \bmod m$. Furthermore we generalize a dozen known identities for the totient function, to identities which involve the discrete Fourier transform of the greatest common divisor, including its partial sums, and its Lambert series.

1. Introduction

In [15] discrete Fourier transforms of functions of the greatest common divisor were studied, i.e.,

$$\widehat{h}[a](m) := \sum_{k=1}^{m} h(\gcd(k,m))\alpha_m^{ka},$$

where α_m is a primitive m-th root of unity. The main result in that paper is the identity[1] $\widehat{h}[a] = h * c_\bullet(a)$, where

$$c_m(a) := \sum_{\substack{k=1 \\ \gcd(k,m)=1}}^{m} \alpha_m^{ka} \tag{1}$$

[1]Similar results in the context of r-even function were obtained earlier; see [10] for details.

is Ramanujan's sum, and $*$ denotes Dirichlet convolution, i.e.,

$$\widehat{h}[a](m) = \sum_{d|m} h(\frac{m}{d}) c_d(a). \tag{2}$$

Ramanujan's sum generalizes both Euler's totient function $\phi = c_\bullet(0)$ and the Möbius function $\mu = c_\bullet(1)$. Thus, identity (2) generalizes the formula

$$\sum_{k=1}^{m} h(\gcd(k,m)) = (h * \phi)(m), \tag{3}$$

already known to Cesàro in 1885. The formula (2) shows that $\widehat{h}[a]$ is multiplicative if h is multiplicative (because $c_\bullet(a)$ is multiplicative and Dirichlet convolution preserves multiplicativity).

Here we will take $h = \mathrm{id}$ to be the identity function (of the greatest common divisor) and study its Fourier transform. Obviously, as $\mathrm{id}(n) := n$ is multiplicative, the function $\widehat{\mathrm{id}}[a]$ is multiplicative, for all a. Two special cases are well-known. Taking $a = 0$ we have $\widehat{\mathrm{id}}[0] = \mathcal{P}$, where

$$\mathcal{P}(m) := \sum_{k=1}^{m} \gcd(k,m) \tag{4}$$

is known as Pillai's arithmetical function or the gcd-sum function [4]. Secondly, by taking $a = 1$ in (2), we find that $\widehat{\mathrm{id}}[1] = \mathrm{id} * \mu$ equals ϕ, by Möbius inversion of Euler's identity $\phi * u = \mathrm{id}$, where $u = \mu^{-1}$ is the unit function defined by $u(m) := 1$.

Let $\mathcal{F}_a^m$ denote the set of ordered pairs of congruence classes (i, j) such that $i \cdot j \equiv a \bmod m$, the set of *factorizations of a modulo m*. We claim that $\widehat{\mathrm{id}}[a](m)$ counts its number of elements. Let us first consider two special cases.

$a = 0$ For given $i \in \{1, 2, \ldots, m\}$ the congruence $i \cdot j \equiv 0 \bmod m$ yields

$$\frac{i}{\gcd(i,m)} j \equiv 0 \mod \frac{m}{\gcd(i,m)},$$

which has a unique solution modulo $m/\gcd(i,m)$, and so there are $\gcd(i,m)$ solutions modulo m. Hence, the total number of elements in $\mathcal{F}_0^m$ is $\mathcal{P}(m)$.

$a = 1$ The totient function $\phi(m)$ counts the number of invertible congruence classes modulo m. As for every invertible congruence class i modulo m there is a unique $j = i^{-1} \bmod m$ such that $i \cdot j \equiv 1 \bmod m$, it counts the number of elements in the set $\mathcal{F}_1^m$.

To prove the general case we employ a Kluyver-like formula for $\widehat{\mathrm{id}}[a]$, that is, a formula similar to the formula for the Ramanujan sum function:

$$c_k(a) = \sum_{d|\gcd(a,k)} d\mu\left(\frac{k}{d}\right). \tag{5}$$

attributed to Kluyver [8]. Together the identities (2) and (5) imply, cf. Section 3,

$$\widehat{\mathrm{id}}[a](m) = \sum_{d \mid \gcd(a,m)} d\phi\left(\frac{m}{d}\right), \tag{6}$$

and we will show, in the next section, that the number $\#\mathcal{F}_a^m$ of factorizations of a mod m is given by the same sum.

The right-hand sides of (5) and (6) are particular instances of so-called generalized Ramanujan sums [1], and both formulas follow as consequence of a general formula for the Fourier coefficients of such sums [2, 3]. In Section 3 we provide simple proofs for some of the nice properties of these sums. In particular we interpret the sums as a generalization of Dirichlet convolution. This interpretation lies at the heart of many of the generalized totient identities we establish in Section 4.

