Lax representations for integrable maps

Omar Rojas, Peter H van der Kamp and G R W Quispel

Department of Mathematics and Statistics, La Trobe University, Victoria, 3086, Australia

E-mail: `o.rojas@latrobe.edu.au`

Abstract. For a partial difference equation on the lattice, we present a two-parameter family of discrete travelling wave reductions, from which ordinary difference equations, or mappings, corresponding to solutions of such problem, are obtained. Based on the staircase method, for which we give an algebraic description as a set of reduced variables, we derive a Lax representation to obtain integrals of such mappings, whenever a Lax pair for the partial difference equation exists. Finally, from a reduction of the lattice sine-Gordon equation, we present a four-dimensional mapping that is integrable, since it is symplectic and possesses two functionally independent integrals in involution.

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1. Introduction

The study of integrable systems has witnessed an ongoing shift of direction towards the discrete setting which, according to many, seems to be of fundamental nature due to its rich structure. Integrable partial difference equations (PΔEs) on a planar lattice were obtained in the 80s under integrable discretizations from known integrable PDEs [6, 7, 8, 14]. Some years later, an elegant and efficient algorithmic method, dubbed *the staircase method*, was applied to the discrete potential Korteweg-De Vries equation (pKdV) [11], in order to reduce the PΔE to a family of ordinary difference equations (OΔEs) or, more or less equivalently, integrable mappings, via discrete travelling waves, and therefore solve the periodic initial value problem of the PΔE. The method took advantage of the Lax matrices for the PΔE, which carried an *spectral parameter*, to construct a monodromy or transfer matrix, the trace of which yielded *integrals of motion* of the obtained mapping. Subsequently, it was shown that these integrals were in involution with respect to a symplectic structure [4, 9], establishing the complete integrability, in the Liouville-Arnold-Veselov (LAV) sense, of the mappings in question, cf. [3, 20].

Recently, there has been a renewed interest in the study of integrable PΔEs, specifically since [1], where a classification of integrable equations on quad-graphs, possessing the three-dimensional consistency property, was given. The relevance of this property is such that it gives a way [1, 10] to obtain the Lax representation of the integrable PΔE under study.

As for integrable maps, an 18-parameter family of integrable planar mappings (nowadays known as the QRT-mappings) was presented in [15, 16]. Such discovery was based on the observation that simple solutions to many soliton equations lead to integrable mappings. Higher dimensional integrable mappings, derived from integrable discretizations of sine-Gordon and KdV-type equations, were presented in [12], where a characterization of possible travelling wave reductions was given in terms of two natural numbers s_1 and s_2, corresponding to different periodicity conditions

$$u_{l,m} = u_{l+ks_1,m+ks_2}, \qquad \text{for all } k \in \mathbb{Z}, \tag{1}$$

along a descending staircase on the $\mathbb{Z}^2$-lattice. The case where $s_1 = z_2$ and $s_2 = -z_1$, where z_1 and z_2 are relatively prime numbers was treated extensively in that paper. This complemented the original staircase considered in [4, 9, 11], which corresponds to the case $s_1 = -s_2$. In the present paper we unify the two extremes and provide an algebraic description of the standard staircase for any $s := (s_1, s_2) \in \mathbb{Z}^2$. Our algebraic description of such s-reduction, which poses well-defined initial value problems for PΔEs on a square [2, 18], has the advantage of, above its generallity, being algorithmic, which enables one to produce mappings of any phase-space's dimension (i.e., order of the OΔE), with the aid of computer software. The procedure presented here applies to periodic initial value problems of PΔEs, not necessarily integrable. By this method, an integrable PΔE reduces to a system of q coupled OΔEs, where q is the greatest common divisor of $|s_1|$ and $|s_2|$. If these two integers were coprime, the reduction goes into a single equation. Reductions for open-ended initial value problems are also possible, as was shown in [5] with respect to the KdV equation.

We mentioned already that, for integrable PΔEs that belong to the Adler-Bobenko-Suris (ABS) classification [1], a Lax pair can be obtained from the equation itself. We have also mentioned how, starting from a periodic initial value problem of a PΔE, one obtains a mapping for each s-travelling wave reduction, generating a two-parameter dependent family of mappings for each chosen equation from, but not restricted to, the ABS list. Then, if all integrable systems possess a Lax representation, as it is widely believed [17], the following questions arise: do the mappings obtained by s-reductions possess a Lax representation? And, if so, how does one obtain it? In this paper, based on the staircase method, we provide an affirmative answer to the first question, and provide explicit formulas for the Lax-matrices of the system of OΔEs in terms of the Lax-matrices of the original PΔE. This generalizes the observation made in [13] that, in the case where $s_2 = 1$, the Lax matrix $\mathcal{L}$ coincides with the monodromy matrix defined on a standard staircase, whereas the matrix $\mathcal{M}$ appears to be a particular factor of it. The Lax pairs for either an equation or a system of equations can be obtained from our result.

The structure of this paper is as follows: In section 2 we briefly review the notion of Lax representations in the settings of PΔEs, OΔEs, and systems of OΔEs ‡. In section 3 we will describe the two parameter family of travelling wave reductions in detail. In section 4 we describe the standard staircase, and show how it can be stated as a set of reduced variables. In section 5 we show how to obtain the Lax representation of mappings obtained by an s-reduction. Finally, in section 6, we derive, from a $(4, -2)$-reduction of the lattice sine-Gordon equation, a new four-dimensional integrable mapping which is symplectic, reversible and possesses two functionally independent integrals in involution, from which it follows that the considered mapping is completely integrable.

