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Closed-form expressions for integrals of traveling wave reductions of integrable lattice equations

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Abstract

We give a method to calculate closed-form expressions in terms of multi-sums of products for integrals of ordinary difference equations which are obtained as traveling wave reductions of integrable partial difference equations. Important ingredients are the staircase method, a non-commutative Vieta formula and certain splittings of the Lax matrices. The method is applied to all equations of the Adler–Bobenko–Suris classification, with the exception of Q_4.

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1. Introduction

Two main classes of discrete integrable systems that may be distinguished are ordinary difference equations (O∆Es) and partial difference equations (P∆Es). By imposing periodic initial conditions, a P∆E reduces to an O∆E. The staircase method which was introduced in [7, 8] provides us with a tool to construct integrals of O∆Es, or mappings, derived as reductions of integrable P∆Es. These integrals are obtained by expanding the trace of a monodromy matrix in powers of the spectral parameter.

Recently in [6], elegant and succinct formulae for integrals of sine-Gordon and modified Korteweg–de Vries (mKdV) maps have been given for the first time. The integrals are expressed in terms of multi-sums of products, Θ, which are defined in section 2. These multi-sums of products were discovered by inspection and proved by induction. Properties of these multi-sums of products were used to prove the invariance of the integrals independently of the staircase method and make it possible to prove functional independence and involutivity of the integrals directly.

In a recent paper by Adler, Bobenko and Suris (ABS), multi-linear equations on quad-graphs are classified with respect to consistency around the cube [1]. It is also known that for P∆Es on quad-graphs which satisfy the consistency property, we can obtain a Lax pair algorithmically [2, 3]. Hence, the staircase method can be applied to obtain integrals of traveling wave reductions of the equations in the ABS list.

Some questions arise here: is it possible to give similar explicit expressions for these integrals as was done for mKdV and sine-Gordon? If so, is there any method to obtain closed-form expressions for integrals directly, rather than using inspection and induction? This paper answers both these questions in the affirmative. Based on a non-commutative Vieta formula, and certain splittings of the Lax matrices, we explain how multi-sums of products emerge and give a method to actually derive closed-form expressions for the integrals.

2. Outline of this paper

In this paper, we restrict ourselves to the so-called $(1, z_2)$ traveling wave reductions ($z_2 \in \mathbb{N}^+$) which we now explain. With $(l, m) \in \mathbb{Z} \times \mathbb{Z}$, we consider a two-dimensional PΔE with field variable v,

$$f\left(v_{l,m}, v_{l+1,m}, v_{l,m+1}, v_{l+1,m+1}; \alpha\right) = 0, \tag{1}$$

and parameters $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Now we introduce a reduction $v_{l,m} = v_n$, where $n = l + mz_2$. Then, $v_{l,m}$ satisfies the periodicity $v_{l,m} = v_{l+z_2,m-1}$, and the PΔE reduces to an OΔE:

$$f\left(v_n, v_{n+1}, v_{n+z_2}, v_{n+z_2+1}; \alpha\right) = 0. \tag{2}$$

We suppose that our equation (1) arises as the compatibility condition of two linear equations, that is, it has a Lax pair. A Lax pair $L_{l,m}, M_{l,m}$ for a PΔE (1) is a pair of matrices that satisfy (cf [9])

$$L_{l,m} M_{l,m}^{-1} - M_{l+1,m}^{-1} L_{l,m+1} = 0. \tag{3}$$

Similarly, an OΔE has a Lax pair $\mathcal{L}_n, \mathcal{M}_n$ if they are non-singular matrices that satisfy

$$\mathcal{M}_n \mathcal{L}_n - \mathcal{L}_{n+1} \mathcal{M}_n = 0. \tag{4}$$

Using the $(1, z_2)$ traveling wave reduction, the Lax pair for a PΔE reduces to matrices L_n, M_n which satisfy the following equation:

$$L_n M_n^{-1} - M_{n+1}^{-1} L_{n+z_2} = 0. \tag{5}$$

The monodromy matrix $\mathcal{L}_n$ for the $(1, z_2)$ reduction is given by (cf [8])

$$\mathcal{L}_n = M_n^{-1} \overset{\curvearrowleft}{\prod_{i=0}^{z_2-1}} L_{i+n}, \tag{6}$$

where the inversely ordered product is

$$\overset{\curvearrowleft}{\prod_{i=a}^{b}} L_i := L_b L_{b-1} \ldots L_{a+1} L_a. \tag{7}$$

Taking $\mathcal{M}_n = L_n$, we obtain a Lax pair $\mathcal{L}_n, \mathcal{M}_n$ for the reduced OΔE (2) from the Lax pair of the corresponding PΔE (1) cf [9]. Recently, a description of (z_1, z_2) reduction, with general $(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}$, is given in [4, 11]. This generalizes the reductions in [8] where the case $(z_1, z_2) \in \mathbb{N} \times \mathbb{N}$ with z_1, z_2 co-prime was considered. In [11], it was proved that for any (z_1, z_2) reduction, there is a $\mathcal{M}$ with whom the monodromy matrix $\mathcal{L}$ forms a Lax pair for the reduction, and explicit formulae for $\mathcal{L}, \mathcal{M}$ in terms of the reduced Lax matrices L, M were provided.

It follows from equation (4) that the trace of $\mathcal{L}_n$ is invariant under the map obtained from an OΔE. Since the reduced Lax matrices generally depend on a spectral parameter, integrals for the OΔE (2) are obtained by expanding the trace of the monodromy matrix in powers of the spectral parameter. Therefore, to give explicit expressions for these integrals, we need to expand a product of L matrices. We split the L matrices in the form $L_i = r_i(\lambda X_i + Y_i)$, where λ is (a function of) the spectral parameter. Next, we consider the formal expansion of the matrix product in a non-commutative Vieta formula:

$$\prod_{i=a}^{\widehat{b}}(\lambda X_i + Y_i) = \sum_{r=0}^{b-a+1} \lambda^{b-a+1-r} Z_r^{a,b} \tag{8a}$$

$$= \sum_{r=0}^{b-a+1} \lambda^r \widetilde{Z}_r^{a,b}, \tag{8b}$$

where

$$Z_r^{a,b} = \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b} X_b X_{b-1}\cdots X_{i_r+1} Y_{i_r} X_{i_r-1}\cdots X_{i_1+1} Y_{i_1} X_{i_1-1}\cdots X_a, \tag{9a}$$

$$\widetilde{Z}_r^{a,b} = \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b} Y_b Y_{b-1}\cdots Y_{i_r+1} X_{i_r} Y_{i_r-1}\cdots Y_{i_1+1} X_{i_1} Y_{i_1-1}\cdots Y_a. \tag{9b}$$

This is a generalization of the Vieta expansion with commutative variables:

$$\prod_{i=a}^{\widehat{b}}(\lambda + f_i) = \sum_{r=0}^{b-a+1} \lambda^{b-a+1-r} s_r^{a,b}, \tag{10}$$

where the multi-sums of products $s_r^{a,b}$ are the elementary symmetric functions,

$$s_r^{a,b}\{f_i\} := \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b} \prod_{j=1}^{r} f_{i_j}. \tag{11}$$

In this paper, we will consider two different forms of X_i and Y_i with special properties such that the elements of the matrix $Z_r^{a,b}$ or $\widetilde{Z}_r^{a,b}$ can be expressed in terms of multi-sums of products Θ, respectively Φ.

For the reduced mKdV equation [6],

$$\alpha_1\left(v_n v_{n+z_2} - v_{n+1} v_{n+z_2+1}\right) + \alpha_2 v_n v_{n+1} - \alpha_3 v_{n+z_2} v_{n+z_2+1} = 0, \tag{12}$$

and for the reduced sine-Gordon equation [6],

$$\beta_1\left(v_n v_{n+z_2+1} - v_{n+1} v_{n+z_2}\right) + \beta_2 v_n v_{n+1} v_{n+z_2} v_{n+z_2+1} - \beta_3 = 0, \tag{13}$$

the L matrix can be written as $L_i = \lambda X_i + Y_i$, where

$$X_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: J, \qquad Y_i = \begin{pmatrix} v_i/v_{i+1} & 0 \\ 0 & v_{i+1}/v_i \end{pmatrix}, \tag{14}$$

and λ is the spectral parameter. Substituting (14) in equations (9*a*) and (9*b*), we derive multi-sums of products, Θ, with different arguments:

$$\Theta_{r,\epsilon}^{a,b}\left\{\left(\frac{v_i}{v_{i+1}}\right)^{(-1)^i}\right\} = \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b} \prod_{j=1}^{r}\left(\left(\frac{v_{i_j}}{v_{i_j+1}}\right)^{(-1)^{i_j}}\right)^{(-1)^{j+\epsilon}}, \tag{15}$$

$$\Theta_{r,\epsilon}^{a,b}\{v_i v_{i+1}\} = \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b} \prod_{j=1}^{r}\left(v_{i_j} v_{i_j+1}\right)^{(-1)^{j+\epsilon}}. \tag{16}$$

See lemmas 1 and 4. The latter one is the definition of Θ given in [6]. The general definition of Θ,

$$\Theta_{r,\epsilon}^{a,b}\{f_i\} := \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b} \prod_{j=1}^{r} (f_{i_j})^{(-1)^{j+\epsilon}}, \tag{17}$$

where f is a function on $[a, b] \subset \mathbb{Z}$, is obtained by replacing Y_i with

$$D_i = \begin{pmatrix} f_i^{(-1)^i} & 0 \\ 0 & f_i^{(-1)^{i+1}} \end{pmatrix}. \tag{18}$$

Substituting $X_i = J$ and $Y_i = D_i$ into the Vieta expansion (9*a*), we obtain

$$Z_r^{a,b} = J^{b-a+1-r}\Theta_r^{a,b}, \tag{19}$$

where

$$\Theta_r^{a,b} = \begin{pmatrix} \Theta_{r,a-1}^{a,b}\{f_i\} & 0 \\ 0 & \Theta_{r,a}^{a,b}\{f_i\} \end{pmatrix}. \tag{20}$$