2. The Number of Factorizations of a mod m

We are after the number of pairs (i, j) of congruence classes modulo m such that

$$i \cdot j \equiv a \mod m. \tag{7}$$

For given $i, m \in \mathbb{N}$, let g denote $\gcd(i, m)$. If the congruence $i \cdot j \equiv a$ mod m has a solution j, then $g \mid a$ and $j \equiv i^{-1}a/g$ is unique mod m/g, so mod m there are g solutions. This yields

$$\#\mathcal{F}_a^m = \sum_{\substack{i=1 \\ \gcd(i,m) \mid a}}^{m} \gcd(i, m),$$

which can be written as

$$\#\mathcal{F}_a^m = \sum_{d \mid a} \sum_{\substack{i=1 \\ \gcd(i,m)=d}}^{m} d. \tag{8}$$

If $d \nmid m$ then the sum

$$\sum_{\substack{i=1 \\ \gcd(i,m)=d}}^{m} 1$$

is empty. Now let $d \mid m$. The only integers i which contribute to the sum are the multiples of d, kd, where $\gcd(k, m/d) = 1$. There are exactly $\phi(m/d)$ of them. Therefore the right-hand sides of formulae (6) and (8) agree, and hence $\#\mathcal{F}_a^m = \widehat{\mathrm{id}}[a](m)$.

3. A Historical Remark and Generalized Ramanujan Sums

It is well-known that Ramanujan was not the first who considered the sum $c_m(a)$. Kluyver proved his formula (5) in 1906, twelve years before Ramanujan published the novel idea of expressing arithmetical functions in the form of a series $\sum_s a_s c_s(n)$ [11]. It is not well-known that Kluyver actually showed that $c_m(a)$ equals Von Sterneck's function, introduced in [13], i.e.,

$$c_m(a) = \frac{\mu\Big(\dfrac{m}{\gcd(a,m)}\Big)\phi(m)}{\phi\Big(\dfrac{m}{\gcd(a,m)}\Big)}. \tag{9}$$

This relation is referred to in the literature as Hölder's relation, cf. the remark on page 213 in [1]. However, Hölder published this relation (9) thirty years after Kluyver [7]. We refer to [1, Theorem 2], or [2, Theorem 8.8] for a generalization of (9). The so-called generalized Ramanujan sums,

$$f *_a g(m) := \sum_{d|\gcd(a,m)} f(d) g\big(\frac{m}{d}\big), \tag{10}$$

were introduced in [1]. The notation $*_a$ is new, the sums are denoted $S(a;m)$ in [1], $s_m(a)$ in [2], and $S_{f,h}(a,m)$ in [3]. In the context of r-even functions [10] the sums are denoted $S_{f,g}(a)$, and considered as sequences of m-even functions, with argument a. We consider the sums as a sequence of functions with argument m, labeled by a. We call $f *_a g$ the **a-convolution of f and g**.

The concept of a-convolution provides a generalization of Dirichlet convolution as $f *_0 g = f * g$. As we will see below shortly, the function $f *_a g$ is multiplicative (for all a) if f and g are, and the following inter-associative property holds, cf. [3, Theorem 4]:

$$(f *_a g) * h = f *_a (g * h). \tag{11}$$

We also adopt the notation $f_a := \mathrm{id} *_a f$, and call this **the Kluyver, or a-extension of f**. Thus, we have $f_0 = \mathrm{id} * f$, $f_1 = f$, and formulas (5) and (6) become $c_m(a) = \mu_a(m)$, and $\widehat{\mathrm{id}}[a] = \phi_a$, respectively.

In terms of the Iverson bracket, with P a logical statement,

$$[P] := \begin{cases} 1 & \text{if } P, \\ 0 & \text{if not } P, \end{cases}$$

the identity function I is given by $I(k) := [k = 1]$. It is the identity function in the algebra where convolution is the multiplication, i.e., $I * f = f * I = f$. Let us consider the function $f *_a I$. It is

$$f *_a I(k) = \sum_{d|\gcd(a,k)} f(d)[d = k] = [k \mid a] f(k).$$

Since the function $k \to [k \mid a]$ is multiplicative, the function $f *_a I$ is multiplicative if f is multiplicative. We may write, cf. [3, eq. (9)],

$$f *_a g(m) = \sum_{d \mid m} [d \mid a] f(d) g\big(\frac{m}{d}\big) = (f *_a I) * g(m), \tag{12}$$

which shows that $f *_a g$ is multiplicative if f and g are. Using equation (12), the inter-associativity property (11) follows easily from the associativity of Dirichlet convolution.