2. Lax representation

Consider a PΔE on a two-dimensional lattice

$$f_{l,m} := f(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+1,m+1}) = 0, \tag{2}$$

for fields $u_{l,m}$ defined at the sites (l, m) of a two dimensional lattice $\mathbb{Z}^2$. We assume that (2) is an affine linear function, which allows to solve for any of its variables. The PΔE (2) possesses a *Lax representation* if there are matrices L and M, usually depending on a spectral parameter λ, such that, given the linear problem $\Psi_{l+1,m} = L_{l,m}\Psi_{l,m}$, $\Psi_{l,m+1} = M_{l,m}\Psi_{l,m}$, their compatibility condition (or discrete zero-curvature condition)

‡ It is common to use indistinctly the term *Lax representation* or *zero-curvature condition*, the main difference being that the first aludes to ODEs, whereas the last to PDEs. In this paper we use the term *Lax representation* for both ordinary and partial equations, always mentioning to which of them we are refering.

yields

$$L_{l,m}M_{l,m}^{-1} - M_{l+1,m}^{-1}L_{l,m+1} = f_{l,m}N_{l,m} \tag{3}$$

where $N_{l,m}$ is a matrix that is non-singular on solutions of (2). Attempts to give a systematic way of finding Lax pairs for P∆Es have been made, but it seems an efficient and algorithmic method exists only for equations that are *three-dimensionally consistent* [1, 10].

An order-d ordinary difference equation (O∆E), with $n, d \in \mathbb{N}$,

$$f_n := f(v_n, v_{n+1}, \ldots, v_{n+d}) = 0, \tag{4}$$

or the corresponding d-dimensional mapping

$$(v_n, v_{n+1}, \ldots, v_{n+d-1}) \mapsto (v_{n+1}, v_{n+2}, \ldots, v_{n+d}), \tag{5}$$

where v_{n+d} is given by the solution of $f_n = 0$, admits a *Lax representation* if there exists non-trivial matrices $\mathcal{L}, \mathcal{M}, \mathcal{N}$ such that $\mathcal{M}$ is non-degenerate, and

$$\mathcal{M}_n\mathcal{L}_n - \mathcal{L}_{n+1}\mathcal{M}_n = f_n\mathcal{N}_n. \tag{6}$$

The significance of having a Lax representation for an integrable O∆E is apparent. Right-multiplying (6) by $-\mathcal{M}_n^{-1}$ and taking the trace we obtain

$$\mathrm{T}r(\mathcal{L}_{n+1}) - \mathrm{T}r(\mathcal{L}_n) = f_n\Lambda_n, \tag{7}$$

where $\Lambda_n = -\mathrm{T}r(\mathcal{N}_n\mathcal{M}_n^{-1})$. Thus $\mathrm{T}r(\mathcal{L}_n)$ is an invariant and called an *integral*, of the equation $f_n = 0$ with *integrating factor* Λ_n.

The integral $\mathrm{T}r(\mathcal{L}_n)$ will in general still depend on the spectral parameter λ. The coefficients in the λ-expansion of $\mathrm{T}r(\mathcal{L}_n)$ yield integrals of the mapping, whereas the corresponding integrating factors will be given by the λ-expansion of Λ_n. In [19] closed form expressions, in terms of multi-sums of products, for the integrals of $(s_1, -1)$-travelling wave reductions of the mKdV and the sine-Gordon equations were given.

For systems of O∆Es, with $i = 1, \ldots, q$,

$$f_n^i := f^i(v_n^1, v_{n+1}^1, \ldots, v_{n+d_1}^1, v_n^2, \ldots, v_{n+d_q}^q) = 0, \tag{8}$$

corresponding to the $(d_1 + d_2 + \cdots + d_q)$-dimensional mapping

$$v_{n+j}^i \mapsto v_{n+j+1}^i, \quad 0 \le j < d_i, \tag{9}$$

where $v_{n+d_i}^i$ is the solution of $f_n^i = 0$, the Lax representation takes the form

$$\mathcal{M}_n\mathcal{L}_n - \mathcal{L}_{n+1}\mathcal{M}_n = \sum f_n^i\mathcal{N}_n^i \tag{10}$$

which vanishes on the system. Similarly as was done with (6), right-multiplying (10) by $-\mathcal{M}_n^{-1}$ and taking the trace we obtain

$$\mathrm{T}r(\mathcal{L}_{n+1}) - \mathrm{T}r(\mathcal{L}_n) = \sum f_n^i\Lambda_n^i, \tag{11}$$

where $\Lambda_n^i = -\mathrm{T}r(\mathcal{N}_n^i\mathcal{M}_n^{-1})$.

Let $s := (s_1, s_2)$. Then, by the s-travelling wave reduction described in the next section, a P∆E of the form (2) reduces to a system of q O∆Es of order $d_i = a + b$, where q is the greatest common divisor of $|s_1| = aq$ and $|s_2| = bq$, which is equivalent to a $(|s_1| + |s_2|)$-dimensional mapping (9).