Equation (19) can be generalized to $2n \times 2n$ matrices J and D, provided they satisfy the following properties:

$$J^2 = I, \qquad D_i J D_i = J, \qquad \text{and} \qquad D_i \quad \text{is diagonal.}$$

To illustrate the second way of splitting L, we write the L matrices of the mKdV and sine-Gordon equations in a different way with $X_i = H$ and $Y_i = s_i A_i^i$, where H and A_j^i are defined as

$$H := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_j^i = \begin{pmatrix} a_i & a_i b_j \\ 1 & b_j \end{pmatrix}. \tag{21}$$

Then, with special properties of H and A we obtain multi-sums of products called Ψ which are introduced below. In particular, if we take $s_i = f_i$, $a_i = 1/f_i$ and $b_i = 0$, then formula (9*b*) can be evaluated as

$$\widetilde{Z}_r^{a,b} = \begin{pmatrix} \Phi_r^{a,b-1} & \Phi_{r-1}^{a+1,b-1} \\ f_b\Phi_r^{a,b-2} & f_b\Phi_{r-1}^{a+1,b-2} \end{pmatrix}, \tag{22}$$

where multi-sums of products, Φ, are defined as

$$\Phi_r^{a,b}\{f_i\} := \sum_{a\leqslant i_1, i_1+1<i_2, i_2+1<\cdots<i_{r-1}, i_{r-1}+1<i_r\leqslant b} \prod_{j=1}^{r} f_{i_j}, \tag{23}$$

with f as a function on $[a, b] \subset \mathbb{Z}$. More generally, when H and A satisfy

$$H^2 = 0, \qquad A_m^n A_l^k = \alpha_{k,m} A_l^n, \qquad H A_m^n H = H \qquad \text{and} \qquad A_m^n H A_l^k = A_l^n, \tag{24}$$

substituting $X_i = H$ and $Y_i = s_i A_i^i$ in equation (9*b*) yields

$$\widetilde{Z}_r^{a,b} = \Psi_{r-1}^{a+1,b-2} H A_a^{b-1} + \Psi_{r-1}^{a+2,b-1} A_{a+1}^b H + \Psi_{r-2}^{a+2,b-2} H + \Psi_r^{a+1,b-1} A_a^b, \tag{25}$$

where Ψ is

$$\Psi_r^{a,b} := s_{a-1}\Phi_r^{a,b}\left\{\frac{1}{s_{i+1}\alpha_{i-1,i}\alpha_{i,i+1}}\right\}\prod_{i=a}^{b+1}(s_i\alpha_{i-1,i}). \tag{26}$$

The elementary symmetric functions satisfy the following recursive formulae (for all c such that $0 \leqslant c \leqslant b - a + 1$):

$$s_r^{a,b} = \sum_{i=m_1}^{m_2} s_{r-i}^{a+c,b} s_i^{a,a+c-1}, \tag{27a}$$

$$s_r^{a,b} = \sum_{i=m_1}^{m_2} s_i^{b-c+1,b} s_{r-i}^{a,b-c}, \tag{27b}$$

where $m_1 = \max(0, r + a + c - b - 1)$ and $m_2 = \min(r, c)$, or one can take $m_1 = 0, m_2 = r$. These recursive formulae are obtained by using the Vieta expansion (10) and writing

$$\prod_{i=a}^{\overset{\curvearrowleft}{b}} z_i = \prod_{i=a+c}^{\overset{\curvearrowleft}{b}} z_i \prod_{i=a}^{\overset{\curvearrowleft}{a+c-1}} z_i, \tag{28}$$

where $z_i = \lambda + f_i$ in this case. Taking $c = r$, these recursive formulae provide us with an efficient way of producing and storing the elementary symmetric functions.

We will obtain similar recursive formulae for $Z_r^{a,b}$ and $\widetilde{Z}_r^{a,b}$ by using the Vieta expansion and the matrix analogue of (28). Recursive formulae for Θ and Φ are derived from those for $Z_r^{a,b}$ and $\widetilde{Z}_r^{a,b}$, respectively. The recursivity of these multi-sums of products gives us a convenient way of computing these multi-sums of products.

The rest of this paper is organized as follows. In section 3, we explain how Θ arises from equation (9*a*) using properties of J (14) and D_i (18). Then, the recursive formulae for Θ are obtained. We give two closed-form expressions for integrals of the mKdV equation in terms of Θ, one of which coincides with the form discovered in [6], using the two Vieta formulae (9*a*) and (9*b*). In section 4, we show how the multi-sums of products, Φ, emerge as entries of matrix coefficients. The recursive formulae for Φ are given and we prove equation (25). As a first application, we express integrals of the mKdV equation in terms of Ψ. Next, we present a general formula for integrals for all equations whose reduced Lax matrices can be written in the form

$$r_i\left(\lambda H + s_i A_i^i\right). \tag{29}$$

We explicitly write the reduced Lax matrices for the equation in the ABS classification in form (29), with the exception of Q_4. So we obtain closed-form expressions for integrals of these equations from the general formula. The final section compares two sets of integrals expressed in terms of Θ and Ψ, respectively. This section also discusses possible future work.

3. Multi-sums of products, Θ

In this section, we first present the properties of the multi-sums of products, Θ. Then, an application of Θ is given. We derive two closed-form expressions for integrals of mKdV.

3.1. Properties of the multi-sums of products, Θ

The integrals in terms of the multi-sums of products (17) with $f_i = v_i v_{i+1}$, first introduced in [6], were discovered by inspection and proved by induction. We found that the multi-sums of products can actually be derived from (Vieta-like) formulae (9*a*) and (9*b*) and special properties of the reduced Lax matrices. We also show that the following properties, which were used in the proofs in [6][identity (3),(4)],

$$\Theta_{n,\epsilon}^{a,b} = \Theta_{n,\epsilon}^{a+1,b} + f_a^{(-1)^{1+\epsilon}} \Theta_{n-1,\epsilon\pm1}^{a+1,b}, \tag{30}$$

$$\Theta_{n,\epsilon}^{a,b} = \Theta_{n,\epsilon}^{a,b-1} + f_b^{(-1)^{n+\epsilon}} \Theta_{n-1,\epsilon}^{a,b-1}, \tag{31}$$

are special cases of the more general identities (33*a*) and (33*b*) given below. These identities (33*a*) and (33*b*) are similar to the recursive formulae for the symmetric functions (27*a*) and

(27*b*), respectively, and can be used to efficiently compute the multi-sums of products, Θ. They also play an important role in calculating gradients for proving the functional independence and the involutivity of integrals [13, 14].

Lemma 1. *Let J and D_i be as in (14) and (18) respectively. The multi-sums of products $\Theta_{r,\epsilon}^{a,b}$ defined by (17) are the entries in the matrix coefficients in the expansion*

$$\prod_{i=a}^{\overset{\frown}{b}}(\lambda J + D_i) = \sum_{r=0}^{b-a+1}(\lambda J)^{b-a+1-r}\Theta_r^{a,b}, \tag{32}$$

where $\Theta_r^{a,b}$ is defined in (20).

Proof. We use Vieta expansions (8*a*) and (9*a*) to expand the left-hand side of equation (20):

$$\prod_{i=a}^{\overset{\frown}{b}}(\lambda J + D_i) = \sum_{r=0}^{b-a+1}\lambda^{b-a+1-r}Z_r^{a,b}.$$

Using properties of J and D_i such as $D_i J^k = J^k D_i^{(-1)^k}$ and $J^2 = I$, we have

$$\begin{aligned}
Z_r^{a,b} &= \sum_{a\leqslant i_1<i_2\ldots<i_r\leqslant b} JJ\ldots JD_{i_r}J\ldots JD_{i_{r-1}}J\ldots JD_{i_1}J\ldots J\\
&= \sum_{a\leqslant i_1<i_2\ldots<i_r\leqslant b} J^{b-i_r}D_{i_r}J^{i_r-i_{r-1}-1}D_{i_{r-1}}\ldots J^{i_2-i_1-1}D_{i_1}J^{i_1-a}\\
&= \sum_{a\leqslant i_1<i_2\ldots<i_r\leqslant b} J^{b-i_r}D_{i_r}J^{i_r-i_{r-1}-1}D_{i_{r-1}}\ldots J^{i_3-i_2-1}D_{i_2}J^{i_2-a-1}D_{i_1}^{(-1)^{i_1-a}}\\
&= \sum_{a\leqslant i_1<i_2\ldots<i_r\leqslant b} J^{b-i_r}D_{i_r}J^{i_r-i_{r-1}-1}D_{i_{r-1}}\ldots J^{i_3-a-2}D_{i_2}^{(-1)^{i_2-a-1}}D_{i_1}^{(-1)^{i_1-a}}\\
&\ \vdots\\
&= \sum_{a\leqslant i_1<i_2\ldots<i_r\leqslant b} J^{b-a+1-r}D_{i_r}^{(-1)^{i_r-(r-1)-a}}D_{i_{r-1}}^{(-1)^{i_{r-1}-(r-2)-a}}\ldots D_{i_1}^{(-1)^{i_1-a}}.
\end{aligned}$$

Now since

$$D_i^{(-1)^k} = \begin{pmatrix} f_i^{(-1)^{i+k}} & 0\\ 0 & f_i^{(-1)^{i+k+1}}\end{pmatrix},$$

we have

$$\begin{aligned}
\Theta_r^{a,b} &= \sum_{a\leqslant i_1<i_2\ldots<i_r\leqslant b}\\
&\begin{pmatrix} f_{i_r}^{(-1)^{-r-a+1}}f_{i_{r-1}}^{(-1)^{-(r-1)-a+1}}\ldots f_{i_1}^{(-1)^{-1-a+1}} & 0\\ 0 & f_{i_r}^{(-1)^{-r-a+2}}f_{i_{r-1}}^{(-1)^{-(r-1)-a+2}}\ldots f_{i_1}^{(-1)^{-1-a+2}}\end{pmatrix}\\
&= \begin{pmatrix}\Theta_{r,a-1}^{a,b} & 0\\ 0 & \Theta_{r,a}^{a,b}\end{pmatrix}.
\end{aligned}$$

□

Using lemma 1 and the matrix analogue of (28), we obtain the following recursive formulae for Θ.