Proof. We have $(f *_a g) * h = ((f *_a I) * g) * h = (f *_a I) * (g * h) = f *_a (g * h)$. $\square$

We note that the a-convolution product is neither associative, nor commutative. The inter-associativity and the commutativity of Dirichlet convolution imply that

$$f_a * g = (f * g)_a = f * g_a. \tag{13}$$

We provide a simple proof for formula (6), which states that the Fourier transform of the greatest common divisor is the Kluyver extension of the totient function.

Proof. Using (2), (5) and (13) we get $\widehat{\mathrm{id}}[a] = \mathrm{id} * c_\bullet(a) = \mathrm{id} * \mu_a = (\mathrm{id} * \mu)_a = \phi_a$. $\square$

Formula (6) also follows as a special case of the formula for the Fourier coefficients of a-convolutions,

$$f *_a g(m) = \sum_{k=1}^{m} q[k](m) \alpha_m^{ka}, \qquad q[k] := g *_k \frac{f}{\mathrm{id}}, \tag{14}$$

which was given in [1, 2]. Formula (14) combines with (2) and (5) to yield a formula for functions of the greatest common divisor, $\bar{h}[k] : m \to h(\gcd(k, m))$, namely

$$\bar{h}[k] = (h * \mu) *_k u. \tag{15}$$

Proof. The Fourier coefficients of $\widehat{h}[a](m)$ are $\bar{h}[k](m)$. But $\widehat{h}[a] = h * c_\bullet(a) = (\mathrm{id} *_a \mu) * h = \mathrm{id} *_a (\mu * h)$, and so, using (14), the Fourier coefficients are also given by $(h * \mu) *_k u(m)$. $\square$

For a Dirichlet convolution with a Fourier transform of a function of the greatest common divisor we have

$$f * \widehat{g}[a] = \widehat{f * g}[a]. \tag{16}$$

Proof. We have $f * \widehat{g}[a] = f * (g * \mu_a) = (f * g) * \mu_a = \widehat{f * g}[a]$. $\square$

Similarly, for an a-convolution with a Fourier transform of a function of the greatest common divisor, we have

$$f *_a \widehat{g}[b] = \widehat{f *_a g}[b]. \tag{17}$$

Proof. Indeed, $f *_a \widehat{g}[b] = f *_a (g * \mu_b) = (f *_a g) * \mu_b = \widehat{f *_a g}[b]$. $\square$

4. Generalized Totient Identities

The totient function is an important function in number theory, and related fields of mathematics. It is extensively studied, connected to many other notions and functions, and there exist numerous generalization and extensions, cf. the chapter "The many facets of Euler's totient" in [12]. The Kluyver extension of the totient function is a very natural extension, and it is most surprising it has not been studied before. In this section we generalize a dozen known identities for the totient function ϕ, to identities which involve its Kluyver extension ϕ_a, a.k.a. the discrete Fourier transform of the greatest common divisor. This includes a generalization of Euler's identity, the partial sums of ϕ_a, and its Lambert series.

4.1. The Value of ϕ_a at Powers of Primes

We start by providing a formula for the value of ϕ_a at powers of primes. This depends only on the multiplicity of the prime in a. The formulae, with p prime,

$$\mathcal{P}(p^k) = (k+1)p^k - kp^{k-1}, \qquad \phi(p^k) = p^k - p^{k-1},$$

of which the first one is Theorem 2.2 in [4], generalize to

$$\phi_a(p^k) = \begin{cases} (p^k - p^{k-1})(l+1) & \text{if } l < k, \\ (k+1)p^k - kp^{k-1} & \text{if } l \geq k, \end{cases} \tag{18}$$

where l is the largest integer, or infinity, such that $p^l \mid a$.

Proof. We have

$$\begin{aligned} \phi_a(p^k) &= \sum_{d \mid \gcd(p^l, p^k)} d\phi\big(\frac{p^k}{d}\big) \\ &= \sum_{r=0}^{\min(l,k)} p^r \phi(p^{k-r}) \\ &= \begin{cases} \sum_{r=0}^{l} p^k - p^{k-1} & \text{if } l < k, \\ (\sum_{r=0}^{k-1} p^k - p^{k-1}) + p^k & \text{if } l \geq k, \end{cases} \end{aligned}$$

which equals (18). □

4.2. Partial Sums of ϕ_a/id

To generalize the totient identity

$$\sum_{k=1}^{n} \frac{\phi(k)}{k} = \sum_{k=1}^{n} \frac{\mu(k)}{k} \left\lfloor \frac{n}{k} \right\rfloor. \tag{19}$$

to an identity for ϕ_a we first establish

$$\sum_{k=1}^{n} \frac{f_0(k)}{k} = \sum_{k=1}^{n} \frac{f(k)}{k} \left\lfloor \frac{n}{k} \right\rfloor. \tag{20}$$

