J. Phys. A: Math. Theor. **44** (2011) 295206 (13pp) doi:10.1088/1751-8113/44/29/295206

Involutivity of integrals of sine-Gordon, modified KdV and potential KdV maps

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Received 2 December 2010, in final form 19 May 2011
Published 21 June 2011
Online at stacks.iop.org/JPhysA/44/295206

Abstract
Closed form expressions in terms of multi-sums of products have been given in Tran *et al* (2009 *J. Phys A: Math. Theor.* **42** 225201) and van der Kamp *et al* (2007 *J. Phys. A: Math. Theor.* **39** 12789–98) of integrals of sine-Gordon, modified Korteweg–de Vries and potential Korteweg–de Vries maps obtained as so-called $(p, -1)$-travelling wave reductions of the corresponding partial difference equations. We prove the involutivity of these integrals with respect to recently found symplectic structures for those maps. The proof is based on explicit formulae for the Poisson brackets between multi-sums of products.

PACS number: 02.30.Ik

1. Introduction

Integrable systems boast a long and venerable history. The history dates back to the 17th century with the work of Newton on the two-body problem. The notion of integrability was first introduced by Liouville in the 19th century in the context of finite-dimensional continuous Hamiltonian systems. Since then, it has been expanded to classes of nonlinear (partial) differential equations, see for example [4, 5]. More recently, there has been a shift of interest into systems with discrete time, e.g. integrable ordinary difference equations (or maps) and integrable partial difference (or lattice) equations. Some of the first examples of discrete integrable systems appeared in [6, 11]. And a classification of integrable lattice equations defined on a elementary square of the lattice has recently been obtained [1], based on the notion of multi-dimensional consistency. For maps, there is the notion of complete or Liouville–Arnold integrability [2, 8, 18], analogous to the same notion for continuous systems. Briefly speaking, a mapping is said to be completely integrable if it has a sufficient number of functionally independent integrals that are in involution, that is, they Poison commute.

In this paper, we study the involutivity of integrals of a certain class of integrable maps related to the fully discrete sine-Gordon, modified Korteweg–de Vries (mKdV) and potential

Korteweg–de Vries (pKdV) equations. These maps arise as travelling wave reductions from the corresponding lattice equations. Such maps typically come in an infinite family of increasing dimension, and for this reason it is not feasible to calculate Poisson brackets one by one and show that they all vanish. One way to circumvent this problem is to use the so-called Yang–Baxter structure, and that is the approach taken in [3, 9]. This approach was used to prove the involutivity of integrals for the so-called $(p, -p)$-reduction of the lattice pKdV equation. We refer to [10, 16] for the background on (p, q)-travelling wave reductions. In this paper, we study $(p, -1)$-reductions and we take a different approach. Starting from recently found symplectic structures [7, 12] and recently obtained closed-form expressions in terms of multi-sums of products for integrals of our family of sine-Gordon, mKdV and pKdV maps [14, 17], we proceed to prove involutivity of the integrals directly, using explicit formulae for the Poisson brackets between multi-sums of products. These formulae will be proven by induction on the number of variables, that is, on the dimension of the maps.

Recall, cf [2, 7, 18], that a $2n$-dimensional discrete map $L : x \mapsto x'$ is said to be completely integrable if

- there is a $2n \times 2n$ anti-symmetric non-degenerate matrix Ω satisfying the Jacobi identity

$$\sum_l \left(\Omega_{li} \frac{\partial}{\partial x_l} \Omega_{jk} + \Omega_{lj} \frac{\partial}{\partial x_l} \Omega_{ki} + \Omega_{lk} \frac{\partial}{\partial x_l} \Omega_{ij} \right) = 0,$$

  such that $\mathrm{d}L(x)\Omega(x)\mathrm{d}L^T(x) = \Omega(x')$, where $\mathrm{d}L$ is the Jacobian of the map, $\mathrm{d}L_{ij} := \frac{\partial x'_i}{\partial x_j}$.
- There exist n functionally independent integrals $I_1, I_2, \ldots, I_n$ satisfying $\{I_r, I_s\}_x = 0$ for all $1 \leqslant r, s \leqslant n$, where the Poisson bracket is defined by

$$\{f, g\}_x = \nabla_x(f).\Omega.(\nabla_x(g))^T, \tag{1}$$

  with $\nabla_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{2n}}\right)$. Note that we will encounter several (related) Poisson brackets which are distinguished by the label x denoting the coordinates in which the bracket is expressed. Also ∇_x will always have the right number of components.

The families of ordinary difference sine-Gordon, mKdV and pKdV equations are given as follows [14, 17]:

$$\text{sine-Gordon}: \ \alpha_1(v_n v_{n+p+1} - v_{n+1} v_{n+p}) + \alpha_2 v_n v_{n+1} v_{n+p} v_{n+p+1} - \alpha_3 = 0, \tag{2}$$

$$\text{modified KdV}: \ \beta_1(v_n v_{n+p} - v_{n+1} v_{n+p+1}) + \beta_2 v_n v_{n+1} - \beta_3 v_{n+p} v_{n+p+1} = 0, \tag{3}$$

$$\text{potential KdV}: \ (v_n - v_{n+p+1})(v_{n+1} - v_{n+p}) - \gamma = 0. \tag{4}$$

These equations are obtained from the $(p, -1)$-travelling wave reductions of the corresponding partial difference equations of the form

$$f(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+1,m+1}) = 0, \tag{5}$$

where we have taken $v_n = u_{l,m}$ with $n = l + mp$, introducing the periodicity $u_{l,m} = u_{l+p,m-1}$, cf [10, 16].