3. s-travelling wave reductions

Based on [[12], third concluding remark]§ we propose the following travelling wave reduction. We define a new lattice variable n and a modular variable p counting the reduced fields, such that the periodicity condition (1) holds. We may restrict ourselves to taking $s_1 \in \mathbb{N}, s_2 \in \mathbb{Z}$ such that $s_1 \neq 0$ when $s_2 \leq 0$. This is the area in which reductions have been defined since [12]. Let ϵ denote the sign of s_2, i.e., $\mathrm{sgn}(s_2) = \epsilon$. Let q be the greatest common divisor of $s_1 = aq$ and $|s_2| = bq$. We fix $c, d \in \mathbb{N}$ by choosing the smallest c in $0 < c \leq s_1$ such that $bc - ad = 1$. The proposed (s_1, s_2)-reduction is

$$\rho : u_{l,m} \rightsquigarrow v_n^p, \tag{12}$$

where

$$n = bl - \epsilon am, \qquad p \equiv \epsilon cm - dl \bmod q, \tag{13}$$

and $p \in \mathbb{N}_q := \{0, \ldots, q-1\}$. This reduction has the following properties:

- a k-shift on p, leaving n invariant is given by
$$\sigma^k : u_{l,m} \mapsto u_{l+ak,m+\epsilon bk} \rightsquigarrow v_n^{p+k}. \tag{14}$$
- a k-shift on n, leaving p invariant is given by
$$\delta^k : u_{l,m} \mapsto u_{l+ck,m+\epsilon dk} \rightsquigarrow v_{n+k}^p. \tag{15}$$

From the first property we immediately obtain the periodicity condition (1). Under the reduction (12) the P∆E (2) reduces to the following system of q O∆Es

$$f_n^p = f(v_n^p, v_{n+b}^{p+\epsilon d}, v_{n+a}^{p+\epsilon c}, v_{n+a+b}^{p+\epsilon(c+d)}) = 0, \tag{16}$$

which can be solved for any of the variables to give

$$v_{n+a+b}^p = g(v_n^{p-\epsilon(c+d)}, v_{n+b}^{p-\epsilon c}, v_{n+a}^{p-\epsilon d}). \tag{17}$$

This system is equivalent to the mapping $\tau : \mathbb{C}^{|s_1|+|s_2|} \to \mathbb{C}^{|s_1|+|s_2|}$ given by

$$(\xi_0^0, \xi_0^1, \ldots, \xi_0^{q-1}) \mapsto (\xi_1^0, \xi_1^1, \ldots, \xi_1^{q-1}) \tag{18}$$

where $\xi_j^i := (v_j^i, v_{j+1}^i, \ldots, v_{j+a+b-1}^i)$ and v_{a+b}^i in ξ_1^i is given by equation (17) taking $n = 0$. In section 5 we provide a Lax representation for the system (16), in terms of Lax matrices L, M of the original P∆E.

4. The standard staircase

A staircase is a discrete path connecting two points of the lattice. Initial values are given at the points of such staircase, which historically has been taken from the upper left in the plane to the lower right. Provided the P∆E (2) is affine-linear, allowing to solve for $u_{l+1,m+1}$, say, the initial value problem is well-defined. Whenever a periodic initial value problem is given, such that equation (1) is satisfied, periodic solutions of the P∆E can be found by solving the system of O∆Es (16).

§ However, our notation is closer to [12, page 248], with $p_1 = d, p_2 = c$.

The *standard staircase* is defined to be the path from the point $u_{0,0} \rightsquigarrow v_0^0$ to the point $u_{s_1,s_2} \rightsquigarrow v_0^0$ such that the set of points it passes through, reduces to the set of initial values given in the left hand side of equation (18). For any choice of integers s_1, s_2 such a path always exists, and the one defined by the standard staircase is optimal [18]. The staircase for the (q,q)-reduction was first given in [11]. A description of the standard staircase when s_1 and s_2 are co-prime, can be found in [12]. That description, which also includes the case $s_2 = -1$, studied in more detail in [13, 19], is generalised, for any $s_1, s_2 \in \mathbb{Z}$, with the one we will present below.

In order to construct the standard staircase, one could, in principle, build the staircase corresponding to an (a, b)-reduction, starting at the point $u_{0,0}$, and then glue q copies of such staircase, having as the endpoint of the q-th staircase, the point $u_{s_1,s_2} = u_{0,0}$. The staircase leading from $u_{0,0}$ to u_{s_1,s_2} is then repeated periodically by a subsequent shift consisting of s_1 steps in the horizontal direction, combined with s_2 steps in the vertical direction. A simple way to describe the points on a descending staircase, corresponding to a reduction where $s_1, -s_2 \in \mathbb{N}$ where $\gcd(|s_1|, |s_2|) = q$, is as follows:

- Without loss of generality, one can start with the point at $(l, m) = (s_1, s_2) \in \mathbb{Z}^2$ which, by periodicity, corresponds to v_0^0.
- Go one step up to the point $(s_1, s_2 + 1)$. This point corresponds to v_a^i.
- Next go to the left as many steps as possible without getting negative subscripts on the n-index of v_n^p. The last point will be $v^i_{s_1 \text{mod } s_2}$.
- Then go one step up to $v^i_{s_1+(s_1 \text{mod } s_2)}$ and repeat the procedure of going left. The last point will be $v^i_{2s_1 \text{mod } s_2}$.
- The whole procedure of going one step up and to the left is then repeated $|s_2|$ times, after which we arrive at the point v_0^i.
- To complete the staircase the steps described above are repeated q times. The value of i at each site is given by the modular variable $i = dl - \epsilon cm \bmod q$.

A geometric approach to describe the standard staircase can be found in [18]. From the above description, it is apparent that a more general way of constructing the staircase is needed. Before stating our algebraic procedure, we show some typical examples of reductions, with their corresponding standard staircases, below. Of course, these staircases are repeated periodically.

The staircase for the (q,q)-reduction was first given in [11]. A description of the standard staircase for the case $q = 1$ can be found in [12]. The latter of course includes the case $s_2 = -1$, which was studied in more detail in [13, 19].