Proposition 2. *For any $0 \leqslant c \leqslant b - a + 1$, we have*

$$\Theta_{r,\epsilon}^{a,b} = \sum_{i=m_1}^{m_2} \Theta_{i,\epsilon}^{a,a+c-1} \Theta_{r-i,\epsilon+i}^{a+c,b}, \tag{33a}$$

$$\Theta_{r,\epsilon}^{a,b} = \sum_{i=m_1}^{m_2} \Theta_{r-i,\epsilon}^{a,b-c} \Theta_{i,r+\epsilon+i}^{b-c+1,b}, \tag{33b}$$

where $m_1 = \max(0, r+a+c-b-1)$ *and* $m_2 = \min(r, c)$.

Proof. It is easy to see that (33*b*) follows from (33*a*) by substituting $c = b-a-c+1$ in (33*a*), changing variable $i = r - j$ and using the fact that $-j \leqslant \min(x, y) \Longrightarrow j \geqslant \max(-x, -y)$.

By using the Vieta expansion (8*a*) and the product structure (28), the recursive formulae (27*a*) and (27*b*) still hold if we replace s by Z. Therefore, using lemma 1 we have

$$J^{b-a+1-r}\Theta_r^{a,b} = \sum_{i=m_1}^{m_2} J^{b-(a+c)+1-(r-i)}\Theta_{r-i}^{a+c,b} J^{c-i}\Theta_i^{a,a+c-1}. \tag{34}$$

If $b-a+1-r$ is even and using equation (20), we have

$$\begin{pmatrix} \Theta_{r,a-1}^{a,b} & 0 \\ 0 & \Theta_{r,a}^{a,b} \end{pmatrix} = \sum_{m_1\leqslant i\leqslant m_2, c-i\,\mathrm{even}} \begin{pmatrix} \Theta_{r-i,a+c-1}^{a+c,b} & 0 \\ 0 & \Theta_{r-i,a+c}^{a+c,b} \end{pmatrix} \begin{pmatrix} \Theta_{i,a-1}^{a,a+c-1} & 0 \\ 0 & \Theta_{i,a}^{a,a+c-1} \end{pmatrix}$$
$$+ \sum_{m_1\leqslant i\leqslant m_2, c-i\,\mathrm{odd}} \begin{pmatrix} 0 & \Theta_{r-i,a+c}^{a+c,b} \\ \Theta_{r-i,a+c-1}^{a+c,b} & 0 \end{pmatrix} \begin{pmatrix} 0 & \Theta_{i,a}^{a,a+c-1} \\ \Theta_{i,a-1}^{a,a+c-1} & 0 \end{pmatrix}.$$

Therefore, we get

$$\Theta_{r,a-1}^{a,b} = \sum_{i=m_1}^{m_2} \Theta_{i,a-1}^{a,a+c-1}\Theta_{r-i,a+i-1}^{a+c,b}, \qquad \Theta_{r,a}^{a,b} = \sum_{i=m_1}^{m_2} \Theta_{i,a}^{a,a+c-1}\Theta_{r-i,a+i}^{a+c,b}.$$

Since $\{a-1, a\} = \{0, 1\} (\mathrm{mod}\ 2)$, we obtain

$$\Theta_{r,\epsilon}^{a,b} = \sum_{i=m_1}^{m_2} \Theta_{i,\epsilon}^{a,a+c-1}\Theta_{r-i,\epsilon+i}^{a+c,b}.$$

The proof for odd $b-a+1-r$ is similar. □

Note that from the definition of Θ (17), or from equation (32), we have the following properties:

- $\Theta_{r,\epsilon}^{a,b} = 0$ if $r < 0$ or $r > b-a+1$,
- $\Theta_{0,\epsilon}^{a,b} = 1$,
- $\Theta_{1,\epsilon}^{a,a} = f_a^{(-1)^{1+\epsilon}}$.

If we consider these properties as initial values of Θ, then we can efficiently calculate Θ from the recursive formulae (33*a*) by taking $c = b-a$.

3.2. *Applications of* Θ *to the mKdV equation*

The $(1, z_2)$ reductions of the Lax matrices for mKdV are given by

$$L_n = \begin{pmatrix} \frac{v_n}{v_{n+1}} & k \\ k & \frac{v_{n+1}}{v_n} \end{pmatrix} \qquad \text{and} \qquad M_n^{-1} = \begin{pmatrix} \alpha_3\frac{v_{n+z_2}}{v_n} & \alpha_1 k \\ \alpha_1 k & \alpha_2\frac{v_n}{v_{n+z_2}} \end{pmatrix}. \tag{35}$$

Indeed, we have $L_n M_n^{-1} - M_{n+1}^{-1} L_{n+z_2} = \mathcal{F}_{\mathrm{mKdV}} \cdot N_n$, where $\mathcal{F}_{\mathrm{mKdV}}$ is the left-hand side of (12) and

$$N_n = \begin{pmatrix} 0 & \frac{k}{v_{n+1} v_{n+z_2}} \\ \frac{-k}{v_n v_{n+z_2+1}} & 0 \end{pmatrix}.$$

Using lemma 1 with $f_i = (v_i/v_{i+1})^{(-1)^i}$, $a = 0$, $b = z_2 - 1$, and taking the trace of the monodromy matrix (6), we obtain the following result.

Proposition 3. *The closed-form expressions for integrals of the mKdV equation are given as follows:*

$$I_r = \alpha_1 \left(\Theta_{r,0}^{0,z_2-1} + \Theta_{r,1}^{0,z_2-1}\right) + \alpha_2 \frac{v_0}{v_{z_2}} \Theta_{r-1,0}^{0,z_2-1} + \alpha_3 \frac{v_{z_2}}{v_0} \Theta_{r-1,1}^{0,z_2-1}, \tag{36}$$

where $z_2 - r$ is odd.

However, in [6], integrals of the mKdV equation are expressed in terms of Θ with different arguments, namely $f_i = v_i v_{i+1}$. To derive the latter closed-form expressions for integrals of the mKdV equation, the following lemma was used [6].

Lemma 4. *We have*

$$\prod_{i=a}^{\overset{\curvearrowleft}{b-1}} L_i = \sum_{r=0}^{b-a} \widetilde{Z}_r^{a,b-1} \lambda^r, \tag{37}$$

where $f_i = v_i v_{i+1}$, and

$$\widetilde{Z}_r^{a,b-1} = \begin{pmatrix} \frac{v_a}{v_b} \Theta_{r,0}^{a,b-1} & 0 \\ 0 & \frac{v_b}{v_a} \Theta_{r,1}^{a,b-1} \end{pmatrix},$$

when r is even, and

$$\widetilde{Z}_r^{a,b-1} = \begin{pmatrix} 0 & \frac{1}{v_a v_b} \Theta_{r,1}^{a,b-1} \\ v_a v_b \Theta_{r,0}^{a,b-1} & 0 \end{pmatrix},$$

when r is odd.

Whereas for lemma 1 we used the Vieta expansion (9*a*), this lemma is proved by using the Vieta expansion (9*b*); see appendix A.

Multiplying both sides of equation (37) by M_0^{-1} and taking the trace we obtain the following result, which is theorem 3 in [6].

Proposition 5. *The trace of the monodromy matrix of the mKdV equation is*

$$\mathrm{Tr}(\mathcal{L}_0) = \sum_{r=0}^{\lfloor (z_2+1)/2 \rfloor} k^{2r} I_r,$$

where

$$I_r = \alpha_1 \left(v_0 v_{z_2} \Theta_{2r-1,0}^{0,z_2-1} + \frac{1}{v_0 v_{z_2}} \Theta_{2r-1,1}^{0,z_2-1} \right) + \alpha_2 \Theta_{2r,1}^{0,z_2-1} + \alpha_3 \Theta_{2r,0}^{0,z_2-1}, \tag{38}$$

with $f_i = v_i v_{i+1}$.

It follows that the coefficients I_r form a set of integrals for the mKdV equation. Their invariance can also be proven directly. By using the properties (30) and (31) of the multi-sums of products, it was shown in [6] that

$$S(I_r) - I_r = \mathcal{F}_{\mathrm{mKdV}} \cdot \Lambda_r, \tag{39}$$

where S is the shift operator $S(v_i) = v_{i+1}$ and

$$\Lambda_r = \frac{1}{v_0 v_1 v_{z_2} v_{z_2+1}} \Theta^{1,z_2-1}_{2r-1,1} - \Theta^{1,z_2-1}_{2r-1,0} \tag{40}$$

is called an integrating factor.

It is important to note that the set of integrals expressed in Θ with $f_i = (v_i/v_{i+1})^{(-1)^i}$ is the same as that expressed in Θ with $f_i = v_i v_{i+1}$ due to the following identities:

$$\Theta^{a,b}_{r,a+\epsilon}\left\{\left(\frac{v_i}{v_{i+1}}\right)^{(-1)^i}\right\} = \left(\frac{v_a}{v_{b+1}}\right)^{(-1)^{\epsilon+1}} \Theta^{a,b}_{b-a+1-r,\epsilon+1}\{v_i v_{i+1}\},$$

if $b - a + 1 - r$ is even, and

$$\Theta^{a,b}_{r,a+\epsilon}\left\{\left(\frac{v_i}{v_{i+1}}\right)^{(-1)^i}\right\} = (v_a v_{b+1})^{(-1)^{\epsilon+1}} \Theta^{a,b}_{b-a+1-r,\epsilon+1}\{v_i v_{i+1}\},$$

if $b - a + 1 - r$ is odd.