The corresponding $(d = p + 1)$-dimensional maps derived from equations (2), (3) and (4) are $\mathbb{R}^d \to \mathbb{R}^d$:

$$(v_1, v_2, \ldots, v_d) \mapsto (v_2, v_3, \ldots, v_{d+1}), \tag{6}$$

where

$$v_{d+1} = v_1^{-1} \frac{\alpha_1 v_2 v_d + \alpha_3}{\alpha_2 v_2 v_d + \alpha_1}, \qquad v_{d+1} = v_1 \frac{\beta_1 v_d + \beta_2 v_2}{\beta_1 v_2 + \beta_3 v_d}, \qquad v_{d+1} = v_1 - \frac{\gamma}{v_2 - v_d},$$

respectively. The integrals of sine-Gordon and mKdV maps can be expressed in terms of multi-sums of products, which we call Θ:

$$\Theta^{a,b}_{r,\epsilon}(f_a, f_{a+1}, \ldots, f_b) := \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b} \prod_{j=1}^{r} (f_{i_j})^{(-1)^{j+\epsilon}}, \tag{7}$$

with $f_i = v_i v_{i+1}$. In [17], it was shown that $\lfloor d/2\rfloor$ integrals of the sine-Gordon map are given by

$$I^{\mathrm{sG}}_r = \alpha_1\left(\frac{v_d}{v_1}\Theta^{1,d-1}_{2r,1} + \frac{v_1}{v_d}\Theta^{1,d-1}_{2r,0}\right) + \alpha_2\Theta^{1,d-1}_{2r+1,1} + \alpha_3\Theta^{1,d-1}_{2r+1,0}, \quad 0 \leqslant 2r < d-1 \tag{8}$$

and $\lfloor (d-1)/2\rfloor$ integrals of the mKdV map are given by

$$I^{\mathrm{mKdV}}_r = \beta_1\left(v_1 v_d \Theta^{1,d-1}_{2r-1,0} + \frac{1}{v_1 v_d}\Theta^{1,d-1}_{2r-1,1}\right) + \beta_2\Theta^{1,d-1}_{2r,1} + \beta_3\Theta^{1,d-1}_{2r,0}, \qquad 0 < 2r < d. \tag{9}$$

In [14], it was shown that $\lfloor (d-1)/2\rfloor$ integrals of the pKdV map are given by

$$\begin{aligned} I^{\mathrm{pKdV}}_r = {} & \Psi^{2,d-2}_{r-1} + (v_d - v_2)\Psi^{2,d-3}_{r-1} + (v_{d-1} - v_1)\Psi^{3,d-2}_{r-1} \\ & + \Psi^{3,d-3}_{r-2} + ((v_{d-1} - v_1)(v_d - v_2) - \gamma)\Psi^{2,d-2}_{r}, \end{aligned} \tag{10}$$

where $0 \leqslant r < \lfloor (d-1)/2\rfloor$ and

$$\Psi^{a,b}_r(c_a, c_{a+1}, \ldots, c_{b+1}) = \left(\sum_{a\leqslant i_1, i_1+1<i_2, i_2+1,\ldots,<i_r\leqslant b} \prod_{j=1}^{r} \frac{1}{c_{i_j} c_{i_j+1}}\right) \prod_{i=a}^{b+1} c_i, \tag{11}$$

with $c_i = v_{i-1} - v_{i+1}$. In this paper, we will prove that the integrals (8), (9) and (10) are in involution with respect to accompanying symplectic structures.

The paper is organized as follows. In section 2, we prove the involutivity of integrals of the sine-Gordon maps. Firstly, we consider the odd-dimensional maps. We introduce a transformation to reduce the dimension of the map by 1 and we present a symplectic structure of the reduced map. Then, we present properties of Θ with respect to the Poisson bracket associated with this symplectic structure. To prove the involutivity of the integrals, we write the Poisson bracket $\{I_r, I_s\}$ as a polynomial in $\alpha_1, \alpha_2, \alpha_3$. The coefficients of this polynomial all vanish. Secondly, we consider the even-dimensional map. We provide a symplectic structure for it and show that it relates to the symplectic structure for the odd-dimensional map. Therefore, many properties of Θ with respect to the new Poisson bracket can be obtained directly from the ones with respect to the old Poisson bracket. The proof of involutivity is similar to the first case.

In section 3, we present relationships between symplectic structures of the sine-Gordon and mKdV maps. We use these relationships to derive analogous properties of Θ with respect to the Poisson bracket of the mKdV maps. Involutivity of the integrals of the mKdV follows from these properties.

In section 4, we prove that the integrals of the pKdV map are in involution (with respect to the appropriate symplectic structures). We again distinguish even- and odd-dimensional maps and present a relationship of symplectic structures between the two cases. For the even-dimensional map, we present the properties of multi-sums of products, Ψ, which are proved by induction. These properties are used to prove the involutivity of the integrals. For the other case, the properties of Ψ with respect to its symplectic structure are derived from the previous case. The involutivity of integrals (10) is proved by using these properties.

In section 5, we discuss results, obtained in [15], on the functional independence of the sets of integrals (8), (9), (10), and conclude the integrability of the difference equations (2), (3), (4) for any value of the order d.

2. Involutivity of sine-Gordon integrals

In this section, we distinguish two cases: the odd-dimensional and even-dimensional sine-Gordon maps. In [17], it is shown that for the even-dimensional map, we have enough integrals for integrability. For the odd-dimensional map, we need to reduce the dimension of the map by 1. We expand the Poisson bracket between two integrals $\{I_r, I_s\}$ as a quadratic polynomial in the parameters $\alpha_1, \alpha_2, \alpha_3$ and prove the involutivity of integrals (8) by showing its coefficients vanish.

2.1. The case $d = 2n + 1$

Using a reduction $f_i = v_i v_{i+1}$, we obtain a $2n$-dimensional map

$$\mathrm{sG} : (f_1, f_2, \ldots, f_{2n}) \mapsto \left(f_2, f_3, \ldots, f_{2n}, \frac{f_2 f_4 \ldots f_{2n}(\alpha_1 f_2 f_4 \ldots f_{2n} + \alpha_3 f_3 f_5 \ldots f_{2n-1})}{f_1 f_3 \ldots f_{2n-1}(\alpha_2 f_2 f_4 \ldots f_{2n} + \alpha_1 f_3 f_5 \ldots f_{2n-1})} \right). \tag{12}$$

This map has n integrals given by

$$I_r^{\mathrm{sG}} = \alpha_1 \left(\frac{f_2 f_4 \ldots f_{2n}}{f_1 f_3 \ldots f_{2n-1}} \Theta_{2r,1}^{1,2n} + \frac{f_1 f_3 \ldots f_{2n-1}}{f_2 f_4 \ldots f_{2n}} \Theta_{2r,0}^{1,2n} \right) + \alpha_2 \Theta_{2r+1,1}^{1,2n} + \alpha_3 \Theta_{2r+1,0}^{1,2n}, \tag{13}$$

where the argument of Θ is f_i and $0 \leqslant r \leqslant n-1$.