Our description of the standard staircase for arbitrary s_1, s_2 is an algebraic version and generalization of the description given in [12]. It allows to express the standard staircase as a set of reduced variables.

Consider $a, b, c, d, q, s_1, -s_2 \in \mathbb{N}$ as defined in Section 3. For all $0 \le i \le b$ let x_i be the smallest integer such that $x_i b - ia \ge 0$ and define $y_i := x_i b - ia$, $w_i := x_i d - ic$.

Clearly we have $x_0 = y_0 = w_0 = 0$ and $x_b = a, y_b = 0, w_b = -1$.

We have the following

Proposition 1 *The* standard staircase *is the path from the point $u_{0,0}$ to the point u_{s_1,s_2}, defined in terms of the set of reduced variables*

$$\Gamma_{s_1,s_2} := \{v_i^j : i \in \mathbb{N}_{a+b},\ j \in \mathbb{N}_q\} \tag{19}$$

Proposition 2 *The following equalities between sets hold:*

(i) $\{y_0, \ldots, y_{b-1}\} = \mathbb{N}_b$,

(ii) $\bigcup_{i=0}^{b-1}\{x_i b - ia, (x_i + 1)b - ia, \ldots, x_{i+1}b - ia\} = \mathbb{N}_{a+b}$.

Proof:

(i) First of all we have $y_i < b$ because otherwise x_i would not be the smallest integer such that $y_i \in \mathbb{N}$. Secondly, suppose $y_i \equiv y_j \bmod b$. This implies that $(j - i)a \equiv 0 \bmod b$ and hence $i = j$, since we have $\gcd(a, b) = 1$.

(ii) The set on the left consists of $\sum_{k=0}^{b}(x_k - x_{k-1} + 1) = a + b$ numbers of the form $r_i^j = jb - ia$, $x_i \le j \le x_{i+1}$. We know that that $y_{i+1} < b$. Adding a to both sides shows that $r_i^j \le x_{i+1}b - ia < a + b$. Suppose $r_i^j = r_k^l$. Then $r_i^j \equiv r_k^l \bmod b$, which, using the previous equality, implies that $i = k$. And clearly $r_i^j = r_i^l$ implies $j = l$.

□

5. Factorization of the monodromy matrix

The monodromy matrix $\mathcal{L}_n^p$ is defined to be the ordered product (from the right to the left) of Lax-matrices along the standard staircase. For example, see figure **??**, the monodromy matrix for the $(5, -3)$-reduction is

$$\mathcal{L}_n = M_n^{-1} L_{n+2} M_{n+2}^{-1} L_{n+4} L_{n+1} M_{n+1}^{-1} L_{n+3} L_n, \tag{20}$$

where, as with the traveling wave reduction of the PΔE, we have adopted the notation $L_{l,m} \rightsquigarrow L_n$ for the matrices depending on the reduced variables, in this case omitting the redundant variable $p = 0$. The crucial property of the monodromy matrix is that its trace is invariant under all translations on the lattice.

In [13] the $(a, -1)$-staircase is given as the path from the point $u_{1,0} \rightsquigarrow v_1$ to the point $u_{a+1,-1} \rightsquigarrow v_1$, and the following factorization property was given:

$$\mathcal{L}_n = L_n M_n^{-1} A_n L_{n+1}, \quad A_n = L_{n+a-1} L_{n+a-2} \cdots L_{n+2}. \tag{21}$$

It was shown that the shifted monodromy matrix can be expressed as $\mathcal{L}_{n+1} = L_{n+1} M_{n+1}^{-1} L_{n+a} A_n$ and therefore, cf. (3),

$$\mathcal{L}_n L_{n+1}^{-1} - L_{n+1}^{-1} \mathcal{L}_{n+1} = (L_n M_n^{-1} - M_{n+1}^{-1} L_{n+a}) A_n = f_n N_n A_n, \tag{22}$$

which is equivalent to a Lax-representation for the OΔE $f_n = 0$, cf. equation (6). Now it immediately follows that the trace of powers of $\mathcal{L}_n$ is an integral of the equation.

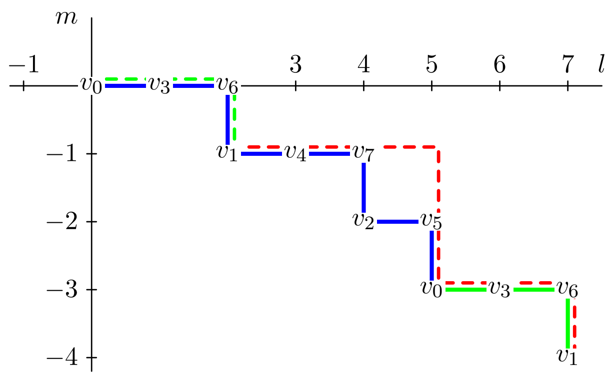

Figure 1. $(5,-3)$-reduction. The product $\mathcal{M}_0\mathcal{L}_0$ corresponds to the solid path from $u_{0,0}$ to $u_{7,-4}$. The product $\mathcal{L}_1\mathcal{M}_0$ corresponds to the dashed path from $u_{2,-1}$ to $u_{7,-4}$. Their difference consists of one elementary square. We have omitted the redundant upper-index $p=0$.

