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Almost integrable evolution equations

Peter H. van der Kamp and Jan A. Sanders

Abstract. We present a 2-component equation with exactly two nontrivial generalized symmetries, a counterexample to Fokas' conjecture that equations with as many symmetries as components are integrable. Furthermore we prove the existence of infinitely many evolution equations with finitely many symmetries. We introduce the concept of almost integrability to describe the situation where one has a finite number of symmetries. The symbolic calculus of Gel'fand–Dikiĭ and p-adic analysis are used to prove our results.

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1. Introduction

The use of symmetries in the study of differential equations was initiated by S. Lie. The idea is to consider a differential equation, ordinary or partial, and study the transformation group in the independent and dependent variables that leaves the equation invariant. In practice one studies the infinitesimal problem and one determines the generators of a near-identity transformation. A Lie point symmetry transforms the dependent and independent variables such that the solution set remains invariant. This idea was generalized by E. Noether, who allowed the infinitesimal symmetry to depend on derivatives of the dependent variables, and proved a one-to-one correspondence between these generalized symmetries and conservation laws in the presence of a nondegenerate Lagrangian [14].

The nonlinear evolution equation $u_t = u_3 + uu_1$ (where we use the abbreviation u_n for the n^{th} derivative of u with respect to x) was derived to describe water waves in shallow canals by Korteweg and de Vries [12]. The equation was rediscovered by Kruskal and Zabusky and numerical computations verified the existence of local and stable waves. A bit of analytical understanding came with the discovery of conservation laws other than the ones governing energy and momentum; infinitely many were found in [13]. Inspired by the result of Noether, infinitely many commuting symmetries of the KdV equation were constructed as well; see [9] and [15]. The existence of infinitely many conservation laws was viewed as integrability, a generalization of this concept for Hamiltonian systems in finite-dimensional manifolds. Since then, several notions of integrability have been introduced, and one cannot even begin to describe the literature of this still expanding field. The relations among these different kinds of integrability are strongly supported by computational evidence and sometimes by theorems. In this paper we say that a system is integrable if it has infinitely many generalized symmetries. A good introduction to this subject and *the* general reference is [16]. The background for some of the techniques we are using in this paper can be found in [18], [3].

In 1980 an observation was made at least twice by different authors. In [7] it is stated:

> Another interesting fact regarding the symmetry structure of evolution equations is that in all known cases the existence of one generalized symmetry implies the existence of infinitely many.

And in [11] the same statement is made together with the footnote

> This is not true for systems of equations. For example, the system $u_t = u_2 + (v^2/2)$, $v_t = 2v_2$ has a nontrivial group, but this group is exhausted by the one-parameter (with parameter τ) group of transformations: $u_\tau = u_3 + 3vv_1$, $v_\tau = 4v_3$.

However, although the remark in the footnote is true, the "counterexample" given there is an integrable system, cf. [1]. In spite of this fact Fokas [8] adapted the remark and formulated the following conjecture in 1987.

Conjecture 1 (Fokas)**.** *If a scalar equation possesses at least one time-independent non-Lie point symmetry, then it possesses infinitely many. Similarly for n-component equations, one needs n symmetries.*

Four years later Bakirov [1] published the first example of a non-integrable equation in the possession of a generalized symmetry. This was a fourth order 2-component equation with a sixth order symmetry and it was shown (with extensive computer algebra computations) that there are no other symmetries up to order 53. The authors of [3] proved in 1998 that the system of Bakirov does not possess another symmetry at any higher order, thereby proving that indeed one symmetry does not imply integrability.

For $n = 1$ the conjecture of Fokas is proven to be true for a large class of equations in [18]. In Section 2 an outline of this proof is given, together with references to the development of the symbolic calculus.

In [20] it is stated that the Bakirov equation seems very exceptional. This is not the case, as we show in Section 3 that there are infinitely many families of non-integrable 2-component equations in the possession of nontrivial generalized symmetries. Therefore these kind of equations are as common (or as rare) as integrable equations. We propose to call them almost integrable. This terminology somehow reflects the idea of the conjecture.

Definition 1. An equation is called (symmetry-)**integrable** if it possesses infinitely many generalized symmetries and **almost integrable of depth (at least, at most) n** if there are exactly (at least, at most) n generalized symmetries. When an equation is almost integrable but not integrable we say that it is **almost integrable of finite depth**.

In Section 4 we show the Fokas conjecture to be false for $n = 2$ by constructing three 9-parameter families of 2-component equations which is almost integrable of depth 2.

These results are of importance both to the somewhat philosophical question "What is integrability" and to the more practical question of how to algorithmically check integrability, since some authors take the conjecture as their definition of integrability (cf. [17]). It is our point of view that this last choice does not suffice, and that it is necessary to produce an explicit proof for the existence of infinitely many symmetries. An important object of research is the recognition and classification of integrable (and almost integrable) equations. Some further developments on this subject are included in Section 5.

We felt it appropriate to give detailed proofs (Appendix A). To convince the reader that the approach is largely constructive we have included our counterexample and both symmetries explicitly. Appendix B contains Maple [6] code that produces them.