A symplectic structure for the map (12) is given by $\Omega_{2n}^{\mathrm{sG}}$, where

$$(\Omega_p^{\mathrm{sG}})_{ij} = \begin{cases} f_i f_j & i < j, \\ 0 & i = j, \\ -f_i f_j & i > j, \end{cases} \tag{14}$$

cf [7, 12]. One can verify that $\mathrm{dsG} \cdot \Omega_{2n}^{\mathrm{sG}} \cdot \mathrm{dsG}^T = \Omega_{2n}^{\mathrm{sG}} \circ \mathrm{sG}$. Let g and h be functions differentiable with respect to the f_i's. The symplectic structure Ω_p^{sG} defines the following Poisson bracket:

$$\begin{aligned} \{g, h\}_f &= \nabla_f(g) \cdot \Omega_p^{\mathrm{sG}} \cdot (\nabla_f(h))^T \\ &= \sum_{i<j} f_i f_j \left(\frac{\partial g}{\partial f_i} \frac{\partial h}{\partial f_j} - \frac{\partial g}{\partial f_j} \frac{\partial h}{\partial f_i} \right). \end{aligned} \tag{15}$$

We will prove that integrals (13) are in involution with respect to the symplectic structure $\Omega_{2n}^{\mathrm{sG}}$, i.e $\{I_r^{\mathrm{sG}}, I_s^{\mathrm{sG}}\}_f = 0$ for all $0 \leqslant r, s \leqslant n-1$. The proof is based on the following explicit expressions for the Poisson bracket between Θ multi-sums.

Lemma 1. *Let $1 \leqslant r, s \leqslant p$ and $\epsilon \in \{0, 1\}$. We have*

$$\{\Theta_{r,\epsilon}^{1,p}, \Theta_{s,\epsilon}^{1,p}\}_f = \begin{cases} 0 & r, s \text{ are both odd or both even}, \\ \sum_{i \geqslant 0} (-1)^i \Theta_{r+i,\epsilon}^{1,p} \Theta_{s-i,\epsilon}^{1,p} & r \text{ even},\ s \text{ odd and } r > s, \\ \sum_{i \geqslant 1} (-1)^{i-1} \Theta_{r-i,\epsilon}^{1,p} \Theta_{s+i,\epsilon}^{1,p} & r \text{ even},\ s \text{ odd and } r < s. \end{cases} \tag{16}$$

Note that the right-hand side of (16) is a finite sum.

Proof. Due to $\Theta_{r,\epsilon}^{a,b}(f_a, f_{a+1}, \ldots, f_b) = \Theta_{r,\epsilon+1}^{a,b}\left(f_a^{-1}, f_{a+1}^{-1}, \ldots, f_b^{-1}\right)$ it suffices to prove the lemma taking $\epsilon = 1$. The proof is done by induction using the recurrence relations given in [14, 17]:

$$\Theta_{r,\epsilon}^{a,b} = \Theta_{r,\epsilon}^{a,b-1} + f_b^{(-1)^{\epsilon+r}} \Theta_{r-1,\epsilon}^{a,b-1}, \tag{17}$$

$$\Theta^{a,b}_{r,\epsilon} = \Theta^{a+1,b}_{r,\epsilon} + f_a^{(-1)^{\epsilon\pm1}} \Theta^{a+1,b}_{r-1,\epsilon\pm1}, \tag{18}$$

and the bracket

$$\left\{\Theta^{1,p}_{r,\epsilon}, f_{p+1}^{(-1)^\delta}\right\}_f = \begin{cases} 0 & r \text{ even}, \\ (-1)^{\delta+\epsilon+1} f_{p+1}^{(-1)^\delta} \Theta^{1,p}_{r,\epsilon} & r \text{ odd}. \end{cases}$$

□

The next proposition provides the Poisson bracket between two Θ multi-sums with different values of ϵ.

Lemma 2. *Let* $1 \leqslant r, s \leqslant p$.

(1) If $r \equiv s \pmod 2$*, we have*

$$\left\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\right\}_f = \begin{cases} \sum_{i\geqslant0} (-1)^i \Theta^{1,p}_{r-1-2\lfloor i/2\rfloor,i} \Theta^{1,p}_{s+1+2\lfloor i/2\rfloor,i+1} & r \leqslant s, \\ \sum_{i\geqslant0} (-1)^i \Theta^{1,p}_{s-1-2\lfloor i/2\rfloor,i} \Theta^{1,p}_{r+1+2\lfloor i/2\rfloor,i+1} & r > s. \end{cases} \tag{19}$$

(2) If $r \not\equiv s \pmod 2$*, we have*

$$\left\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\right\}_f = \begin{cases} \sum_{i\geqslant0} (-1)^i \Theta^{1,p}_{s+i,i+1} \Theta^{0,p}_{r-i,i} & r \text{ odd}, \ s \text{ even}, \\ \sum_{i\geqslant0} (-1)^{i-1} \Theta^{1,p}_{s-i,i+1} \Theta^{1,p}_{r+i,i} & r \text{ even}, \ s \text{ odd}. \end{cases} \tag{20}$$

This lemma is also proven by induction. For the details of this, and of other proofs, we refer to [13].

Using lemma 1 and lemma 2, we have the following corollary.

Corollary 3. *Let r and s be both even or both odd and let* $\epsilon \in \{0, 1\}$. *Then,*

$$\left\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\right\}_f + \left\{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,0}\right\}_f = 0, \tag{21}$$

$$\left\{\Theta^{1,p}_{r-1,\epsilon}, \Theta^{1,p}_{s,\epsilon}\right\}_f + \left\{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon}\right\}_f = \begin{cases} 0, & r, s \text{ even}, \\ \Theta^{1,p}_{r-1,\epsilon}\Theta^{1,p}_{s,\epsilon} - \Theta^{1,p}_{s-1,\epsilon}\Theta^{1,p}_{r,\epsilon}, & r, s \text{ odd}, \end{cases} \tag{22}$$

$$\left\{\Theta^{1,p}_{r-1,\epsilon\pm1}, \Theta^{1,p}_{s,\epsilon}\right\}_f + \left\{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon\pm1}\right\}_f = \begin{cases} 0, & r, s \text{ even}, \\ \Theta^{1,p}_{s-1,\epsilon\pm1}\Theta^{1,p}_{r,\epsilon} - \Theta^{1,p}_{s,\epsilon}\Theta^{1,p}_{r-1,\epsilon\pm1}, & r, s \text{ odd}. \end{cases} \tag{23}$$