4. Multi-sums of products, Φ

Writing the reduced Lax matrices as linear combinations of rank-1 2×2 matrices gives rise to closed-form expressions in terms of multi-sums of products Φ, defined in equation (23). In this section, we first give recursive formulae for Φ. Then, by using the non-commutative Vieta expansion, we give formula (25) for products of $L_i = r_i(\lambda H + s_i A^i_i)$ matrices in terms of Ψ (26) where A^i_j is defined in (21). This formula is first applied to the mKdV equation. At the end, we give the analogous results for nearly all equations in the Adler–Bobenko–Suris list [1].

4.1. *Recursive formulae for the multi-sums of products, Φ*

The lemma below explains how Φ is derived from the Vieta expansions (8*b*) and (9*b*).

Lemma 6. *Let H and F_i be defined by*

$$H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F_i = \begin{pmatrix} 1 & 0 \\ f_i & 0 \end{pmatrix}$$

and let f be a function on $[a, b+1] \subset \mathbb{Z}$. The multi-sums of products, Φ, defined by (23) appear in the entries of the matrix coefficients in the expansion

$$\prod_{i=a}^{\overset{\frown}{b+1}} (\lambda H + F_i) = \sum_{r=0}^{b-a+2} \lambda^r W^{a,b+1}_r,$$

where

$$W^{a,b+1}_r = \begin{pmatrix} \Phi^{a,b}_r & \Phi^{a+1,b}_{r-1} \\ f_{b+1}\Phi^{a,b-1}_r & f_{b+1}\Phi^{a+1,b-1}_{r-1} \end{pmatrix}.$$

Proof. Using the properties $H^2 = 0$, $F_iF_j = F_i$, $F_iHF_j = f_jF_i$, $HF_iH = f_iH$ and the non-commutative Vieta formula (9*b*), we have

$$\begin{aligned}
W_r^{a,b+1} &= \sum_{a+2\leqslant i_1, i_1+1<i_2, i_2+1<\cdots<i_{r-2}\leqslant b-1} HF_b\ldots F_{i_{r-2}+1}HF_{i_{r-2}-1}\ldots F_{i_1+1}HF_{i_1-1}\ldots F_{a+1}H\\
&\quad+\sum_{a+1\leqslant i_1, i_1+1<i_2, i_2+1<\cdots<i_{r-1}\leqslant b-1} HF_b\ldots F_{i_{r-1}+1}HF_{i_{r-1}-1}\ldots F_{i_1+1}HF_{i_1-1}\ldots F_a\\
&\quad+\sum_{a+2\leqslant i_1, i_1+1<i_2, i_2+1<\cdots<i_{r-1}\leqslant b} F_{b+1}\ldots F_{i_{r-1}+1}HF_{i_{r-1}-1}\ldots F_{i_1+1}HF_{i_1-1}\ldots F_{a+1}H\\
&\quad+\sum_{a+1\leqslant i_1, i_1+1<i_2, i_2+1<\cdots<i_r\leqslant b} F_{b+1}\ldots F_{i_r+1}HF_{i_r-1}\ldots F_{i_1+1}HF_{i_1-1}\ldots F_a\\
&= f_b\Phi_{r-2}^{a+1,b-2}\begin{pmatrix}0&1\\0&0\end{pmatrix}+f_b\Phi_{r-1}^{a,b-2}\begin{pmatrix}1&0\\0&0\end{pmatrix}+\Phi_{r-1}^{a+1,b-1}\begin{pmatrix}0&1\\0&f_{b+1}\end{pmatrix}+\Phi_r^{a,b-1}\begin{pmatrix}1&0\\f_{b+1}&0\end{pmatrix}\\
&=\begin{pmatrix}f_b\Phi_{r-1}^{a,b-2}+\Phi_r^{a,b-1} & f_b\Phi_{r-2}^{a+1,b-2}+\Phi_{r-1}^{a+1,b-1}\\ f_{b+1}\Phi_r^{a,b-1} & f_{b+1}\Phi_{r-1}^{a+1,b-1}\end{pmatrix}.
\end{aligned}$$

From the definition of Φ, we have the following properties:

$$\Phi_r^{a,b} = \Phi_r^{a,b-1} + f_b\Phi_{r-1}^{a,b-2}, \tag{41}$$

$$\Phi_r^{a,b} = f_a\Phi_{r-1}^{a+2,b} + \Phi_r^{a+1,b}. \tag{42}$$

Therefore, we obtain

$$W_r^{a,b+1} = \begin{pmatrix}\Phi_r^{a,b} & \Phi_{r-1}^{a+1,b}\\ f_{b+1}\Phi_r^{a,b-1} & f_{b+1}\Phi_{r-1}^{a+1,b-1}\end{pmatrix}.$$

□

Using this lemma, we derive recursive formulae for Φ.

Proposition 7. *With* $a-1\leqslant c\leqslant b+1$*, we have*

$$\Phi_r^{a,b} = \sum_{i=0}^{r}\left(\Phi_{r-i}^{a,c-1}\Phi_i^{c+1,b} + \Phi_{r-i-1}^{a,c-2}\Phi_1^{c,c}\Phi_i^{c+2,b}\right) \tag{43a}$$

$$= \sum_{i=0}^{r}\left(\Phi_{r-i}^{a,c-1}\Phi_1^{c+1,c+1}\Phi_{i-1}^{c+3,b} + \Phi_{r-i}^{a,c}\Phi_i^{c+2,b}\right). \tag{43b}$$

We note that properties (41) and (42) are special cases of these recursive formulae (43*a*) and (43*b*) where $c = b$ and $c = a-1$, respectively.

Proof. We use recursive formula (27*a*) with W replacing s, c replacing $a+c-1$. We have

$$\begin{aligned}
W_r^{a,b+1} &= \sum_{i=0}^{r} W_i^{c+1,b+1}W_{r-i}^{a,c}\\
&= \sum_{i=0}^{r}\begin{pmatrix}\Phi_i^{c+1,b} & \Phi_{i-1}^{c+2,b}\\ f_{b+1}\Phi_i^{c+1,b-1} & f_{b+1}\Phi_{i-1}^{c+2,b-1}\end{pmatrix}\begin{pmatrix}\Phi_{r-i}^{a,c-1} & \Phi_{r-i-1}^{a+1,c-1}\\ f_c\Phi_{r-i}^{a,c-2} & f_c\Phi_{r-i-1}^{a+1,c-2}\end{pmatrix}.
\end{aligned}$$

Equating the entry in the first column and first row, we get

$$\Phi_r^{a,b} = \sum_{i=0}^{r}\left(\Phi_{r-i}^{a,c-1}\Phi_i^{c+1,b} + \Phi_{r-i-1}^{a,c-2}f_c\Phi_i^{c+2,b}\right),$$

which is the first recursive formula (43*a*). The second recursive formula (43*a*) is obtained from the first one by using the following

$$\Phi_i^{c+1,b} = \Phi_i^{c+2,b} + \Phi_1^{c+1,c+1}\Phi_{i-1}^{c+3,b}, \qquad \Phi_{r-i-1}^{a,c-2}\Phi_1^{c,c} = \Phi_{r-i}^{a,c} - \Phi_{r-i}^{a,c-1}. \qquad \square$$

Note that from the definition of Φ (23), we have the following properties:

- $\Phi_r^{a,b} = 0$ when $r < 0$ or $r > \lfloor (b-a+1)/2 \rfloor$,
- $\Phi_0^{a,b} = 1$,
- $\Phi_1^{a,a} = f_a$.

Once again, these properties can be considered as initial values to calculate Φ using the recursive formulae for Φ provided.

4.2. Product of L matrices and multi-sum of products, Ψ

Let A_j^i and H be matrices that are defined in (21). Now using the Vieta expansions (8*b*) and (9*b*) and the properties of H and A, we obtain the following lemma.

Lemma 8. *Let $L_i = r_i\left(\lambda H + s_i A_i^i\right)$. We have*

$$\prod_{i=a}^{\curvearrowleft b} L_i = \left(\sum_{r=0}^{b-a+1} X_r^{a,b}\lambda^r\right)\prod_{i=a}^{b} r_i,$$

with

$$X_r^{a,b} = \Psi_{r-1}^{a+1,b-2} H A_a^{b-1} + \Psi_{r-1}^{a+2,b-1} A_{a+1}^b H + \Psi_{r-2}^{a+2,b-2} H + \Psi_r^{a+1,b-1} A_a^b, \tag{44}$$

where $c_i = s_i(a_{i-1} + b_i)$ and

$$\Psi_r^{a,b} := s_{a-1}\Phi_r^{a,b}\left\{\frac{s_{i+1}}{c_i c_{i+1}}\right\}\prod_{i=a}^{b+1} c_i. \tag{45}$$

The general case (25) is obtained in the same way.

Proof. For brevity of notation, we write $B_i := s_i A_i^i$. Using properties of H and A_m^n, similar to (24) we have

$$B_k B_{k-1} \dots B_l = s_k A_k^k \dots s_l A_l^l = s_l A_l^k \prod_{r=l+1}^{k} c_r,$$

so when $l < i < k$ we have

$$B_k B_{k-1} \dots B_{i+1} H B_{i-1} \dots B_l = s_{i+1}\left(A_{i+1}^k \prod_{r=i+2}^{k} c_r\right) H s_l \left(A_l^{i-1}\prod_{r=l+1}^{i-1} c_r\right)$$

$$= \frac{s_{i+1}}{c_i c_{i+1}} s_l A_l^k \prod_{r=l+1}^{k} c_r = s_l f_i A_l^k \prod_{r=l+1}^{k} c_r.$$

Therefore, we obtain

$$B_k \dots B_{i_r+1} H B_{i_r-1} \dots B_{i_1+1} H B_{i_1-1} \dots B_l = s_l A_l^k \prod_{j=1}^{r} f_{i_j} \prod_{r=l+1}^{k} c_r$$

when $l < i_1, i_1 + 1 < i_2, \dots < i_{i_r} < k$, and it equals 0 when $i_j = i_{j-1} + 1$ for some j.