A similar idea applies to general (s_1,s_2)-reductions. The monodromy matrix coincides with the Lax-matrix $\mathcal{L}$, whereas the first part of the staircase, up to the point v^p_{n+1}, will provide the auxiliary Lax-matrix $\mathcal{M}$. Before proving the general result we first work out two simple examples, one with $q=1$ and one with $q>1$:

- The case of $(5,-3)$-reduction. The auxiliary matrix $\mathcal{M}_n$ is given by the following factor of the monodromy matrix: $\mathcal{M}_n = M^{-1}_{n+1}L_{n+3}L_n$. Indeed, as one easily verifies,

$$\mathcal{M}_n\mathcal{L}_n - \mathcal{L}_{n+1}\mathcal{M}_n = M^{-1}_{n+1}L_{n+3}(L_nM^{-1}_n - M^{-1}_{n+3}L_{n+5})M_n\mathcal{L}_n \qquad (23)$$

$$= f_nM^{-1}_{n+1}L_{n+3}N_nM_n\mathcal{L}_n. \qquad (24)$$

- The case of $(4,-2)$-reduction. We have

$$\mathcal{L}^p_n = (M^p_n)^{-1}L^{p+1}_{n+1}L^{p+1}_n(M^{p+1}_n)^{-1}L^p_{n+1}L^p_n, \qquad \mathcal{M}^p_n = L^p_n \qquad (25)$$

and can be verified that

$$\mathcal{M}^p_n\mathcal{L}^p_n - \mathcal{L}^p_{n+1}\mathcal{M}^p_n = f^p_nN^p_nL^{p+1}_{n+1}L^{p+1}_n(M^{p+1}_n)^{-1}L^p_{n+1}L^p_n \qquad (26)$$

$$+ f^{p+1}_n(M^p_{n+1})^{-1}L^{p+1}_{n+2}L^{p+1}_{n+1}N^{p+1}_nL^p_{n+1}L^p_n. \qquad (27)$$

In general, taking $n=p=0$, the product $\mathcal{M}^0_0\mathcal{L}^0_0$ corresponds to a path from $u_{0,0} \rightsquigarrow v^0_0$ via $u_{s_1,s_2} \rightsquigarrow v^0_0$ to $u_{s_1+c,s_2+\epsilon d} = v^0_1$, whereas the product $\mathcal{L}^0_1\mathcal{M}^0_0$ corresponds to a path from $u_{0,0} \rightsquigarrow v^0_0$ to $u_{s_1+c,s_2+\epsilon d} = v^0_1$ via $u_{c,\epsilon d} \rightsquigarrow v^0_1$. The two paths are almost the same, their difference is illustrated for the $(4,-2)$-reduction in Figure 2. Clearly, due to the compatibility condition (3), this difference vanishes for solutions of the system of O∆Es (16).

Now we present our main result, explicit formulae for the Lax matrices of a (s_1,s_2)-travelling wave reduction in terms of the Lax pair of the P∆E from which the integrable map was obtained. The advantage of having a formula for the Lax representation of the mapping, is that one can compute the integrals directly from the reduction, without having to construct first the corresponding staircase. Before stating the theorem, we give the following

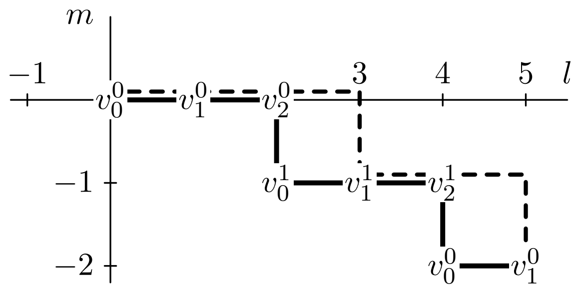

Figure 2. $(4,-2)$-reduction. The product $\mathcal{M}_0^0\mathcal{L}_0^0$ corresponds to the solid path from $u_{0,0}$ to $u_{4,-2}$. The product $\mathcal{L}_1^0\mathcal{M}_0^0$ corresponds to the dashed path from $u_{1,0}$ to $u_{5,-2}$. Their difference consists of $q=2$ elementary squares.

Definition 3 *Given $i, j, x_j, y_j, w_j, b, p \in \mathbb{N}$, with x_j, y_j, w_j defined in Section 4, let $Q_n^{j,i}$ be a matrix such that*

$$Q_n^{j,i} := \Big(\prod_{k=0}^{\overset{\curvearrowleft}{x_{j+1}-x_j-1}} L_{n+y_j+kb}^{p+i+w_j+kd}\Big) \cdot (M_{n+y_j}^{p+i+w_j})^{-1}, \tag{28}$$

where

$$\prod_{k=\alpha}^{\overset{\curvearrowleft}{\beta}} f_k = f_\beta \cdots f_\alpha. \tag{29}$$

Now we state our main result.

Theorem 4 *The following matrices form Lax pairs for the system (16). The monodromy matrix $\mathcal{L}_n^p$ for the (s_1, s_2)-reduction, with $a, b, c, d, q, x_i, y_i, w_i$ defined in sections 3 and 4, is given by*

$$\mathcal{L}_n^p = (M_n^p)^{-1} \cdot \Big(\prod_{i=1}^{\overset{\curvearrowleft}{q}} \prod_{j=0}^{\overset{\curvearrowleft}{b-1}} Q_n^{j,i}\Big) \cdot M_n^p \tag{30}$$

and the auxiliary matrix is $\mathcal{M}_n^p = L_n^p$ if $b=1$, and

$$\mathcal{M}_n^p = (M_{n+1}^p)^{-1} \cdot \Big(\prod_{j=0}^{\overset{\curvearrowleft}{d-1}} Q^{j,i}\Big) \cdot M_n^p, \tag{31}$$

otherwise.

In order to prove the above theorem, we will need the following two lemmas.