Theorem 4. *Let* $0 \leqslant r, s \leqslant n-1$. *Let* $I^{\mathrm{sG}}_r, I^{\mathrm{sG}}_s$ *be given by formula (13). Then*

$$\left\{I^{\mathrm{sG}}_r, I^{\mathrm{sG}}_s\right\}_f = 0.$$

Proof. First of all, we denote

$$F = \frac{f_1 f_3 \dots f_{2n-1}}{f_2 f_4 \dots f_{2n}}.$$

For any $g(f_1, f_2, \dots, f_{2n})$, we find $\{F^{\pm1}, g\}_f = \pm F^{\pm1} E_f g$, where

$$E_f = \sum_{i\geqslant1} f_i \frac{\partial}{\partial f_i}, \tag{24}$$

which scales any homogeneous expression by its total degree. Every term in the multi-sum has total degree 0 if r is even and degree $(-1)^{\epsilon+1}$ if r is odd; hence,

$$\left\{F^{\pm 1}, \Theta^{1,p}_{r,\epsilon}\right\}_f = \begin{cases} 0 & \text{if } r \text{ even,} \\ \mp(-1)^{\epsilon} F^{\pm 1}\Theta^{1,p}_{r,\epsilon} & \text{if } r \text{ odd.} \end{cases} \tag{25}$$

Now we expand $\left\{I^{\mathrm{sG}}_r, I^{\mathrm{sG}}_s\right\}_f$ in terms of polynomials in $\alpha_1, \alpha_2, \alpha_3$ as follows:

$$\left\{I^{\mathrm{sG}}_r, I^{\mathrm{sG}}_s\right\}_f = \alpha_1^2 A_1 + \alpha_2^2 A_2 + \alpha_3^2 A_3 + \alpha_1\alpha_2 A_{12} + \alpha_1\alpha_3 A_{13} + \alpha_2\alpha_3 A_{23},$$

where e.g. $A_1 = \{F^{-1}\Theta^{1,2n}_{2r,1} + F\Theta^{1,2n}_{2r,0}, F^{-1}\Theta^{1,2n}_{2s,1} + F\Theta^{1,2n}_{2s,0}\}_f$. The coefficients A_i are expanded and can be shown to vanish using lemma 1, corollary 3, and equations (16), (21), (22), (23), and (25). □

2.2. The case $d = 2n$

In this section, we consider a $2n$-dimensional map

$$\widetilde{\mathrm{sG}} : (v_1, v_2, \cdots, v_{2n}) \mapsto \left(v_2, v_3, \ldots, v_{2n}, v_1^{-1}\frac{\alpha_1 v_2 v_{2n} + \alpha_3}{\alpha_2 v_2 v_{2n} + \alpha_1}\right). \tag{26}$$

This map has n integrals given by

$$I^{\widetilde{\mathrm{sG}}}_r = \alpha_1\left(\frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2r,1} + \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2r,0}\right) + \alpha_2\Theta^{1,2n-1}_{2r+1,1} + \alpha_3\Theta^{1,2n-1}_{2r+1,0}, \tag{27}$$

where $0 \leqslant r \leqslant n-1$ and $f_i = v_i v_{i+1}$ in the argument of Θ (7). The sine-Gordon map (26) has a symplectic structure $\Omega^{\widetilde{\mathrm{sG}}}_{2n}$, where

$$\left(\Omega^{\widetilde{\mathrm{sG}}}_p\right)_{ij} = \begin{cases} v_i v_j & i < j,\ i+j \text{ odd,} \\ 0 & i+j \text{ even,} \\ -v_i v_j & i > j,\ i+j \text{ odd,} \end{cases} \tag{28}$$

cf [7, 12]. The Poisson bracket $\frac{1}{2}\nabla_v(g)\Omega^{\widetilde{\mathrm{sG}}}_p(\nabla_v(h))^T$ is denoted $\{g, h\}_v$. Before we prove that the integrals (27) are in involution with respect to this bracket, we first establish the following Poisson brackets between Θ multi-sums:

$$\left\{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s,\delta}\right\}_v = \left\{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s,\delta}\right\}_f\Big|_{f_i = v_i v_{i+1}}, \tag{29}$$

where the right-hand side is given by lemmas 1 and 2. Equation (29) follows as a corollary from the next lemma. Consider the map $\mathbb{R}^p \to \mathbb{R}^{p-1}$:

$$G_p : (v_1, v_2, \ldots, v_p) \mapsto (v_1 v_2, v_2 v_3, \ldots, v_{p-1} v_p).$$

Lemma 5. *With g, h differentiable functions on $\mathbb{R}^{p-1}$, we have*

$$\{g \circ G_p, h \circ G_p\}_v = \{g, h\}_{f = G_p(v)},$$

i.e. G_p is a Poisson map.

Proof. The $(p-1) \times p$ Jacobian of the map G_p is

$$(\mathrm{d}G_p)_{ij} = \begin{cases} v_{i+1} & i = j, \\ v_i & i+1 = j, \\ 0 & \text{otherwise.} \end{cases}$$

By direct calculation, we have

$$\mathrm{d}G_p \cdot \Omega^{\widetilde{\mathrm{sG}}}_p \cdot (\mathrm{d}G_p)^T = 2\Omega^{\mathrm{sG}}_{p-1}\big|_{f = G_p(v)}. \tag{30}$$

Applying ∇ to $(g \circ G_p)(v) = g(f)|_{f=G_p(v)}$ (and omitting some arguments), we find

$$\nabla_v(g \circ G_p) = \nabla_f(g) dG_p|_{f=G_p(v)}.$$

Hence, we have

$$\begin{aligned}\{g \circ G_p, h \circ G_p\}_v &= \tfrac{1}{2}\nabla_v(g \circ G_p)\Omega_p^{\widetilde{\mathrm{sG}}}\nabla_v(g \circ G_p)^T \\ &= \tfrac{1}{2}\nabla_f(g)\mathrm{d}G_p|_{f=G_p(v)}\Omega_p^{\widetilde{\mathrm{sG}}}(\nabla_f(h)\mathrm{d}G_p)^T|_{f=G_p(v)} \\ &= \nabla_f(g)\Omega_p^{\mathrm{sG}}(\nabla_f(h))^T|_{f=G_p(v)} \\ &= \{g, h\}_{f=G_p(v)}.\end{aligned}$$

□

Now we will prove the involutivity of the integrals (27) of the sine-Gordon map (26).