Now applying these formulae and formula (9*b*), we have

$$
\begin{aligned}
X_r^{a,b} = & \sum_{a+1\leqslant i_1, i_1+1<i_2,\ldots<i_{r-1}\leqslant b-2} HB_{b-1}\ldots B_{i_{r-1}+1}HB_{i_{r-1}-1}\ldots B_{i_1+1}HB_{i_1-1}\ldots B_a \\
& + \sum_{a+2\leqslant i_1, i_1+1<i_2,\ldots<i_{r-1}\leqslant b-1} B_b\ldots B_{i_{r-1}+1}HB_{i_{r-1}-1}\ldots B_{i_1+1}HB_{i_1-1}\ldots B_{a+1}H \\
& + \sum_{a+2\leqslant i_1, i_1+1<i_2,\ldots<i_{r-2}\leqslant b-2} HB_{b-1}\ldots B_{i_{r-2}+1}HB_{i_{r-2}-1}\ldots B_{i_1+1}HB_{i_1-1}\ldots B_{a+1}H \\
& + \sum_{a+1\leqslant i_1, i_1+1<i_2,\ldots<i_r\leqslant b-1} B_b\ldots B_{i_r+1}HB_{i_r-1}\ldots B_{i_1+1}HB_{i_1-1}\ldots B_a \\
= & \left(\frac{s_a}{c_b}\Phi_{r-1}^{a+1,b-2}HA_a^{b-1} + \frac{s_{a+1}}{c_{a+1}}\Phi_{r-1}^{a+2,b-1}A_{a+1}^bH + \frac{s_{a+1}}{c_{a+1}c_b}\Phi_{r-2}^{a+2,b-2}H + s_a\Phi_r^{a+1,b-1}A_a^b\right)\prod_{i=a+1}^b c_i \\
= & \Psi_{r-1}^{a+1,b-2}HA_a^{b-1} + \Psi_{r-1}^{a+2,b-1}A_{a+1}^bH + \Psi_{r-2}^{a+2,b-2}H + \Psi_r^{a+1,b-1}A_a^b. \qquad \square
\end{aligned}
$$

Note that if $r > \lfloor (b-a+1)/2 \rfloor$, we have $X_r^{a,b} = 0$. We also note that lemma 6 is a special case of this lemma with $a_i = 1/f_i$, $b_i = 0$ and $s_i = f_i$.

4.3. *Application of* Ψ *to the mKdV equation*

In this section, we present closed-form expressions for integrals of mKdV in terms of the multi-sums of products Ψ.

The reduced Lax pair for the mKdV equation given by (35) gives rise to the multi-sums of products, Θ. Now we use a gauge transformation to obtain a new reduced Lax pair which gives the multi-sums of products, Ψ. Recall that for a PΔE (1) with a Lax pair $(L_{l,m}, M_{l,m})$, a gauge matrix $G_{l,m}$ gives us a new Lax pair for the equation, that is,

$$\widetilde{L}_{l,m} = G_{l+1,m}L_{l,m}G_{l,m}^{-1},\ \widetilde{M}_{l,m} = G_{l,m+1}M_{l,m}G_{l,m}^{-1}.$$

These matrices reduce to matrices $\widetilde{L}$, $\widetilde{M}$ of the corresponding OΔE.

Using the gauge matrix

$$G_{l,m} = \begin{pmatrix} 0 & 1/v_{l,m} \\ -1/k & 0 \end{pmatrix},$$

we have a new reduced Lax pair for the mKdV equation:

$$\widetilde{L}_n = \begin{pmatrix} \frac{v_{n+1}}{v_n} & \frac{-k^2}{v_n} \\ -v_{n+1} & 1 \end{pmatrix} \qquad \text{and} \qquad \widetilde{M}_n^{-1} = \begin{pmatrix} \alpha_2\frac{v_n}{v_{n+z_2}} & -\alpha_1\frac{k^2}{v_{n+z_2}} \\ -\alpha_1 v_n & \alpha_3 \end{pmatrix}. \tag{46}$$

We note that the trace of the monodromy matrix is invariant under gauge transformations.

Now we write the reduced Lax pair for the mKdV equation as follows:

$$\widetilde{L}_i = r_i\left(s_iA_i^i + \lambda H\right), \qquad (\widetilde{M}_i)^{-1} = \breve{r}_i\left(\breve{s}_i\breve{A}_i^i + \breve{\lambda}H\right), \tag{47}$$

where

$$\breve{A}_j^i = \begin{pmatrix} \breve{a}_i & \breve{a}_i\breve{b}_j \\ 1 & \breve{b}_j \end{pmatrix},$$

and

$$\lambda = 1-k^2, \qquad r_i = \frac{1}{v_i}, \qquad s_i = -v_iv_{i+1}, \qquad a_i = -\frac{1}{v_i}, \qquad b_i = -\frac{1}{v_{i+1}},$$

$$\breve{\lambda} = \lambda + \frac{\alpha_2\alpha_3 - \alpha_1^2}{\alpha_1}, \qquad \breve{r}_i = \frac{1}{v_{i+z_2}}, \qquad \breve{s}_i = -\alpha_1v_iv_{i+z_2}, \qquad \breve{a}_i = \frac{-\alpha_2}{\alpha_1v_{i+z_2}}, \qquad \breve{b}_i = \frac{-\alpha_3}{\alpha_1v_i}.$$

Using lemma 8, we expand the trace of the monodromy matrix in powers of λ. After multiplying with $\widetilde{M}_0^{-1}$ and taking the trace, we get

$$\mathrm{Tr}\widetilde{\mathcal{L}}_0 = \sum_{i=0}^{z_2+1} \lambda^i \widetilde{I}_i,$$

where

$$\widetilde{I}_r = \left(\alpha_1 \Psi_{r-1}^{1,z_2-2} + \left(\alpha_2 v_0 + \frac{\alpha_1 v_0 v_{z_2}}{v_1}\right)\Psi_{r-1}^{1,z_2-3} + \left(\alpha_3 v_{z_2} + \frac{\alpha_1 v_0 v_{z_2}}{v_{z_2-1}}\right)\Psi_{r-1}^{2,z_2-2} - \alpha_1 v_0 v_{z_2} \Psi_{r-2}^{2,z_2-3}\right.$$
$$\left. - \left(\frac{\alpha_2 v_0}{v_{z_2-1}} + \frac{\alpha_3 v_{z_2}}{v_1} + \frac{\alpha_1 v_0 v_{z_2}}{v_1 v_{z_2-1}} + \alpha_1\right)\Psi_r^{1,z_2-2}\right)\prod_{i=0}^{z_2} v_i^{-1}, \tag{48}$$

with

$$\Psi_r^{a,b} = -v_{a-1} v_a \Phi_r^{a,b}\left\{\frac{v_{i-1} v_{i+2}}{(v_{i-1}+v_{i+1})(v_i+v_{i+2})}\right\}\prod_{i=a}^{b+1}\frac{v_i(v_{i-1}+v_{i+1})}{v_{i-1}} \tag{49}$$

Note that if $r > \lfloor (z_2+1)/2 \rfloor$, then $\widetilde{I}_r = 0$. It is clear by the staircase method that $\widetilde{I}_r$ is an integral of the corresponding mKdV map. However, using properties of Ψ one can also show that $\widetilde{I}_r$ is invariant under the mKdV map. By doing so, we obtain an integrating factor [6] which we do not get from the staircase.

Proposition 9. *For $0 \leqslant r \leqslant \lfloor (z_2+1)/2 \rfloor$, $\widetilde{I}_r$ is an integral of the mKdV map with the integrating factor*

$$\Lambda_r = \frac{1}{v_0 v_1 v_{z_2} v_{z_2+1}}\left(\Psi_r^{1,z_2-1} + v_1 v_{z_2}\Psi_{r-1}^{2,z_2-2} - v_1 \Psi_r^{2,z_2-1} - v_{z_2}\Psi_r^{1,z_2-2}\right)\prod_{i=1}^{z_2} v_i^{-1},$$

where Ψ is given by (49).

The proof of this proposition is given in appendix B.

4.4. General closed-form expressions for integrals of all equations in the ABS list, except Q_4

In this section, we give similar results as we did for the mKdV equation for almost all equations in the ABS list. We present a general formula for closed-form expressions for integrals of all equations whose reduced Lax pairs can be written in form (47). Then, we write the reduced Lax matrices of ABS equations with the exception of Q_4 in this form, so that we can apply the general expressions to obtain integrals.

4.4.1. General closed-form expressions for integrals. Assume that the reduced Lax matrices are written in form (47), with $\breve{\lambda} = g\lambda + h$. We now expand the trace of the monodromy matrix $\mathcal{L}_0$ in terms of powers of λ.

Theorem 10. *Let $\mathcal{L}_0$ be given by (6), where L, M^{-1} are matrices which can be written in form (47). Then, we have*

$$\mathrm{Tr}(\mathcal{L}_0) = \sum_{i=0}^{z_2+1} \lambda^r I_r,$$

where I_r is expressed in terms of Ψ (45):

$$I_r = \big(g\Psi_{r-1}^{1,z_2-2} + h\Psi_r^{1,z_2-2} + \check{s}_0(\check{a}_0 + b_0)\Psi_{r-1}^{1,z_2-3} + \check{s}_0\big(\check{b}_0 + a_{z_2-1}\big)\Psi_{r-1}^{2,z_2-2} + \check{s}_0\Psi_{r-2}^{2,z_2-3} + \check{s}_0(\check{a}_0 + b_0)\big(a_{z_2-1} + \check{b}_0\big)\Psi_r^{1,z_2-2}\big)\check{r}_0\prod_{i=0}^{z_2-1} r_i. \tag{50}$$

Proof. We use lemma 8 to expand the monodromy matrix. We have

$$\mathcal{L}_0 = \big(\check{s}_0\check{A}_0^0 + (g\lambda + h)H\big)\left(\sum_{r=0}^{z_2}\lambda^r X_r^{0,z_2-1}\right)\check{r}_0\prod_{i=0}^{z_2-1} r_i = \left(\sum_{r=0}^{z_2+1}\lambda^r W_r\right)\check{r}_0\prod_{i=0}^{z_2-1} r_i, \tag{51}$$

where

$$W_r = \big(\check{s}_0\check{A}_0^0 + hH\big)X_r^{0,z_2-1} + gHX_{r-1}^{0,z_2-1}.$$

Using properties of H and A, we have

$$\mathrm{Tr}\big(HX_r^{0,z_2-1}\big) = \Psi_r^{1,z_2-2},$$
$$\mathrm{Tr}\big(HX_{r-1}^{0,z_2-1}\big) = \Psi_{r-1}^{1,z_2-2},$$
$$\mathrm{Tr}\big(\check{A}_0^0 X_r^{0,z_2-1}\big) = (\check{a}_0 + b_0)\Psi_{r-1}^{1,z_2-3} + \big(\check{b}_0 + a_{z_2-1}\big)\Psi_{r-1}^{2,z_2-2} + \Psi_{r-2}^{2,z_2-3} + (\check{a}_0 + b_0)\big(a_{z_2-1} + \check{b}_0\big)\Psi_r^{1,z_2-2},$$

which we use to evaluate the trace of (51). □

Now from this theorem, if I_r do not depend on λ, then from the staircase method we have that I_r is invariant. Hence, we have the following corollary.