Lemma 1
$$Q^{j,i} = \begin{cases} Q_1^{j+a-c,i+1} & \text{when } 0 < j < c, \\ Q_1^{j-c,i} & \text{when } c \le j < a-1, \end{cases}$$

As a corollary, Theorem (4) implies that, for solutions of the system (8), the trace of the monodromy matrix $\mathrm{Tr}(\mathcal{L}_n^p)$ is invariant under shifts in n. But clearly, from the explicit formula, this trace is also invariant under shifts in p. Therefore, $\mathrm{Tr}(\mathcal{L}_n^p)$ is in fact an integral of any mapping which corresponds to a shift on the lattice.

6. A new four-dimensional integrable map

In this section, we provide the integrals for the (4,-2)-reduction of the sine-Gordon P∆E, which is a 6-dimensional mapping that can be reduced to a 4-dimensional mapping. The two functionally independent integrals survive this reduction and we show they are in involution with respect to a symplectic structure, from what follows that the mapping is LAV-completely integrable.

Consider the sine-Gordon equation defined on the $\mathbb{Z}^2$-lattice:

$$\alpha_1(u_{l,m}u_{l+1,m+1} - u_{l+1,m}u_{l,m+1}) + \alpha_2 u_{l,m}u_{l+1,m}u_{l,m+1}u_{l+1,m+1} - \alpha_3 = 0. \tag{32}$$

Solving the above equation for $u_{l+1,m+1}$ yields

$$u_{l+1,m+1} = g(u_{l,m}, u_{l+1,m}, u_{l,m+1}) = \frac{1}{u_{l,m}}\frac{\alpha_1 u_{l+1,m}u_{l,m+1} + \alpha_3}{\alpha_1 + \alpha_2 u_{l+1,m}u_{l,m+1}}. \tag{33}$$

Equation (32) arises as the compatibility condition $L_{l,m}M^{-1}_{l,m} - M^{-1}_{l+1,m}L_{l,m+1} = f_{l,m}N_{l,m}$ of the Lax representation given by the matrices

$$M^{-1}_{l,m} = \begin{pmatrix} \alpha_1\frac{u_{l,m}}{u_{l,m+1}} & -\alpha_3 k^{-2}\frac{1}{u_{l,m+1}} \\ -\alpha_2 u_{l,m} & \alpha_1 \end{pmatrix}, \; L_{l,m} = \begin{pmatrix} 1 & -u_{l+1,m} \\ -k^2\frac{1}{u_{l,m}} & \frac{u_{l+1,m}}{u_{l,m}} \end{pmatrix}. \tag{34}$$

and

$$N_{l,m} = \begin{pmatrix} \frac{1}{u_{l,m+1}u_{l+1,m+1}} & 0 \\ 0 & -\frac{1}{u_{l,m}u_{l,m+1}} \end{pmatrix}. \tag{35}$$

Performing a $(4,-2)$-travelling wave reduction, as explained in section 3, a system of 2 O∆Es of order 3 can be obtained from equation (32). The periodicity condition $u_{l,m} = u_{l+4,m-2}$ on the initial value problem allows for the coupled system to be written as a six-dimensional map. We have that $q = 2, a = 2, b = 1$. Thus, the smallest integers c, d such that $2d - c = 1$ are $c = -1, d = 0$. The travelling wave reduction is $u_{l,m} \mapsto v^p_n$, where $n = l + 2m$ and $p \equiv -m \bmod 2$. In this way, the mapping

$$\Delta_0 : (v^0_0, v^0_1, v^0_2, v^1_0, v^1_1, v^1_2) \mapsto (v^0_0, v^0_1, v^0_2, v^1_0, v^1_1, v^1_2)' \tag{36}$$

is given by

$$v^{0'}_0 = v^0_1 \tag{37}$$
$$v^{0'}_1 = v^0_2 \tag{38}$$
$$v^{0'}_2 = g(v^1_0, v^1_1, v^0_2) \tag{39}$$
$$v^{1'}_0 = v^1_1 \tag{40}$$
$$v^{1'}_1 = v^1_2 \tag{41}$$
$$v^{1'}_2 = g(v^0_0, v^0_1, v^1_2) \tag{42}$$

where

$$g(v^p_n, v^{p+c}_{n+a}, v^{p+d}_{n+b}) = \frac{1}{v^p_n}\frac{\alpha_1 v^{p+c}_{n+a}v^{p+d}_{n+b} + \alpha_3}{\alpha_2 v^{p+c}_{n+a}v^{p+d}_{n+b} + \alpha_1} \tag{43}$$

A convenient relabeling of the variables $v_n^p = v_{n+(a+b)p}$ allows for the map to be written

$$\Delta_0 : (v_0, v_1, v_2, v_3, v_4, v_5) \mapsto (v_0, v_1, v_2, v_3, v_4, v_5)' \tag{44}$$

where

$$v_0' = v_1 \tag{45}$$
$$v_1' = v_2 \tag{46}$$
$$v_2' = g(v_3, v_4, v_2) \tag{47}$$
$$v_3' = v_4 \tag{48}$$
$$v_4' = v_5 \tag{49}$$
$$v_5' = g(v_0, v_1, v_5) \tag{50}$$

Right-multiplying (26) at $n = p = 0$ by $-\mathcal{M}_0^0$ and taking the trace leads to

$$\mathrm{Tr}(\mathcal{L}_1^0) - \mathrm{Tr}(\mathcal{L}_0^0) = f_0^0 \Lambda_0 + f_0^1 \Lambda_1 \tag{51}$$

where the Λ_i integrating factors are

$$\Lambda_0 := -\mathrm{Tr}(L_0 N_0 L_4 L_3 M_3^{-1} L_1 L_0) \tag{52}$$
$$\Lambda_1 := -\mathrm{Tr}(L_0 M_1^{-1} L_5 L_4 N_3 L_1 L_0). \tag{53}$$