Theorem 6. *Let $I_r^{\widetilde{\mathrm{sG}}}$ and $I_s^{\widetilde{\mathrm{sG}}}$, with $0 \leqslant r, s \leqslant n-1$, be given by formula (27). Then, we have*

$$\{I_r^{\widetilde{\mathrm{sG}}}, I_s^{\widetilde{\mathrm{sG}}}\}_v = 0.$$

Proof. With $V = v_1/v_{2n}$ we have

$$\left\{V^{\pm 1}, \Theta_{r,\epsilon}^{1,p}\right\}_v = V^{\pm 1}E_v\Theta_{r,\epsilon}^{1,p} = \begin{cases}0 & \text{if } r \text{ even,}\\ \mp(-1)^\epsilon V^{\pm}\Theta_{r,\epsilon}^{1,p} & \text{if } r \text{ odd.}\end{cases}$$

The Poisson bracket between two integrals is expanded as

$$\left\{I_r^{\widetilde{\mathrm{sG}}}, I_s^{\widetilde{\mathrm{sG}}}\right\}_v = \alpha_1^2B_1 + \alpha_2^2B_2 + \alpha_3^2B_3 + \alpha_1\alpha_2B_{12} + \alpha_1\alpha_3B_{13} + \alpha_2\alpha_3B_{23},$$

where the coefficients B_I are similar to the A_I given in section 2.1, replacing F by V and $2n$ by $2n-1$. The rules for simplification are also similar. Therefore, $\{I_r^{\widetilde{\mathrm{sG}}}, I_s^{\widetilde{\mathrm{sG}}}\}_v = 0$. □

3. Involutivity of mKdV integrals

We consider the d-dimensional mKdV map

$$(v_1, v_2, \dots, v_d) \mapsto \left(v_2, v_3, \dots, v_d, v_1\frac{\beta_1v_d + \beta_2v_2}{\beta_1v_2 + \beta_3v_d}\right). \tag{31}$$

As shown in [17], this map has $\lfloor (d-1)/2 \rfloor$ integrals given by formula (9) with $0 < 2r < d$. If $d = 2n+1$, the map (31) reduces to a $2n$-dimensional map with exactly n integrals via a reduction $z_i = v_{i+1}/v_i$. For the other case, where $d = 2n+2$, the map (31) reduces to a $2n$-dimensional map with exactly n integrals via the reduction $z_i = v_{i+2}/v_i$. We will show that the integrals of these reduced maps are in involution. In each case, we present a relationship between the relevant symplectic structures and the symplectic structures of the sine-Gordon map in the even case (14). This relation can be used to derive properties of Θ with new symplectic structures.

3.1. The case $d = 2n+1$

Using the reduction $z_i = v_{i+1}/v_i$, we obtain the map

$$\mathrm{mKdV} : (z_1, z_2, \dots, z_{2n}) \mapsto \left(z_2, z_3, \dots, z_{2n}, \frac{1}{z_1z_2\dots z_{2n}}\cdot\frac{\beta_1z_2z_3\dots z_{2n} + \beta_2}{\beta_1 + \beta_3z_2z_3\dots z_{2n}}\right). \tag{32}$$

The integrals of this map are given by

$$I_r^{\mathrm{mKdV}} = \beta_1\left(z_1z_2\dots z_{2n}\Theta_{2r-1,0}^{1,2n} + \frac{1}{z_1z_2\dots z_{2n}}\Theta_{2r-1,1}^{1,2n}\right) + \beta_2\Theta_{2r,1}^{1,2n} + \beta_3\Theta_{2r,0}^{1,2n}, \tag{33}$$

where arguments for Θ are $f_i = z_1^2 z_2^2 \dots z_{i-1}^2 z_i$. Here we have used an 'inverse reduction', $v_i = v_1 z_1 z_2 \cdots z_{i-1}$ to express $f_i = v_i v_{i+1}$ in terms of the z_j and we omitted the v_1 dependence as both the integral and the map do not depend on it.

We obtain a symplectic structure $\Omega_{2n}^{\mathrm{mKdV}}$ for the map (32), with entries

$$\left(\Omega_p^{\mathrm{mKdV}}\right)_{ij} = \begin{cases} -(-1)^{i+j} z_i z_j & i < j, \\ 0 & i = j, \\ (-1)^{i+j} z_i z_j & i > j, \end{cases} \tag{34}$$

cf [7, 12]. This gives us a Poisson bracket $\{g, h\}_z = \nabla_z(g)\Omega_{2n}^{\mathrm{mKdV}}(\nabla_z(h))^T$. As before, we can express the z-Poisson brackets between Θ multi-sums in terms of the corresponding f-Poisson brackets. Consider the map

$$M_p : (z_1, z_2, \dots, z_p) \mapsto \left(z_1, z_1^2 z_2, \dots, z_1^2 z_2^2 \cdots z_{p-1}^2 z_p\right).$$

We have the following result.

Lemma 7. *With g, h differentiable functions on $\mathbb{R}^p$ we have*

$$\{g \circ M_p, h \circ M_p\}_z = \{g, h\}_{f=M_p(z)},$$

i.e. M_p is a Poisson map.

Proof. The $p \times p$ Jacobian of the map M_p is

$$\mathrm{d}M_p = \begin{cases} 0 & i < j, \\ \prod_{k=1}^{i-1} z_k^2 & i = j, \\ 2 z_i z_j^{-1} \prod_{k=1}^{i-1} z_k^2 & i > j \end{cases}$$

and a calculation shows

$$\mathrm{d}M_p.\Omega_p^{\mathrm{mKdV}}.\mathrm{d}M_p^T = \Omega_p^{\mathrm{sG}}\mid_{f=M_p(z)}. \tag{35}$$

The argument is finished along the lines of the proof for lemma 5. □

We are now ready to prove the following theorem.