Proof. Since there are $\lfloor n/d \rfloor$ multiples of d in the range $[1, n]$ it follows that

$$\sum_{k=1}^{n} \frac{f * \mathrm{id}(k)}{k} = \sum_{k=1}^{n} \sum_{d|k} \frac{f(d)}{d} = \sum_{d=1}^{n} \frac{f(d)}{d} \left\lfloor \frac{n}{d} \right\rfloor.$$

□

As a corollary we obtain

$$\sum_{k=1}^{n} \frac{f *_a g_0(k)}{k} = \sum_{k=1}^{n} \frac{f *_a g(k)}{k} \left\lfloor \frac{n}{k} \right\rfloor. \tag{21}$$

Proof. Employing (11) we find

$$\sum_{k=1}^{n} \frac{f *_a (g * \mathrm{id})(k)}{k} = \sum_{k=1}^{n} \frac{(f *_a g) * \mathrm{id}(k)}{k} = \sum_{k=1}^{n} \frac{f *_a g(k)}{k} \left\lfloor \frac{n}{k} \right\rfloor.$$

□

Now taking $f = \mathrm{id}$ and $g = \mu$ in (21) we find

$$\sum_{k=1}^{n} \frac{\phi_a(k)}{k} = \sum_{k=1}^{n} \frac{c_k(a)}{k} \left\lfloor \frac{n}{k} \right\rfloor. \tag{22}$$

4.3. Partial Sums of $\mathcal{P}_a/\mathrm{id}$ Expressed in Terms of ϕ_a

Taking $f = \mathrm{id}$ and $g = \phi$ in (21) we find

$$\sum_{k=1}^{n} \frac{\mathcal{P}_a(k)}{k} = \sum_{k=1}^{n} \frac{\phi_a(k)}{k} \left\lfloor \frac{n}{k} \right\rfloor. \tag{23}$$

Note that by taking either $a = 0$ in (22), or $a = 1$ in (23), we find an identity involving the totient function and the gcd-sum function,

$$\sum_{k=1}^{n} \frac{\mathcal{P}(k)}{k} = \sum_{k=1}^{n} \frac{\phi(k)}{k} \left\lfloor \frac{n}{k} \right\rfloor. \tag{24}$$

4.4. Partial Sums of ϕ_a

To generalize the totient identity, with $n > 0$,

$$\sum_{k=1}^{n} \phi(k) = \frac{1}{2}\left(1 + \sum_{k=1}^{n} \mu(k) \left\lfloor \frac{n}{k} \right\rfloor^2\right), \tag{25}$$

we first establish

$$\sum_{k=1}^{n} f_0(k) = \frac{1}{2}\left(\sum_{k=1}^{n} f(k) \left\lfloor \frac{n}{k} \right\rfloor^2 + \sum_{k=1}^{n} f * u(k)\right). \tag{26}$$

Proof. By changing variables, $k = dl$, we find

$$\begin{aligned} \sum_{k=1}^{n}(2f * \mathrm{id} - f * u)(k) &= \sum_{k=1}^{n} \sum_{d|k} f(d)(\frac{2k}{d} - 1) \\ &= \sum_{d=1}^{n} \sum_{l=1}^{\lfloor n/d \rfloor} f(d)(2l - 1) \\ &= \sum_{d=1}^{n} f(d) \left\lfloor \frac{n}{d} \right\rfloor^2 . \end{aligned}$$

□

Note that this gives a nice proof of (25), taking $f = \mu$, as $\sum_{k=1}^{n} I(k) = [n > 0]$. When $f = \mu_a$, then (13) implies $f * \mathrm{id} = \phi_a$, and $f * u = I_a$, and therefore as a special case of (26) we obtain

$$\sum_{k=1}^{n} \phi_a(k) = \frac{1}{2}\left(\sum_{k|a} k[k \leq n] + \sum_{k=1}^{n} c_k(a) \left\lfloor \frac{n}{k} \right\rfloor^2\right). \tag{27}$$

We remark that when $n \geq a$ we have $\sum_{k|a} k[k \leq n] = \sigma(a)$, where $\sigma = \mathrm{id} * u$ is the sum of divisors function.

4.5. Generalization of Euler's Identity

Euler's identity, $\phi * u = \mathrm{id}$, generalizes to

$$\sum_{d|m} \phi_a(d) = \tau(\gcd(a, m))m, \tag{28}$$

where $\tau = u * u$ is the number of divisors function.