The integrals of the map (45) are the coefficients of the Laurent expansion of the trace of the monodromy matrix

$$\mathrm{Tr}\mathcal{L}_0 = \mathrm{Tr}\left(M_0^{-1} L_4 L_3 M_3^{-1} L_1 L_0\right) = 2\alpha_2\alpha_3 k^{-2} + I + J k^2 + 2\alpha_1^2 \tag{54}$$

where, for $\mathbf{v} = (v_0, v_1, v_2, v_3, v_4, v_5)$, the two independent integrals $I := I(\mathbf{v})$ and $J := J(\mathbf{v})$ are in involution $\{I(\mathbf{v}), J(\mathbf{v})\} = 0$ with respect to the symplectic matrix

$$\Omega(\mathbf{v}) := \begin{pmatrix} 0 & v_0v_1 & 0 & v_0v_3 & 0 & v_0v_5 \\ -v_0v_1 & 0 & v_1v_2 & 0 & v_1v_4 & 0 \\ 0 & -v_1v_2 & 0 & v_2v_3 & 0 & v_2v_5 \\ -v_0v_3 & 0 & -v_2v_3 & 0 & v_3v_4 & 0 \\ 0 & -v_1v_4 & 0 & -v_3v_4 & 0 & v_4v_5 \\ -v_0v_5 & 0 & -v_2v_5 & 0 & -v_4v_5 & 0 \end{pmatrix} \tag{55}$$

As we saw in section **??**, one may perform different travelling wave reductions leading to maps that are conjugate to the translated map Δ_0. A different set of solutions to the equation $d - 2c = 1$ is $(c_1, d_1) := (c, d) + (1, 2) = (1, 3)$ that corresponds to the mapping $u_{l,m} \mapsto u_{l+3,m-1}$ on the lattice. We denote by $v_n^{p_1}$ the variables under this new reduction and therefore $n = l + 2m$ and $p_1 \equiv l + 3m \bmod 2$. A mapping obtained from this reduction is $\Delta_1 : (v_0^0, v_1^0, v_2^0, v_0^1, v_1^1, v_2^1) \mapsto (v_0^0, v_1^0, v_2^0, v_0^1, v_1^1, v_2^1)'$ given by

$$v_0^{0'} = v_1^0 \tag{56}$$
$$v_1^{0'} = v_2^0 \tag{57}$$
$$v_2^{0'} = g(v_0^0, v_1^1, v_2^1) \tag{58}$$
$$v_0^{1'} = v_1^1 \tag{59}$$
$$v_1^{1'} = v_2^1 \tag{60}$$
$$v_2^{1'} = g(v_0^1, v_1^0, v_2^0) \tag{61}$$

To show the relationship between the maps Δ_1 and Δ_0 we start by defining some transformations. Let $u_{l,m} \in U$, $(v_0^p, v_1^p, \ldots, v_5^p) \in V_0$ and $(v_0^{p_1}, v_1^{p_1}, \ldots, v_5^{p_1}) \in V_1$ where $U, V_0, V_1 \subset \mathbb{R}^6$, and we have set $v_n^{p_0} = v_n^p$. Thus, let $\delta_0 : u_{l,m} \mapsto u_{l+1,m}$, $\delta_1 : u_{l,m} \mapsto u_{l+3,m-1}$ and $\sigma : u_{l,m} \mapsto u_{l+2,m-1}$. Furthermore, let $S : (v_0^0, v_1^0, v_2^0, v_0^1, v_1^1, v_2^1) \mapsto (v_0^1, v_1^1, v_2^1, v_0^0, v_1^0, v_2^0)$ and $T : (v_0^0, v_1^0, v_2^0, v_0^1, v_1^1, v_2^1) \mapsto (v_0^0, v_1^1, v_2^0, v_0^1, v_1^0, v_2^1)$. Therefore, system (56) can be obtained from (45) from the relationship $\Delta_1 = T_1^{-1} \circ S_0 \circ \Delta_0 \circ T_1$ since

$$\begin{pmatrix} v_0^0 \\ v_1^0 \\ v_2^0 \\ v_0^1 \\ v_1^1 \\ v_2^1 \end{pmatrix} T_1 \begin{pmatrix} v_0^0 \\ v_1^1 \\ v_2^0 \\ v_0^1 \\ v_1^0 \\ v_2^1 \end{pmatrix} \Delta_0 \begin{pmatrix} v_1^1 \\ v_2^0 \\ g(v_0^1, v_1^0, v_2^0) \\ v_1^0 \\ v_2^1 \\ g(v_0^0, v_1^1, v_2^1) \end{pmatrix} S_0 \begin{pmatrix} v_1^0 \\ v_2^1 \\ g(v_0^0, v_1^1, v_2^1) \\ v_1^1 \\ v_2^0 \\ g(v_0^1, v_1^0, v_2^0) \end{pmatrix} T_1^{-1} \begin{pmatrix} v_1^0 \\ v_2^0 \\ g(v_0^0, v_1^1, v_2^1) \\ v_1^1 \\ v_2^1 \\ g(v_0^1, v_1^0, v_2^0) \end{pmatrix}$$

The integrals of (56) can be obtained from the integrals of (45) performing the transformation $I_1 = I \circ T$ and $J_1 = J \circ T$.