Theorem 8. *Let I_r^{mKdV} and I_s^{mKdV} be given by formula (33) with $1 \leqslant r, s \leqslant n$. Then, we have*

$$\left\{I_r^{\mathrm{mKdV}}, I_s^{\mathrm{mKdV}}\right\}_z = 0. \tag{36}$$

Proof. With $Z = (z_1 z_2 \dots z_{2n})^{-1}$ we have $F^{\pm 1} \circ M_{2n} = Z^{\pm 1}$. Thus, lemma 7 implies

$$\left\{Z^{\pm 1}, \Theta_{r,\epsilon}^{1,2n}\right\}_z = \begin{cases} 0 & \text{if } r \text{ even}, \\ \mp(-1)^\epsilon Z^{\pm 1}\Theta_{r,\epsilon}^{1,2n} & \text{if } r \text{ odd}. \end{cases} \tag{37}$$

Writing the left-hand side of equation (36) as

$$\left\{I_r^{\mathrm{mKdV}}, I_s^{\mathrm{mKdV}}\right\}_z = \beta_1^2 P_1 + \beta_2^2 P_2 + \beta_3^2 P_3 + \beta_1\beta_2 P_{12} + \beta_1\beta_3 P_{13} + \beta_2\beta_3 P_{23} \tag{38}$$

yields coefficients P_I similar to the A_I given in section 2.1, replacing F by Z, $2r$ by $2r-1$, and $2s$ by $2s-1$. Now that we know the brackets between Z, Z^{-1}, and $\Theta_{2s,1}^{1,2n}$, we can expand the coefficients and show they vanish, using lemma 7 in conjunction with equations (16), (21) and (37). □

3.2. The case $d = 2n + 2$

Now using a reduction $w_i = v_{i+2}/v_i$, we obtain the map

$$\widetilde{\text{mKdV}}: \ (w_1, w_2, \dots, w_{2n}) \mapsto \left(w_2, w_3, \dots, w_{2n}, \frac{1}{w_1 w_3 \dots w_{2n-1}} \cdot \frac{\beta_1 w_2 w_4 \dots w_{2n} + \beta_2}{\beta_1 + \beta_3 w_2 w_4 \dots w_{2n}} \right). \tag{39}$$

Integrals of this map are given by

$$I_r^{\widetilde{\text{mKdV}}} = \alpha_1 \left(w_2 w_4 \dots w_{2n} \Theta^{1,2n+1}_{2r-1,0} + \frac{1}{w_2 w_4 \dots w_{2n}} \Theta^{1,2n+1}_{2r-1,1} \right) + \alpha_2 \Theta^{1,2n+1}_{2r,1} + \alpha_3 \Theta^{1,2n+1}_{2r,0}, \tag{40}$$

where $\Theta = \Theta[e_i]$ with $e_i = f_{i-1}$, with $f_0 = 1$ and $f_i = w_1 w_2 \dots w_i$ $(i > 0)$. Note that we have changed the notation in order to relate the next Poisson bracket to the bracket $\{,\}_f$; the argument of $\Theta^{a,b}$ (7) is $(e_a, \dots, e_b)$ with $e_i = v_i v_{i+1}$. In the 'inverse reduction', we have

$$v_n = \begin{cases} v_1 \prod_{j=1}^{i} w_{2j-1} & n = 2i+1, \\ v_2 \prod_{j=1}^{i-1} w_{2j} & n = 2i. \end{cases}$$

Therefore (similar to the case $d = 2n + 1$), both the reduced map and the reduced integrals depend on the variables w_i. Using (18), we obtain

$$\begin{aligned} \Theta^{1,2n+1}_{s,\epsilon}[e_i] &= \Theta^{2,2n+1}_{s,\epsilon}[e_i] + \Theta^{2,2n+1}_{s-1,\epsilon+1}[e_i] \\ &= \Theta^{1,2n}_{s,\epsilon}[f_i] + \Theta^{1,2n}_{s-1,\epsilon+1}[f_i]. \end{aligned}$$

Let $K_p : (w_1, w_2, \dots, w_p) \mapsto (w_1, w_1 w_2, \dots, w_1 w_2 \cdots w_p)$ and $W = w_2 w_4 \dots w_{2n}$. Then, the integrals can be written as

$$\begin{aligned} I_r^{\widetilde{\text{mKdV}}} = \alpha_1 \big(& W^{-1} \big(\Theta^{1,2n}_{2r-1,0} + \Theta^{1,2n}_{2r-2,1} \big) + W \big(\Theta^{1,2n}_{2r-1,1} + \Theta^{1,2n}_{2r-2,0} \big) \big) \\ & + \alpha_2 \big(\Theta^{1,2n}_{2r,1} + \Theta^{1,2n}_{2r-1,0} \big) + \alpha_3 \big(\Theta^{1,2n}_{2r,0} + \Theta^{1,2n}_{2r-1,1} \big), \end{aligned} \tag{41}$$

where $\Theta = \Theta[f_i]$ with $f = K_p(w)$.

The map (39) has a symplectic structure $\Omega_{2n}^{\widetilde{\text{mKdV}}}$, where

$$\big(\Omega_p^{\widetilde{\text{mKdV}}} \big)_{ij} = \begin{cases} \pm w_i w_j & j = i \pm 1, \\ 0 & j \neq i \pm 1. \end{cases} \tag{42}$$

This gives us a Poisson bracket $\{g, h\}_w = \nabla_w(g) \Omega_{2n}^{\widetilde{\text{mKdV}}} (\nabla_w(h))^T$. Once again we can express the w-Poisson brackets between Θ multi-sums in terms of the corresponding f-Poisson brackets.

Lemma 9. *With g, h differential functions on $\mathbb{R}^p$, we have*

$$\{g \circ K_p, h \circ K_p\}_w = \{g, h\}_{f = K_p(w)},$$

i.e. K_p is a Poisson map.

Proof. This follows from

$$\mathrm{d}K_p \Omega_p^{\widetilde{\text{mKdV}}} \mathrm{d}K_p^T = \Omega_p^{\text{sG}} \mid_{f = K_p(w)} . \tag{43}$$

$\square$

Because $F^{\pm 1} \circ K_{2n} = W^{\pm 1}$, this lemma implies that $\{W, W^{-1}\}_w = 0$:

$$\big\{ W^{\pm 1}, \Theta^{1,2n}_{r,\epsilon} \big\}_w = \begin{cases} 0 & \text{if } r \text{ even}, \\ \mp (-1)^\epsilon W^{\pm 1} \Theta^{1,2n}_{r,\epsilon} & \text{if } r \text{ odd}, \end{cases}$$

and we can also evaluate the brackets between $\Theta^{1,2n}_{r,\epsilon}$. Thus, the following theorem can be proven by mechanical expansion and evaluation of the bracket.