Proof. We have $\phi_a * u = (\phi * u)_a = \mathrm{id}_a$, where

$$\mathrm{id}_a(m) = \sum_{d|\gcd(a,m)} d\frac{m}{d} = m\tau(\gcd(a, m)). \tag{29}$$

□

4.6. Partial Sums of $\mathcal{P}_a$ Expressed in Terms of ϕ_a (and τ)

If $f = \phi_a$, then $f * \mathrm{id} = \mathcal{P}_a$, and (26) becomes, using (28),

$$\sum_{k=1}^{n} \mathcal{P}_a(k) = \frac{1}{2}\Big(\sum_{k=1}^{n} \tau(\gcd(a,k))k + \sum_{k=1}^{n} \phi_a(k) \left\lfloor \frac{n}{k} \right\rfloor^2 \Big). \tag{30}$$

4.7. Four Identities From Césaro

According to Dickson [5] the following three identities were obtained by Césaro:

$$\sum_{d|n} d\phi\big(\frac{n}{d}\big) = \mathcal{P}(n), \tag{31}$$

$$\sum_{d|n} \frac{d}{n}\phi(d) = \sum_{j=1}^{n} \frac{1}{\gcd(j,n)}, \tag{32}$$

$$\sum_{d|n} \phi(d)\phi\big(\frac{n}{d}\big) = \sum_{j=1}^{n} \phi(\gcd(j,n)). \tag{33}$$

Identity (31), which is Theorem 2.3 in [4], is obtained by taking $a = 0$ in (6), or $h = \mathrm{id}$ in (3). It generalizes to

$$\sum_{d|n} d\phi_a\big(\frac{n}{d}\big) = \mathcal{P}_a(n). \tag{34}$$

Proof. Take $f = \phi$ and $g = \mathrm{id}$ in (13). $\square$

Identity (32) is obtained by taking $h = 1/\mathrm{id}$ in (3) and generalizes to

$$\sum_{d|n} \frac{d}{n}\phi_a(d) = \sum_{j=1}^{n} \sum_{d|\gcd(a,n)} \frac{1}{\gcd\big(j, \frac{n}{d}\big)}. \tag{35}$$

Proof. Take $f = \phi$ and $g = 1/\mathrm{id}$ in (13). $\square$

Identity (33) is also a special case of (3), with $h = \phi$. It generalizes to

$$\sum_{d|n} \phi_a(d)\phi_b\big(\frac{n}{d}\big) = \sum_{j=1}^{n} \sum_{d|\gcd(a,n)} \phi_b\big(\gcd\big(j, \frac{n}{d}\big)\big). \tag{36}$$

Proof. We have

$$(\mathrm{id} *_a \phi) * (\mathrm{id} *_b \phi) = \mathrm{id} *_a (\mathrm{id} *_b (\phi * \phi)) = \mathrm{id} *_a (\mathrm{id} *_b \widehat{\phi}[0]) = \mathrm{id} *_a \widehat{\phi}_b[0],$$

and evaluation at m yields

$$\sum_{d|\gcd(a,m)} d \sum_{j=1}^{m/d} \phi_b\big(\gcd\big(j, \frac{m}{d}\big)\big) = \sum_{d|\gcd(a,m)} \sum_{j=1}^{m} \phi_b\big(\gcd\big(j, \frac{m}{d}\big)\big).$$

□

The more general identity (3) generalizes to

$$\sum_{k=1}^{m} h_a(\gcd(k,m)) = h * \phi_a(m). \tag{37}$$

4.8. Three Identities From Liouville

Dickson [5, p.285-286] states, amongst many others identities that were presented by Liouville in the series [9], the following identities:

$$\sum_{d|m} \phi(d)\tau\big(\frac{m}{d}\big) = \sigma(m), \tag{38}$$

$$\sum_{d|m} \phi(d)\sigma[n+1]\big(\frac{m}{d}\big) = m\sigma[n](m), \tag{39}$$

$$\sum_{d|m} \phi(d)\tau\Big(\frac{m^2}{d^2}\Big) = \sum_{d|m} d\theta\big(\frac{m}{d}\big), \tag{40}$$

where $\sigma[n] = \mathrm{id}[n] * u$, $\mathrm{id}[n](m) := m^n$, and $\theta(m)$ is the number of decompositions of m into two relatively prime factors. All three are of the form $\phi * f = g$ and therefore they gain significance due to (3), though Liouville might not have been aware of this. For example, (3) and (38) combine to yield

$$\sum_{k=1}^{m} \tau(\gcd(k,m)) = \sigma(m).$$

The three identities are easily proven by substituting $\tau = u * u$, $\sigma[n] = \mathrm{id}[n] * u$, $\tau \circ \mathrm{id}[2] = \theta * u$, $\phi = \mu * \mathrm{id}$, and using $\mu * u = I$. They generalize to