Corollary 11. *Suppose that $a_{i-1} + b_i$, $\check{a}_0 + b_0$, $a_{z_2-1} + \check{b}_0$, r_i, s_i, $\check{r}_i$, $\check{s}_i$, g and h do not depend on the spectral parameter k. Then I_r given by (50) is an integral.*

Here, we give a direct proof that I_r is an integral of the equation derived from the Lax equation $L_0M_0^{-1} = M_1^{-1}L_{z_2}$.

Proof. Since $L_0M_0^{-1} = M_1^{-1}L_{z_2}$, we have

$$r_0\check{r}_0\big(s_0A_0^0 + \lambda H\big)\big(\check{s}_0\check{A}_0^0 + (g\lambda + h)H\big) = \check{r}_1r_{z_2}\big(\check{s}_1\check{A}_1^1 + (g\lambda + h)H\big)\big(s_{z_2}A_{z_2}^{z_2} + \lambda H\big).$$

If $a_i, \check{a}_i, b_i, \check{b}_i$ do not depend on k (the cases H_1 and H_3), equating coefficients of λ on both sides we obtain $E_i = 0$, $i = 1, 2, \ldots, 6$, with

$$E_1 := r_0\check{r}_0\check{s}_0 - \check{r}_1r_{z_2}gs_{z_2},$$
$$E_2 := r_0\check{r}_0gs_0 - \check{r}_1r_{z_2}\check{s}_1,$$
$$E_3 := r_0\check{r}_0(\check{s}_0\check{b}_0 + gs_0a_0) - \check{r}_1r_{z_2}\big(\check{s}_1\check{a}_1 + gs_{z_2}b_{z_2}\big),$$
$$E_4 := \check{r}_0r_0s_0\check{s}_0(b_0 + \check{a}_0) - r_{z_2}\check{r}_1\check{s}_1s_{z_2}\big(a_{z_2} + \check{b}_1\big),$$
$$E_5 := r_0\check{r}_0s_0(\check{s}_0(b_0 + \check{a}_0)\check{b}_0 + h) - \check{r}_1r_{z_2}\check{s}_1s_{z_2}b_{z_2}\big(a_{z_2} + \check{b}_1\big),$$
$$E_6 := \check{r}_0r_0s_0a_0\check{s}_0(b_0 + \check{a}_0) - \check{r}_1r_{z_2}s_{z_2}\big(\check{s}_1\big(a_{z_2} + \check{b}_1\big)\check{a}_1 + h\big).$$

For the case where $a_i, b_i, \check{a}_i, \check{b}_i$ depend on k, we were able to check that the identities $E_i = 0$ hold for the cases Q_1 and Q_3^0. For the rest of the equations, calculations get complicated as we have to deal with square roots.

Using the fourth and fifth identities, we have $h = \check{s}_0(b_0 + \check{a}_0)\big(b_{z_2} - b_0\big)$. Similarly, using the fourth and sixth identities, we have $h = \check{s}_1(a_0 - \check{a}_1)\big(a_{z_2} + \check{b}_1\big)$. We have

$$S(I_r) = \big(g\Psi_{r-1}^{2,z_2-1} + h\Psi_r^{2,z_2-1} + \check{s}_1(\check{a}_1 + b_1)\Psi_{r-1}^{2,z_2-2} + \check{s}_1\big(\check{b}_1 + a_{z_2}\big)\Psi_{r-1}^{3,z_2-1}$$
$$+ \check{s}_1\Psi_{r-2}^{3,z_2-2} + s_1(\check{a}_1 + b_1)\big(a_{z_2} + \check{b}_1\big)\Psi_r^{2,z_2-1}\big)\check{r}_1\prod_{i=1}^{z_2} r_i.$$

Now we write $S(I_r) - I_r = (A + B)\prod_{i=1}^{z_2-1} r_i$, where

$$A := \check{r}_1 r_{z_2}\big(g\Psi_{r-1}^{2,z_2-1} + \check{s}_1(\check{a}_1 + b_1)\Psi_{r-1}^{2,z_2-2} + \check{s}_1\Psi_{r-2}^{3,z_2-2}\big)$$
$$- r_0\check{r}_0\big(g\Psi_{r-1}^{1,z_2-2} + \check{s}_0\big(\check{b}_0 + a_{z_2-1}\big)\Psi_{r-1}^{2,z_2-2} + \check{s}_0\Psi_{r-2}^{2,z_2-3}\big)$$
$$B := \check{r}_1 r_{z_2}\big(\big(h + \check{s}_1(\check{a}_1 + b_1)\big(a_{z_2} + \check{b}_1\big)\big)\Psi_r^{2,z_2-1} + \check{s}_1\big(\check{b}_1 + a_{z_2}\big)\Psi_{r-1}^{3,z_2-1}\big)$$
$$- \check{r}_0 r_0\big(\big(h + \check{s}_0(\check{a}_0 + b_0)\big(a_{z_2-1} + \check{b}_0\big)\big)\Psi_r^{1,z_2-2} + \check{s}_0(\check{a}_0 + b_0)\Psi_{r-1}^{1,z_2-3}\big).$$

From the properties of Φ (41) and (42), we obtain the following properties of Ψ:

$$\Psi_r^{n,m} = s_{m+1}(a_m + b_{m+1})\Psi_r^{n,m-1} + s_{m+1}\Psi_{r-1}^{n,m-2}, \tag{52}$$

$$\Psi_r^{n,m} = s_{n-1}(a_{n-1} + b_n)\Psi_r^{n+1,m} + s_{n-1}\Psi_{r-1}^{n+2,m}, \tag{53}$$

where $n \leqslant m$ and $0 \leqslant r$. Using these properties (52) and (53), we get

$$A = \big(\check{r}_1 r_{z_2}\big(gs_{z_2}\big(a_{z_2-1} + b_{z_2}\big) + \check{s}_1(\check{a}_1 + b_1)\big) - \check{r}_0 r_0\big(gs_0(a_0 + b_1) + \check{s}_0\big(\check{b}_0 + a_{z_2-1}\big)\big)\big)$$
$$\times \Psi_{r-1}^{2,z_2-2} + \big(\check{r}_1 r_{z_2} gs_{z_2} - r_0\check{r}_0\check{s}_0\big)\Psi_{r-2}^{2,z_2-3} + \big(\check{r}_1 r_{z_2}\check{s}_1 - r_0\check{r}_0 gs_0\big)\Psi_{r-2}^{3,z_2-2}$$
$$= \big(a_{z_2-1}\Psi_{r-1}^{2,z_2-2} + \Psi_{r-2}^{2,z_2-3}\big)\big(\check{r}_1 r_{z_2} gs_{z_2} - r_0\check{r}_0\check{s}_0\big) + \big(b_1\Psi_{r-1}^{2,z_2-2} + \Psi_{r-2}^{3,z_2-2}\big).$$
$$\times \big(\check{r}_1 r_{z_2}\check{s}_1 - \check{r}_0 r_0 gs_0\big) + \big(\check{r}_1 r_{z_2}\big(gs_{z_2}a_{z_2} + \check{s}_1\check{a}_1\big) - r_0\check{r}_0\big(gs_0 a_0 + \check{s}_0\check{b}_0\big)\big)\Psi_{r-1}^{2,z_2-2}$$
$$= -\big(a_{z_2-1}\Psi_{r-1}^{2,z_2-2} + \Psi_{r-2}^{2,z_2-3}\big)E_1 - \big(b_1\Psi_{r-1}^{2,z_2-2} + \Psi_{r-2}^{3,z_2-2}\big)E_2 - \Psi_{r-1}^{2,z_2-2}E_3$$
$$= 0.$$

We also have

$$B = \check{r}_1 r_{z_2}\big(h + \check{s}_1\big(a_{z_2} + \check{b}_1\big)(a_0 - \check{a}_0)\big)\Psi_r^{2,z_2-1} - \check{r}_0 r_0\big(h + s_0(\check{a}_0 + b_0)\big(\check{b}_0 - b_{z_2}\big)\big)\Psi_r^{1,z_2-2}$$
$$- \frac{E_4}{s_0 s_{z_2}}\Psi_r^{1,z_2-1}$$
$$= 0.$$

This proves our statement. □

From this proof, one can derive an integrating factor by dividing E_1, E_2, E_3, E_4 and $h + \check{s}_1\big(a_{z_2} + \check{b}_1\big)(a_0 - \check{a}_0)$, $h + s_0(\check{a}_0 + b_0)\big(\check{b}_0 - b_{z_2}\big)$ by the corresponding equation.

4.4.2. Application to equations in the ABS list In table 1, we give a table for writing the reduced Lax matrices for equations H_1, H_2, H_3 and Q_1, Q_2, Q_3 (see [1]) in form (47). All the reduced Lax pairs given satisfy the conditions in corollary 11, so that we obtain closed-form expressions for the integrals from formula (50).