Theorem 10. *Let $\widetilde{I^{\mathrm{mKdV}}_r}$ and $\widetilde{I^{\mathrm{mKdV}}_s}$ be given by formula (40). Then,*

$$\left\{\widetilde{I^{\mathrm{mKdV}}_r}, \widetilde{I^{\mathrm{mKdV}}_s}\right\}_w = 0.$$

4. Involutivity of pKdV integrals

In this section, we prove the involutivity of the integrals of order-reduced pKdV maps. Similar to the sine-Gordon map, we consider two cases depending on whether the dimension d of the map (4) is even or odd. Here, in both cases, there are not enough integrals for integrability, and therefore we perform reductions. We present symplectic structures for the reduced maps in both cases and give a relationship between these symplectic structures. For the case where d is even, properties of multi-sums of products, Ψ, with respect to its symplectic structure are proved in appendix B. For the case where d is odd, the Poisson bracket between Ψ multi-sums are derived from those in the even case and the relationship between the two symplectic structures.

4.1. The case $d = 2n + 2$

We have a $(2n + 2)$-dimensional map (6). The number of integrals I_r of this map given by (10) with $0 \leqslant r \leqslant n - 1$ is not enough for integrability in the sense of Liouville–Arnold. We are not aware of any additional functionally independent integrals. Therefore, we use a reduction $c_i = v_i - v_{i+2}$ to reduce the dimension of the map by 2, while preserving the number of integrals. From equation (4), we obtain the following map:

$$\mathrm{pKdV}: \ (c_1, c_2, \dots, c_{2n}) \mapsto \left(c_2, c_3, \dots, c_{2n}, \frac{\gamma}{c_2 + c_4 + \cdots + c_{2n}} - c_1 - c_3 - \cdots - c_{2n-1}\right). \tag{44}$$

This map has exactly n integrals given by

$$I^{\mathrm{pKdV}}_r = \Psi^{1,2n-1}_{r-1} - (c_2 + c_4 + \cdots + c_{2n})\Psi^{1,2n-2}_{r-1} - (c_1 + c_3 + \cdots + c_{2n-1})\Psi^{2,2n-1}_{r-1} + \Psi^{2,2n-2}_{r-2} + ((c_1 + c_3 + \cdots + c_{2n-1})(c_2 + c_4 + \cdots + c_{2n}) - \gamma)\,\Psi^{1,2n-1}_r, \tag{45}$$

with $r = 0, 1, \dots, n - 1$. The map is symplectic; we have $\mathrm{dpKdV} \cdot \Omega^{\mathrm{pKdV}}_{2n} \cdot \mathrm{dpKdV}^T = \Omega^{\mathrm{pKdV}}_{2n} \circ \mathrm{pKdV}$, where

$$\left(\Omega^{\mathrm{pKdV}}_p\right)_{ij} = \begin{cases} \pm 1 & j = i \pm 1, \\ 0 & j \neq i \pm 1. \end{cases} \tag{46}$$

which is given in [7, 12]. The corresponding Poisson bracket is denoted by $\{g, h\}_c = \nabla_c(g)\Omega^{\mathrm{pKdV}}_{2n}(\nabla_c(h))^T$. We prove that the integrals of the map pKdV are in involution with respect to this Poisson bracket. The proof is based on the knowledge of the Poisson brackets between two Ψ multi-sums which is given as follows.

Lemma 11. *Let $p \geqslant 1$ and $0 \leqslant r, s \leqslant \lfloor (p + 1)/2 \rfloor$. Then, we have the following identities:*

$$\left\{\Psi^{1,p}_r, \Psi^{1,p}_s\right\}_c = 0, \tag{47}$$

$$\left\{\Psi^{1,p}_r, \Psi^{1,p-1}_s\right\}_c + \left\{\Psi^{1,p-1}_r, \Psi^{1,p}_s\right\}_c = 0. \tag{48}$$

This lemma can be proven by induction using the following recurrence relations:

$$\Psi_r^{a,b} = c_{b+1}\Psi_r^{a,b} + \Psi_{r-1}^{a,b-1} \text{ and } \Psi_r^{a,b} = c_a\Psi_r^{a+1,b} + \Psi_{r-1}^{a+2,b}, \tag{49}$$

and the following is implied.

Corollary 12. *Let $p \geqslant 1$ and let $r, s \in \mathbb{Z}$. Then,*

(1) $\{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{1,p}\Psi_s^{1,p-1} - \Psi_s^{1,p}\Psi_r^{1,p-1}$,
(2) $\{\Psi_r^{a,b}, \Psi_s^{a,b}\}_c = 0$ *with* $0 \leqslant r, s \leqslant \lfloor (b-a)/2 \rfloor + 1$,
(3) $\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}_c = \Psi_s^{1,n}\Psi_r^{2,p} - \Psi_r^{1,p}\Psi_s^{2,p}$,
(4) $\{\Psi_r^{1,p}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p}\}_c = 0$,
(5) $\{\Psi_r^{1,p}, \Psi_s^{2,p+1}\}_c + \{\Psi_r^{2,p+1}, \Psi_s^{1,p}\}_c = 0$,
(6) $\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{2,p}\Psi_s^{1,p-1} - \Psi_s^{2,p}\Psi_r^{1,p-1}$,
(7) $\{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{1,p}\Psi_s^{1,p-1} - \Psi_s^{1,p}\Psi_r^{1,p-1}$,
(8) $\{\Psi_r^{1,p+1}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p+1}\}_c = 0$,
(9) $\{\Psi_r^{1,p}, \Psi_{s-2}^{2,p-1}\}_c + \{\Psi_{r-2}^{2,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{2,p}\Psi_{s-1}^{1,p-1} - \Psi_s^{2,p}\Psi_{r-1}^{1,p-1} + \Psi_{r-1}^{2,p}\Psi_s^{1,p-1} - \Psi_{s-1}^{2,p}\Psi_r^{1,p-1}$.