$$\sum_{d|m} \phi_a(d)\tau\big(\frac{m}{d}\big) = \sigma_a(m), \tag{41}$$

$$\sum_{d|m} \phi_a(d)\sigma[n+1]\big(\frac{m}{d}\big) = mu *_a \sigma[n](m), \tag{42}$$

$$\sum_{d|m} \phi_a(d)\tau\Big(\frac{m^2}{d^2}\Big) = \sum_{d|m} d\tau(\gcd(a,d))\theta\big(\frac{m}{d}\big). \tag{43}$$

These generalizations are proven using the same substitutions, together with (13), or for the latter identity, (11) and (29). We note that n need not be integer valued in (39) and (42).

4.9. One Identity from Dirichlet

Dickson [5] writes that Dirichlet [6], by taking partial sums on both sides of Euler's identity, obtained

$$\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \phi(k) = \binom{n+1}{2}.$$

By taking partial sums on both sides of equation (28) we obtain

$$\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \phi_a(k) = \sum_{d|a} d \binom{\left\lfloor \frac{n}{d} \right\rfloor + 1}{2}. \tag{44}$$

Proof. Summing the left-hand side of (28) over m yields

$$\sum_{m=1}^{n} \sum_{d|m} \phi_a(d) = \sum_{d=1}^{n} \left\lfloor \frac{n}{d} \right\rfloor \phi_a(d)$$

and summing the right-hand side of (28) over m yields

$$\sum_{m=1}^{n} \tau(\gcd(a,m))m = \sum_{m=1}^{n} \sum_{d|\gcd(a,m)} m = \sum_{d|a} \sum_{k=1}^{\lfloor n/d \rfloor} dk = \sum_{d|a} d \left\lfloor \frac{n}{d} \right\rfloor \Big(\left\lfloor \frac{n}{d} \right\rfloor + 1 \Big)/2.$$

□

4.10. The Lambert Series of ϕ_a

As shown by Liouville [9], cf. [5, p.120], the Lambert series of the totient function is given by

$$\sum_{m=1}^{\infty} \phi(m) \frac{x^m}{1-x^m} = \frac{x}{(1-x)^2}.$$

The Lambert series for ϕ_a is given by

$$\sum_{m=1}^{\infty} \phi_a(m) \frac{x^m}{1-x^m} = p[a](x) \frac{x}{(1-x^a)^2}, \tag{45}$$

where the coefficients of $p[a](x) = \sum_{k=1}^{2a} c[a](k) x^{k-1}$ are given by $c[a] = \mathrm{id}_a \circ t[a]$, and $t[a]$ is the piece-wise linear function $t[a](n) = a - |n-a|$.

Proof. Cesàro proved

$$\sum_{n=1}^{\infty} f(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} x^n \sum_{d|n} f(d),$$

cf. [5] and exercise 31 to chapter 2 in [14]. By substituting (28) in this formula we find

$$\sum_{n=1}^{\infty} \phi_a(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} x^n \tau(\gcd(a,n))n.$$

Multiplying the right-hand side by $(1 - 2x^a + x^{2a})$ yields

$$\begin{aligned}
&(\sum_{n=1}^{\infty} x^n \tau(\gcd(a,n))n) - 2(\sum_{n=a+1}^{\infty} x^n \tau(\gcd(a,n))(n-a)) \\
&+ (\sum_{n=1+2a}^{\infty} x^n \tau(\gcd(a,n))(n-2a)) = \sum_{n=1}^{\infty} c[a](n) x^n,
\end{aligned}$$

where

$$c[a](n) = \begin{cases} \tau(\gcd(a,n))n & 0 < n \le a, \\ \tau(\gcd(a,n))(n - 2(n-a)) = \tau(\gcd(a,n))(2a-n) & a < n \le 2a, \\ \tau(\gcd(a,n))(n - 2(n-a) + n - 2a) = 0 & n > 2a. \end{cases}$$

Rewriting, using (29), the fact that $\gcd(a, a+k) = \gcd(a, a-k)$, and dividing by x, yields the result. □

The polynomials $p[a]$ seem to be irreducible over $\mathbb{Z}$ and their zeros are in some sense close to the a-th roots of unity; see Figures 4.10 and 4.10.