Table 1. Reduced Lax pairs for equations in the ABS list.

E	λ	g	a_i	b_i	r_i	s_i
		h	$\breve{a}_i$	$\breve{b}_i$	$\breve{r}_i$	$\breve{s}_i$
$\bar{H}_1$	$p-k$	1	v_i	$-v_{i+1}$	1	1
		$q-p$	v_{i+z_2}	$-v_i$	1	1
H_2	$p-k$	1	$p-k+v_i$	$-(p-k+v_{i+1})$	$2\sqrt{p+v_i+v_{i+1}}$	$2/r_i^2$
		$q-p$	$q-k+v_{i+z_2}$	$-(q-k+v_i)$	$2\sqrt{q+v_i+v_{i+z_2}}$	$2/\breve{r}_i^2$
H_3	$\frac{k^2-p^2}{k^2}$	$\frac{q^2}{p^2}$	$\frac{v_i}{p}$	$\frac{-v_{i+1}}{p}$	$\frac{\sqrt{\delta p+v_i v_{i+1}}}{p}$	$1/r_i^2$
		$\frac{p^2-q^2}{p^2}$	$\frac{v_{i+z_2}}{p}$	$\frac{-v_i}{q}$	$\frac{\sqrt{\delta q+v_i v_{i+z_2}}}{q}$	$1/\breve{r}_i^2$
Q_1	$\frac{p-k}{k}$	$\frac{q}{p}$	$\frac{pv_{i+1}/k-v_{i+1}+v_i}{p}$	$-\frac{pv_i/k-v_i+v_{i+1}}{p}$	$\frac{v_i-v_{i+1}-\delta}{p}$	$\frac{p^2}{(v_i-v_{i+1}+p\delta)(v_i-v_{i+1}-p\delta)}$
		$\frac{q-p}{p}$	$\frac{qv_i/k-v_i+v_{i+z_2}}{q}$	$-\frac{qv_{i+z_2}/k-v_{i+z_2}+v_i}{q}$	$\frac{v_i-v_{i+z_2}+\delta q}{q}$	$\frac{q^2}{(v_i-v_{i+z_2}-\delta q)(v_i-v_{i+z_2}+\delta q)}$
Q_2	$\frac{p-k}{k}$	$\frac{q}{p}$	$\frac{(k-p)(\delta pk-v_{i+1})+kv_i}{kp}$	$-\frac{(k-p)(\delta pk-v_i)+kv_{i+1}}{kp}$	$\frac{\sqrt{\delta p^2(\delta p^2-2v_i-2v_{i+1})+(v_i-v_{i+1})^2}}{p}$	$1/r_i^2$
		$\frac{q-p}{p}$	$\frac{(k-q)(\delta qk-v_i)+kv_{i+z_2}}{kq}$	$-\frac{(k-q)(\delta qk-v_{i+z_2})+kv_i}{kq}$	$\frac{\sqrt{\delta q^2(\delta q^2-2v_i-2v_{i+z_2})+(v_i-v_{i+z_2})^2}}{q}$	$1/\breve{r}_i^2$
Q_3^0	$\frac{k^2-p^2}{k^2-1}$	$\frac{q^2-1}{p^2-1}$	$\frac{(p^2-k^2)v_{i+1}+p(k^2-1)v_i}{(k^2-1)(p^2-1)}$	$-\frac{(p^2-k^2)v_i+p(k^2-1)v_{i+1}}{(k^2-1)(p^2-1)}$	$\frac{pv_{i+1}-v_i}{p^2-1}$	$\frac{(p^2-1)^2}{(pv_{i+1}-v_i)(pv_i-v_{i+1})}$
		$\frac{p^2-q^2}{p^{-}1}$	$\frac{(q^2-k^2)v_i+q(k^2-1)v_{i+z_2}}{(k^2-1)(q^2-1)}$	$-\frac{(q^2-k^2)v_{i+z_2}+q(k^2-1)v_i}{(k^2-1)(q^2-1)}$	$\frac{qu_i-v_{i+z_2}}{q^2-1}$	$\frac{(q^2-1)^2}{(qv_{i+z_2}-v_i)(qv_i-v_{i+z_2})}$
Q_3	$\frac{p^2-k^2}{k^2-1}$	$\frac{q^2-1}{p^2-1}$	$\frac{p(k^2-1)v_i+(p^2-k^2)v_{i+1}}{(k^2-1)(p^2-1)}$	$-\frac{p(k^2-1)v_{i+1}+(p^2-k^2)v_i}{(k^2-1)(p^2-1)}$	$\frac{\sqrt{\delta^2(1-p^2)^2+4pv_i v_{i+1}(1-p^2)+4p^2(v_i^2-v_{i+1}^2)}}{(p^2-1)4p}$	$\frac{1}{4pr_i^2}$
		$\frac{q^2-p^2}{p^{-}1}$	$\frac{(q^2-k^2)v_i+q(k^2-1)v_{i+z_2}}{(k^2-1)(q^2-1)}$	$-\frac{(q^2-k^2)v_{i+z_2}+q(k^2-1)v_i}{(k^2-1)(q^2-1)}$	$\frac{\sqrt{\delta^2(1-q^2)^2+4qv_i v_{i+z_2}(1-q^2)+4q^2(v_i^2-v_{i+z_2}^2)}}{4q(q^2-1)}$	$\frac{1}{4q\breve{r}_i^2}$

5. Discussion

5.1. Comparing the two sets of integrals of the mKdV equation

We have obtained two sets of integrals for the mKdV equation in equations (38) and (48), which are expressed in terms of the multi-sums of products Θ and Ψ, respectively. This is because we can expand the trace of the monodromy matrix in powers of either k or $\lambda = 1-k^2$. Therefore, we have

$$\mathrm{Tr}(\mathcal{L}_0) = \sum_{i=0}^{\lfloor((z_2+1)/2)\rfloor} k^{2i} I_i = \sum_{i=0}^{\lfloor(z_2+1)/2\rfloor} (1-k^2)^i \widetilde{I}_i.$$

Equating the coefficient of k^{2r}, we have

$$I_r = \sum_{i=r}^{\lfloor(z_2+1)/2\rfloor} (-1)^r \binom{i}{i-r} \widetilde{I}_i.$$

In particular, we have

$$I_{\lfloor(z_2+1)/2\rfloor} = (-1)^{\lfloor(z_2+1)/2\rfloor} \widetilde{I}_{\lfloor(z_2+1)/2\rfloor}$$

and

$$I_0 = \alpha_2 + \alpha_3 = \sum_{i=0}^{\lfloor(z_2+1)/2\rfloor} \widetilde{I}_i.$$

The latter equation means that the set of non-constant integrals $\{\widetilde{I}_r\}$ is not functionally independent. Explicitly taking $z_2 = 3$, we have

$$\begin{aligned}
I_0 &= \alpha_2 + \alpha_3,\\
I_1 &= \alpha_1 \left(\frac{v_0}{v_2} + \frac{v_0 v_3}{v_1 v_2} + \frac{v_3}{v_1} + \frac{v_2}{v_0} + \frac{v_1 v_2}{v_0 v_3} + \frac{v_1}{v_3}\right) + \alpha_2 \left(\frac{v_1}{v_3} + \frac{v_0 v_1}{v_2 v_3} + \frac{v_0}{v_2}\right) + \alpha_3 \left(\frac{v_3}{v_1} + \frac{v_2 v_3}{v_0 v_1} + \frac{v_2}{v_0}\right),\\
I_2 &= 2\alpha_1
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{I}_0 &= \alpha_1 \left(2 + \frac{v_0}{v_2} + \frac{v_0 v_3}{v_1 v_2} + \frac{v_3}{v_1} + \frac{v_2}{v_0} + \frac{v_1 v_2}{v_0 v_3} + \frac{v_1}{v_3}\right) + \alpha_2 \left(1 + \frac{v_1}{v_3} + \frac{v_0 v_1}{v_2 v_3} + \frac{v_0}{v_2}\right)\\
&\quad + \alpha_3 \left(1 + \frac{v_3}{v_1} + \frac{v_2 v_3}{v_0 v_1} + \frac{v_2}{v_0}\right),\\
\widetilde{I}_1 &= -\alpha_1 \left(4 + \frac{v_0}{v_2} + \frac{v_0 v_3}{v_1 v_2} + \frac{v_3}{v_1} + \frac{v_2}{v_0} + \frac{v_1 v_2}{v_0 v_3} + \frac{v_1}{v_3}\right) - \alpha_2 \left(\frac{v_1}{v_3} + \frac{v_0 v_1}{v_2 v_3} + \frac{v_0}{v_2}\right)\\
&\quad - \alpha_3 \left(\frac{v_3}{v_1} + \frac{v_2 v_3}{v_0 v_1} + \frac{v_2}{v_0}\right),\\
\widetilde{I}_2 &= 2\alpha_1.
\end{aligned}$$

It seems that the set of non-constant integrals I_r expressed in Θ is simpler and shorter than the integrals $\widetilde{I}_r$ expressed in Ψ. However, it is interesting to know that with our Maple program, it took much more time to calculate the integrals in terms of Θ than in terms of Ψ.

Similarly, for the sine-Gordon mapping the set of non-constant integrals expressed in terms of Ψ is not functionally independent and also longer than that expressed in terms of Θ.

5.2. Future work

We have presented a tool to obtain closed-form expressions for integrals in terms of the multi-sums of products Θ and Ψ. This is a first step to prove the integrability of a discrete map in the Liouville–Arnold sense [10, 12] (a map has sufficiently many functionally independent integrals in involution). The recursive formulae for the multi-sums of products make it possible for us to prove functional independence and involutivity which we hope to publish elsewhere [13, 14].

We have given closed-form expressions for integrals of all equations in the ABS list but Q_4. It would be worth studying this exceptional case, as Q_4 is the most general equation in the ABS list.

In this paper, we have considered $(1, z_2)$ traveling wave reductions. It would be interesting to study more general traveling wave reductions (cf [8, 11]).