If we define

$$C_1 = c_1 + c_3 + \cdots + c_{2n-1}, \qquad C_2 = c_2 + c_4 + \cdots + c_{2n},$$

and $g(c_1, c_2, \ldots, c_{2n})$ is a differentiable function on $\mathbb{R}^{2n}$, then

$$\{g, C_1\}_c = -\frac{\partial g}{\partial c_{2n}}, \qquad \{g, C_2\}_c = \frac{\partial g}{\partial c_1}. \tag{50}$$

Theorem 13. *For all $0 \leqslant r, s \leqslant n-1$, we have $\{I_r^{\mathrm{pKdV}}, I_s^{\mathrm{pKdV}}\}_c = 0$, where $I_r^{\mathrm{pKdV}}, I_s^{\mathrm{pKdV}}$ are given by (45).*

Proof. It is proved by expansion of the bracket, using lemma 11, corollary 12 and formulas (49) and (50). □

4.2. The case $d = 2n+1$

We introduce a reduction $u_i = v_i - v_{i+1}$. We obtain a $2n$-dimensional map

$$\widetilde{\mathrm{pKdV}}: \ (u_1, u_1, \ldots, u_{2n}) \mapsto \left(u_2, u_3, \ldots, u_{2n}, \frac{\gamma}{u_2 + u_3 + \cdots + u_{2n}} - u_1 - u_2 - \cdots - u_{2n}\right) \tag{51}$$

with n integrals $(0 \leqslant r \leqslant n-1)$

$$\begin{aligned} I_r^{\widetilde{\mathrm{pKdV}}} = {} & \Psi_{r-1}^{1,2n-2} - (u_2 + u_3 + \cdots + u_{2n})\Psi_{r-1}^{1,2n-3} - (u_1 + u_2 + \cdots + u_{2n-1})\Psi_{r-1}^{2,2n-2} \\ & + \Psi_{r-2}^{2,2n-3} + ((u_2 + u_3 + \cdots + u_{2n})(u_1 + u_2 + \cdots + u_{2n-1}) - \gamma)\,\Psi_r^{1,2n-2}, \end{aligned} \tag{52}$$

where the argument of Ψ is $f_i = 1/(c_i c_{i+1})$ with $c_i := u_i + u_{i+1}$. Based on the method given in [12], we obtain a symplectic structure $\Omega_{2n}^{\widetilde{\mathrm{pKdV}}}$ for the map (51), where

$$\left(\Omega_p^{\widetilde{\mathrm{pKdV}}}\right)_{ij} = \begin{cases} -(-1)^{i+j} & j > i, \\ 0 & j = i, \\ (-1)^{i+j} & j < i. \end{cases} \tag{53}$$

The Poisson bracket is denoted $\{g, h\}_u = \nabla_u(g)\widetilde{\Omega_{2n}^{\mathrm{pKdV}}}(\nabla_u(h))^T$. Next we present a relationship between the two symplectic structures (46) and (53) and the corresponding Poisson brackets. Consider the map

$$Q_p : (u_1, u_2, \ldots, u_p) \mapsto (u_1 + u_2, u_2 + u_3, \ldots, u_{p-1} + u_p).$$

Lemma 14. *The map Q_p is a Poisson map, i.e.*

$$\{f \circ Q_p, g \circ Q_p\}_u = \{f, g\}_{c=Q_p(u)}, \tag{54}$$

where $f(c)$ and $g(c)$ are differentiable functions.

Proof. By calculation, we obtain

$$dQ_p\widetilde{\Omega_p^{\mathrm{pKdV}}}dQ_p^T = \Omega_{p-1}^{\mathrm{pKdV}}. \tag{55}$$

□

Theorem 15. *Let $\widetilde{I_r^{\mathrm{pKdV}}}$, $\widetilde{I_s^{\mathrm{pKdV}}}$ be given by (52). Then, for all $0 \leqslant r, s \leqslant n-1$ we have*

$$\{\widetilde{I_r^{\mathrm{pKdV}}}, \widetilde{I_s^{\mathrm{pKdV}}}\}_u = 0.$$

Proof. As the following formulas hold:

$$\{g, u_2 + u_3 + \cdots + u_{2n}\}_u = \frac{\partial g}{\partial u_1}, \tag{56}$$

$$\{g, u_1 + u_2 + \cdots + u_{2n-1}\}_u = -\frac{\partial g}{\partial u_{2n}}, \tag{57}$$

and the properties of Ψ with respect to the bracket $\{,\}_u$ which are the same as those with respect to the bracket $\{,\}_c$, one can prove the involutivity of the integrals (52) similarly to what we did for the case $d = 2n + 2$. □

5. Conclusion

In this paper, we have proved the involutivity of integrals of sine-Gordon, pKdV and mKdV maps directly by using induction and recently found symplectic structures of these maps. In order to prove that these maps are completely integrable in the sense of Liouville–Arnold [2, 18], one also needs to prove the functional independence of their integrals. We now briefly discuss some results that are based on different techniques which fall outside the scope of this paper and will be published elsewhere [15].

To prove functional independence, due to the analyticity of the integrals it suffices to prove linear independence of the gradients of the integrals at a certain point. It turns out that we can evaluate the multi-sums of products at certain points in terms of binomial coefficients (counting the number of terms in the multi-sums of products). Also using certain recursive formulas, we can find the gradients at these points. The proof of functional independence thus boils down to LU-decomposition of a matrix whose entries are expressed in binomials coefficients. This has been performed in [15]. For the sine-Gordon, the integrals are independent at $v_1 = v_2 = \cdots = v_d = c$ when $\alpha_2 c^4 \neq \alpha_3$ from which the result follows by varying c. For the mKdV map, we have proved functional independence when $\beta_2 \neq \beta_3$. And for the pKdV map the functional independence has been established for the generic case where $\gamma \neq \frac{(d-2)^2}{4} + \frac{d-2}{2}$. In these cases we conclude the integrability of equations (2), (3) and (4) for arbitrary order $d = p + 1$.

We also note that the integrals of maps obtained as $(p, -1)$-reductions of the equations in the ABS list [1], with the exception of Q_4, can be expressed in terms of multi-sums of products, Ψ [14]. It would be interesting to study their symplectic structures and furthermore their complete integrability.

Acknowledgments

This research has been funded by the Australian Research Council through the Centre of Excellence for Mathematics and Statistics of Complex Systems. DTT acknowledges the support of two scholarships, one from La Trobe University and the other from the Endeavour IPRS programme.

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