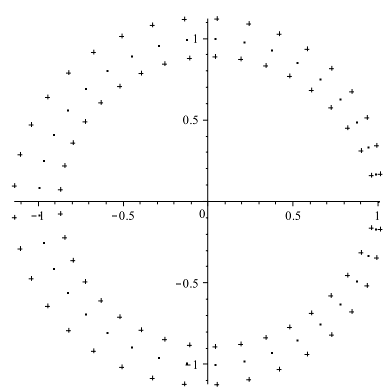

Figure 1: The roots of $p[37]$ are depicted as crosses and the 37^{th} roots of unity as points. This figure shows that when a is prime the roots of $p[a]$ that are close to 1 are closer to a^{th} roots of unity.

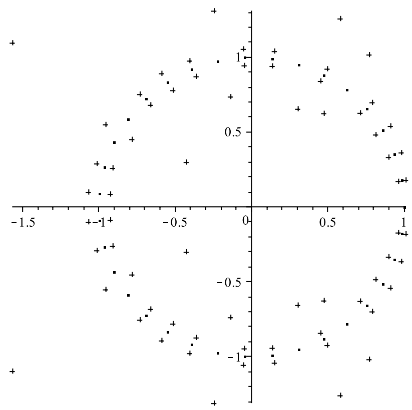

Figure 2: The roots of $p[35]$ are depicted as crosses and the 35^{th} roots of unity as points. This figure shows that roots of $p[a]$ are closest to primitive a^{th} roots of unity.

4.11. A Series Related to the Lambert Series of ϕ_a

Liouville [9] also showed

$$\sum_{m=1}^{\infty} \phi(m) \frac{x^m}{1+x^m} = (1+x^2)\frac{x}{(1-x^2)^2}.$$

We show that

$$\sum_{m=1}^{\infty} \phi_a(m) \frac{x^m}{1+x^m} = q[a](x)\frac{x}{(1-x^{2a})^2}, \tag{46}$$

where $q[a](x) = \sum_{k=1}^{4a} b[a](k)x^{k-1}$, with $b[a] = h[a] \circ t[2a]$, and $h[a](k) = \mathrm{id}_a(k) - 2[2 \mid k]\mathrm{id}_a(k/2)$.

Proof. As the left-hand side of (46) is obtained from the left-hand side of (45) by subtracting twice the same series with x replaced by x^2, the same is true for the right-hand side. Thus it follows that $q[a](x) = p[a](x)(1+x^a)^2 - 2p[a](x^2)x$, and hence, that

$$b[a](k) = \begin{cases} \mathrm{id}_a(k) - 2[2 \mid k]\mathrm{id}_a(k/2) & k \le a, \\ \mathrm{id}_a(2a-k) + 2\mathrm{id}_a(k-a) - 2[2 \mid k]\mathrm{id}_a(k/2) & a < k \le 2a, \\ 2\mathrm{id}_a(3a-k) + \mathrm{id}_a(k-2a) - 2[2 \mid k]\mathrm{id}_a(2a-k/2) & 2a < k \le 3a, \\ \mathrm{id}_a(4a-k) - 2[2 \mid k]\mathrm{id}_a(2a-k/2) & 3a < k \le 4a. \end{cases}$$

The result follows due to the identities

$$\mathrm{id}_a(2a-k) + 2\mathrm{id}_a(k-a) = \mathrm{id}_a(k), \quad 2\mathrm{id}_a(3a-k) + \mathrm{id}_a(k-2a) = \mathrm{id}_a(4a-k),$$

which are easily verified using (29). $\square$

We can express the functions $b[a]$ and $h[a]$ in terms of an interesting fractal function. Let a function κ of two variables be defined recursively by

$$\kappa[a](n) = \begin{cases} 0 & 2 \mid n, 2 \nmid a, \text{ or } n = 0, \\ \kappa[a/2](n/2) & 2 \mid n, 2 \mid a, \\ \tau(\gcd(a,n)) & 2 \nmid n. \end{cases} \tag{47}$$

The following properties are easily verified using the definition. We have

$$\kappa[a](2a+n) = \kappa[a](2a-n), \tag{48}$$

and, with $\gcd(a,b) = 1$,

$$\kappa[an](bn) = \begin{cases} 0 & 2 \mid b, \\ \alpha(n) & 2 \nmid b, \end{cases} \tag{49}$$

where α denotes the number of odd divisors function, i.e., for all k and odd m

$$\alpha(2^k m) = \tau(m). \tag{50}$$

Property (49) is a quite remarkable fractal property; from the origin in every direction we see either the zero sequence, or α, at different scales.

We claim that

$$h[a] = \kappa[a]\mathrm{id} \tag{51}$$

follows from (47), (50), and (29). From (51) and (48) we obtain

$$b[a] = \kappa[a]t[2a]. \tag{52}$$

We now prove that $1 + x^2$ divides $q[a]$ when a is odd.