Another direction of interest is studying more general equations, and systems of equations, which are not necessarily defined on elementary squares (cf [4, 5]).

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Appendix A. Proving lemma 4

Here we give a proof of lemma 4.

Proof. We write $L_i = kJ + Y_i$ as in (14). So we have

$$Y_{i+k}Y_{i+k-1}\dots Y_i = \begin{pmatrix} \frac{v_i}{v_{i+k+1}} & 0 \\ 0 & \frac{v_{i+k+1}}{v_i} \end{pmatrix}.$$

Using the Vieta formula (9*b*) and the above formula, we obtain

$$\widetilde{Z}_r^{a,b-1} = \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b-1} Y_{b-1}\dots Y_{i_r+1}JY_{i_r-1}\dots Y_{i_{r-1}+1}J\dots Y_{i_1+1}JY_{i_1-1}\dots Y_a$$
$$= \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b-1}$$
$$\begin{pmatrix} \frac{v_{i_r+1}}{v_b} & 0 \\ 0 & \frac{v_b}{v_{i_r+1}} \end{pmatrix} J \begin{pmatrix} \frac{v_{i_{r-1}+1}}{v_{i_r}} & 0 \\ 0 & \frac{v_{i_r}}{v_{i_{r-1}+1}} \end{pmatrix} J \dots \begin{pmatrix} \frac{v_{i_1+1}}{v_{i_2}} & 0 \\ 0 & \frac{v_{i_2}}{v_{i_1+1}} \end{pmatrix} J \begin{pmatrix} \frac{v_a}{v_{i_1}} & 0 \\ 0 & \frac{v_{i_1}}{v_a} \end{pmatrix}.$$

If r is even and using the properties $JY = Y^{-1}J$ and $J^2 = I$, then we have

$$\widetilde{Z}_r^{a,b-1} = \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b-1}$$
$$\begin{pmatrix} \frac{v_{i_r+1}}{v_b} & 0 \\ 0 & \frac{v_b}{v_{i_r+1}} \end{pmatrix} \begin{pmatrix} \frac{v_{i_r}}{v_{i_{r-1}+1}} & 0 \\ 0 & \frac{v_{i_{r-1}+1}}{v_{i_r}} \end{pmatrix} \cdots \begin{pmatrix} \frac{v_{i_2}}{v_{i_1+1}} & 0 \\ 0 & \frac{v_{i_1+1}}{v_{i_2}} \end{pmatrix} \begin{pmatrix} \frac{v_a}{v_{i_1}} & 0 \\ 0 & \frac{v_{i_1}}{v_a} \end{pmatrix}$$

$$= \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b-1}$$

$$\begin{pmatrix} \frac{v_a}{v_b}\frac{v_{i_r}v_{i_r+1}}{v_{i_{r-1}}v_{i_{r-1}+1}}\frac{v_{i_{r-2}}v_{i_{r-2}+1}}{v_{i_{r-3}}v_{i_{r-3}+1}}\cdots\frac{v_{i_2}v_{i_2+1}}{v_{i_1}v_{i_1+1}} & 0 \\ 0 & \frac{v_b}{v_a}\frac{v_{i_{r-1}}v_{i_{r-1}+1}}{v_{i_r}v_{i_r+1}}\frac{v_{i_{r-3}}v_{i_{r-3}+1}}{v_{i_{r-2}}v_{i_{r-2}+1}}\cdots\frac{v_{i_1}v_{i_1+1}}{v_{i_2}v_{i_2+1}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{v_a}{v_b}\Theta_{r,0}^{a,b-1} & 0 \\ 0 & \frac{v_b}{v_a}\Theta_{r,1}^{a,b-1} \end{pmatrix}.$$

Similarly, if r is odd, then we have

$$\widetilde{Z}_r^{a,b-1} = \sum_{a\leqslant i_1<i_2<\cdots<i_r\leqslant b-1}$$

$$J\begin{pmatrix} \frac{v_b}{v_{i_r+1}} & 0 \\ 0 & \frac{v_{i_{i_r+1}}}{v_b} \end{pmatrix}\begin{pmatrix} \frac{v_{i_{r-1}+1}}{v_{i_r}} & 0 \\ 0 & \frac{v_{i_r}}{v_{i_{r-1}+1}} \end{pmatrix}\cdots\begin{pmatrix} \frac{v_{i_2}}{v_{i_1+1}} & 0 \\ 0 & \frac{v_{i_1+1}}{v_{i_2}} \end{pmatrix}\begin{pmatrix} \frac{v_a}{v_{i_1}} & 0 \\ 0 & \frac{v_{i_1}}{v_a} \end{pmatrix}$$

$$= J\begin{pmatrix} v_a v_b\Theta_{r,0}^{a,b-1} & 0 \\ 0 & \frac{1}{v_a v_b}\Theta_{r,1}^{a,b-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{v_a v_b}\Theta_{r,1}^{a,b-1} \\ v_a v_b\Theta_{r,0}^{a,b-1} & 0 \end{pmatrix}.$$

□

Appendix B. Direct proof of invariance of integrals of the mKdV equation in terms of Ψ (proposition 9)

Here we give a proof of proposition 9.

Proof. Applying a shift operator on the integral $\widetilde{I}_r$, we obtain

$$S(\widetilde{I}_r) = \left(\alpha_1\Psi_{r-1}^{2,z_2-1} + \left(\alpha_2 v_1 + \frac{\alpha_1 v_1 v_{z_2+1}}{v_2}\right)\Psi_{r-1}^{2,z_2-2} + \left(\alpha_3 v_{z_2+1} + \frac{\alpha_1 v_1 v_{z_2+1}}{v_{z_2}}\right)\Psi_{r-1}^{3,z_2-1}\right.$$
$$\left. - \alpha_1 v_1 v_{z_2+1}\Psi_{r-2}^{3,z_2-2} - \left(\frac{\alpha_2 v_1}{v_{z_2}} + \frac{\alpha_3 v_{z_2+1}}{v_2} + \frac{\alpha_1 v_1 v_{z_2+1}}{v_2 v_{z_2}} + \alpha_1\right)\Psi_r^{2,z_2-1}\right)\prod_{i=1}^{z_2+1} v_i^{-1}.$$

Now we write $S(\widetilde{I}_r) - \widetilde{I}_r = (A+B)\prod_{i=1}^{z_2} v_i^{-1}$, where

$$A = \left(\alpha_1\Psi_{r-1}^{2,z_2-1} + \left(\alpha_2 v_1 + \frac{\alpha_1 v_1 v_{z_2+1}}{v_2}\right)\Psi_{r-1}^{2,z_2-2} - \alpha_1 v_1 v_{z_2+1}\Psi_{r-2}^{3,z_2-2}\right)v_{z_2+1}^{-1}$$
$$- \left(\alpha_1\Psi_{r-1}^{1,z_2-2} + \left(\alpha_3 v_{z_2} + \frac{\alpha_1 v_0 v_{z_2}}{v_{z_2-1}}\right)\Psi_{r-1}^{2,z_2-2} - \alpha_1 v_0 v_{z_2}\Psi_{r-2}^{2,z_2-3}\right)v_0^{-1}$$

$$B = \left(\left(\alpha_3 v_{z_2+1} + \frac{\alpha_1 v_1 v_{z_2+1}}{v_{z_2}}\right)\Psi_{r-1}^{3,z_2-1} - \left(\frac{\alpha_2 v_1}{v_{z_2}} + \frac{\alpha_3 v_{z_2+1}}{v_2} + \frac{\alpha_1 v_1 v_{z_2+1}}{v_2 v_{z_2}} + \alpha_1\right)\Psi_r^{2,z_2-1}\right)v_{z_2+1}^{-1}$$
$$- \left(\left(\alpha_2 v_0 + \frac{\alpha_1 v_0 v_{z_2}}{v_1}\right)\Psi_{r-1}^{1,z_2-3} - \left(\frac{\alpha_2 v_0}{v_{z_2-1}} + \frac{\alpha_3 v_{z_2}}{v_1} + \frac{\alpha_1 v_0 v_{z_2}}{v_1 v_{z_2-1}} + \alpha_1\right)\Psi_r^{1,z_2-2}\right)v_0^{-1}.$$

Using properties (52) and (53), we have

$$\Psi_r^{2,z_2-1} = v_{z_2}v_{z_2+1}\left(\frac{1}{v_{z_2-1}} + \frac{1}{v_{z_2+1}}\right)\Psi_{r-1}^{2,z_2-2} - v_{z_2}v_{z_2+1}\Psi_{r-2}^{2,z_2-3}$$

$$\Psi_{r-1}^{1,z_2-2} = v_0 v_1\left(\frac{1}{v_0} + \frac{1}{v_2}\right)\Psi_{r-1}^{2,z_2-2} - v_0 v_1\Psi_{r-2}^{3,z_2-2}.$$

Substituting these formulae into A, we obtain

$$A = \frac{\Psi_{r-1}^{2,z_2-2}}{v_0 v_{z_2+1}}.\mathcal{F}_{\mathrm{mKdV}}.$$

Applying the properties (52) and (53), we have

$$\Psi_r^{3,z_2-1} = \left(\frac{1}{v_0} + \frac{1}{v_2}\right)\Psi_r^{2,z_2-1} - \frac{\Psi_r^{1,z_2-1}}{v_0 v_1}$$

$$\Psi_{r-1}^{1,z_2-3} = \left(\frac{1}{v_{z_2-1}} + \frac{1}{v_{z_2+1}}\right)\Psi_r^{1,z_2-2} - \frac{\Psi_r^{1,z_2-1}}{v_{z_2} v_{z_2+1}}.$$

Substituting these formulae into B, we get

$$B = \frac{1}{v_0 v_1 v_{z_2} v_{z_2+1}}\left(\Psi_r^{1,z_2-1} - v_1 \Psi_r^{2,z_2-1} - v_{z_2}\Psi_r^{1,z_2-2}\right)\mathcal{F}_{\mathrm{mKdV}}.$$

This proves the statement. □

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