In any case, we have two independent integrals in involution for a six dimensional map, which is not enough to prove integrability of the map in the Liouville-Arnold sense. However, it is possible to reduce the dimension of the map by two, performing the following change of variable. Let $w_0 = v_0 v_4, w_1 = v_1 v_5, w_2 = v_1 v_3, w_3 = v_2 v_4$. Then the system (45) is brought to the four dimensional map

$\tau : (w_0, w_1, w_2, w_3) \mapsto (w_0', w_1', w_2', w_3')$ where

$$w_0' = w_1 \tag{62}$$
$$w_1' = \frac{w_3}{w_0} \frac{\alpha_1 w_1 + \alpha_3}{\alpha_2 w_1 + \alpha_1} \tag{63}$$
$$w_2' = w_3 \tag{64}$$
$$w_3' = \frac{w_1}{w_2} \frac{\alpha_1 w_3 + \alpha_3}{\alpha_2 w_3 + \alpha_1} \tag{65}$$

and the independent integrals are expressed in terms of the new variables as

$$\begin{aligned}
\mathrm{I(w)} := & \left(\tfrac{w_0 w_2}{w_1 w_3} + \tfrac{w_1 w_3}{w_0 w_2}\right) \alpha_1^2 \\
+ & \left(w_0 + w_1 + w_2 + w_3 + \tfrac{w_1 w_3}{w_0} + \tfrac{w_0 w_2}{w_1} + \tfrac{w_1 w_3}{w_2} + \tfrac{w_0 w_2}{w_3}\right) \alpha_1 \alpha_2 \\
+ & \left(\tfrac{1}{w_0} + \tfrac{1}{w_1} + \tfrac{1}{w_2} + \tfrac{1}{w_3} + \tfrac{w_0}{w_1 w_3} + \tfrac{w_1}{w_0 w_2} + \tfrac{w_2}{w_1 w_3} + \tfrac{w_3}{w_0 w_2}\right) \alpha_1 \alpha_3 \\
+ & \left(\tfrac{w_0}{w_3} + \tfrac{w_3}{w_0} + \tfrac{w_2}{w_1} + \tfrac{w_1}{w_2}\right) \alpha_2 \alpha_3 \\
+ & \left(w_0 w_2 + w_0 w_1 + w_1 w_3 + w_2 w_3\right) \alpha_2^2 \\
+ & \left(\tfrac{1}{w_0 w_2} + \tfrac{1}{w_0 w_1} + \tfrac{1}{w_2 w_3} + \tfrac{1}{w_1 w_3}\right) \alpha_3^2
\end{aligned}$$

and

J(w):= $\left(\frac{w_0}{w_1}+\frac{w_1}{w_0}+\frac{w_2}{w_3}+\frac{w_3}{w_2}+\frac{w_0}{w_2}+\frac{w_2}{w_0}+\frac{w_1}{w_2}+\frac{w_2}{w_1}+\frac{w_1}{w_3}+\frac{w_3}{w_1}+\frac{w_0}{w_3}+\frac{w_3}{w_0}\right)\alpha_1^2$
$+\left(w_0+w_1+w_2+w_3+\frac{w_2w_3}{w_0}+\frac{w_0w_1}{w_3}+\frac{w_0w_1}{w_2}+\frac{w_2w_3}{w_1}\right)\alpha_1\alpha_2$
$+\left(\frac{1}{w_0}+\frac{1}{w_1}+\frac{1}{w_2}+\frac{1}{w_3}+\frac{w_0}{w_2w_3}+\frac{w_3}{w_0w_1}+\frac{w_2}{w_0w_1}+\frac{w_1}{w_2w_3}\right)\alpha_1\alpha_3$
$+\left(\frac{w_2w_3}{w_0w_1}+\frac{w_0w_1}{w_2w_3}\right)\alpha_2\alpha_3.$
where $\mathbf{w}=(w_0,w_1,w_2,w_3)$. From the Poisson bracket $\{v_i,v_j\}=\Omega_{ij}$ given in (55) between the v_i-variables we obtain the symplectic structure of the map (62), leading to

$$\Omega(\mathbf{w})=\begin{pmatrix} 0 & 2w_0w_1 & 0 & 0 \\ -2w_0w_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2w_2w_3 \\ 0 & 0 & -2w_2w_3 & 0 \end{pmatrix}. \tag{66}$$

System (62) is a symplectomorphism since $(d\tau(\mathbf{w}))\Omega(\mathbf{w})(d\tau(\mathbf{w}))^T=\Omega(\mathbf{w}')$. It is measure-preserving since the determinant of the Jacobian matrix is det $(d\tau(\mathbf{w}))=\mu(\mathbf{w}')/\mu(\mathbf{w})$ with measure $\mu(\mathbf{w})=w_0w_1w_2w_3$. The mapping (62) is reversible since it can be written as the composition $\tau=i_1\circ i_2$ of the involutions

$$i_1:(w_0,w_1,w_2,w_3)\mapsto\left(w_2,\frac{w_0}{w_3}\frac{\alpha_1w_2+\alpha_3}{\alpha_2w_2+\alpha_1},w_3,\frac{w_2}{w_1}\frac{\alpha_1w_0+\alpha_3}{\alpha_2w_0+\alpha_1}\right) \tag{67}$$

$$i_2:(w_0,w_1,w_2,w_3)\mapsto(w_3,w_2,w_1,w_0). \tag{68}$$

The integrals of the map are in involution $\{I(\mathbf{w}),J(\mathbf{w})\}=(dI(\mathbf{w}))\Omega(\mathbf{w})(dJ(\mathbf{w}))^T=0$ with respect to (66). Therefore, we conclude that the four dimensional map (62) is integrable in the Liouville-Arnold sense.

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