Proof. Noting that $b[a](2a+k) = b[a](2a-k)$ and, when $2 \nmid a$, $b[a](2k) = 0$, we therefore have

$$\begin{aligned} q[a](x) &= \sum_{n=1}^{2a} b[a](2n-1)x^{2n-2} \\ &= \sum_{m=1}^{a} b[a](2a-2m+1)x^{2a-2m} + b[a](2a+2m-1)x^{2a+2m-2} \\ &= \sum_{m=1}^{a} b[a](2a-2m+1)x^{2a-2m}(1+x^{4m-2}), \end{aligned}$$

which vanishes at the points where $x^2 = -1$. □

Apart from the factor $1 + x^2$ when a is odd, the polynomial $q[a]$ seems to be irreducible over $\mathbb{Z}$ and its zeros are in some sense close to the $(2a)^{\text{th}}$ roots of -1 or, to the $(a+1)^{\text{st}}$ roots of unity, see Figure 4.11.

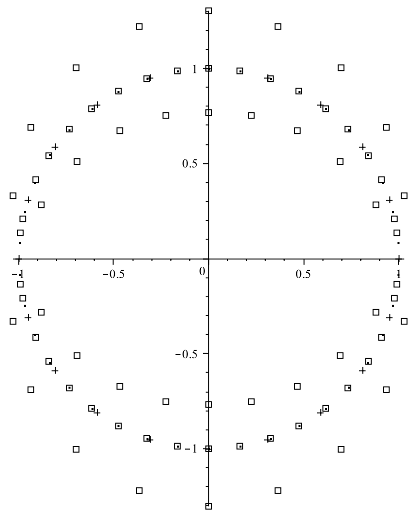

Figure 3: The roots of $q[19]$ are depicted as boxes, the 38^{th} roots of -1 as points, and the 20^{th} roots of unity as crosses.

4.12. A Perfect Square

Our last identity generalizes the faint fact that $\phi(1) = 1$. We have

$$\sum_{a=1}^{n} \phi_a(n) = n^2. \tag{53}$$

Proof. For any lattice point (i, j) in the square $[1, n] \times [1, n]$ the product $i \cdot j \bmod n$ is congruent to some a in the range $[1, n]$. □

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References

[1] D.R. Anderson and T.M. Apostol, *The evaluation of Ramanujan's sum and generalizations*, Duke Math. J. **20** (1953), 211–216.

[2] T.M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag (1976).

[3] T.M. Apostol, *Arithmetical properties of generalized Ramanujan sums*, Pacific J. Math. 41 (1972), 281–293.

[4] K.A. Broughan, *The gcd-sum function*, Journal of Integer Sequences **4** (2001), Art. 01.2.2.

[5] L.E. Dickson, History of the theory of numbers, Carnegie Inst. (1919); reprinted by Chelsea (1952).

[6] G.L. Dirichlet, *Über die Bestimmung der mittleren Werte in der Zahlentheorie*, Abh. Akad. Wiss. Berlin (1849), 78–81; also in Werke, vol. 2 (1897), 60–64.

[7] O. Hölder, *Zur Theorie der Kreisteilungsgleichung* $K_m(x) = 0$, Prace Matematyczno Fizyczne **43** (1936), 13–23.

[8] J.C. Kluyver, *Some formulae concerning the integers less than* n *and prime to* n, in: KNAW, Proceedings 9 I, Amsterdam (1906), 408–414.

[9] J. Liouville, *Sur quelques séries et produits infinis*, J. Math. Pures Appl. **2** (1857), 433–440.

[10] L. Tóth and P. Haukkanen, *The discrete Fourier transform of* r*-even functions*, Acta Univ. Sapientiae Math. **3** (2011), 5–25.

[11] S. Ramanujan, *On Certain Trigonometric Sums and their Applications in the Theory of Numbers*, Transactions of the Cambridge Philosophical Society **22** (1918), 259–276; also in: Collected papers of Srinivasa Ramanujan, Ed. G.H. Hardy et al, Chelsea Publ. Comp. (1962), 179–199.

[12] J. Sandor and B. Crstici, Handbook of Number Theory II, Kluwer Acad. Publ. (2004).

[13] R. Daublebsky Von Sterneck, *Ein Analogon zur additiven Zahlentheorie*, Sitzber, Akad. Wiss. Wien, Math. Naturw. Klasse, vol. I ll (Abt. IIa) (1902), 1567–1601.

[14] H. S. Wilf, Generatingfunctionology, Academic Press, 2nd edition (1994).

[15] W. Schramm, *The Fourier transform of functions of the greatest common divisor*, Integers **8** (2008) A50.