2. A classification of scalar equations

The class of homogeneous scalar equations with nonnegative scaling parameter of the form

$$u_t = u_m + f(u, u_1, \dots, u_{m-1})$$

is classified with respect to symmetries in [18]. To find equations with symmetries one usually derives obstructions by requiring that the Lie bracket vanishes. We indicate how Sanders and Wang were able to:

1. deduce an obstruction for the general equation to possess a symmetry on an arbitrary order;
2. reduce the number of obstructions to be solved.

The first is done in the symbolic calculus as developed by Gel'fand and Dikiĭ in [10]. See also [24] where the calculus was used in connection with symmetries. There is a one-to-one correspondence between functions like f and symmetric polynomials in the symbol $\xi = (\xi_1, \xi_2)$, i.e.,

$$\widehat{u_i u_j} = \frac{\xi_1^i \xi_2^j + \xi_1^j \xi_2^i}{2} u^2.$$

In the symbolic calculus the obstructions become divisibility conditions, which can be solved for infinitely many orders at once.

For the second aim, an implicit function theorem was formulated and proved. This theorem states that under certain technical conditions the existence of one symmetry ensures us that the number of obstructions to be solved for proving the existence of the other symmetries is finite and independent of the order of the symmetries.

Relative primeness of certain elementary polynomials (so called $\mathcal{G}$-functions) was proved using Diophantine approximation theory [2]. This ultimately provided the proof of Fokas' conjecture for $n = 1$.

3. $\mathcal{B}$-equations

The class $\mathcal{B}$ consists of all equations $\mathcal{B}_a^n(K)$ having the form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} a u_n + K(v, v_1, \dots, v_{n-1}) \\ v_n \end{pmatrix}, \qquad a \in \mathbb{C},$$

where K is a quadratic polynomial with complex coefficients and the order of the equation (n) exceeds the number of derivatives for each term of K. We refer to the parameter a as the eigenvalue of the problem. In general one would have two eigenvalues to deal with, but we assume one of them to be nonzero and scale it to one. The class $\mathcal{B}$ contains many integrable equations as well as almost integrable equations of finite depth. The property which makes $\mathcal{B}$ nice is that all symmetries of almost all $\mathcal{B}_a^n(K)$ are in $\mathcal{B}$. The exceptions are $a = 1$ or $n = 1$, cf. [1], [22].

We use the symbol η to translate v and its derivatives. In the symbolic calculus the condition for $\mathcal{B}_b^m(S)$ to be symmetry of the $\mathcal{B}_a^n(K)$ is (cf. [3], [23])

$$\mathcal{G}_n[a]\ \widehat{S} = \mathcal{G}_m[b]\ \widehat{K}, \tag{1}$$

where the $\mathcal{G}$-functions are defined as follows:

$$\mathcal{G}_n[a](\eta) \stackrel{\text{def}}{=} a(\eta_1 + \eta_2)^n - \eta_1^n - \eta_2^n.$$

Consequently, if we find some b and m such that $\mathcal{G}_a[n]$ divides the product $\mathcal{G}_m[b]\widehat{K}$, then $\widehat{S}$ is a polynomial in η and we have found a symmetry explicitly. Notice that we do not need the implicit function theorem here.

Because the $\mathcal{G}$-functions are symmetric in η, we use

$$X \stackrel{\text{def}}{=} \eta_1 + \eta_2 \text{ and } Y \stackrel{\text{def}}{=} \eta_1\eta_2$$

and remark that all symmetric functions can be written in terms of X and Y. For example every $\mathcal{G}_2[a]$ is written $Y - cX^2$ for some $c \in \mathbb{C}$. It follows from Lemma 4 that we can always find some polynomial $B_m(c)$ such that $Y - cX^2$ divides $\mathcal{G}_m[B_m(c)](X,Y)$. Therefore any $\mathcal{B}_a^2(K)$ is integrable with symmetries on every order.

Consider $\mathcal{B}_a^4(K)$. This system has a symmetry if $\mathcal{G}_4[a]$ divides $\mathcal{G}_m[b]\widehat{K}$. It is possible that some second degree factor $\widehat{F}$ of $\mathcal{G}_4[a]$ divides $\widehat{K}$. But then $\mathcal{G}_4[a]/\widehat{F}$ can be written as $c\mathcal{G}_2[d]$ for some c, d. We see that in this case $\mathcal{B}_a^4(K)$ is a symmetry of $\mathcal{B}_d^2(\widehat{K}/c\widehat{F})$,

$$\mathcal{G}_2[d]\ \widehat{K} = \mathcal{G}_4[a]\ \frac{\widehat{K}}{c\widehat{F}}.$$

In the following, since we want to produce equations that are almost integrable of finite depth, we are going to search for all a for which there exist some b such that $\mathcal{G}_4[a]$ divides $\mathcal{G}_m[b]$ at some order m. Observe that for such pairs $(a,4)$ and (b,m), the result is valid for any (not necessarily homogeneous) K and thus the number of parameters in the family of almost integrable $\mathcal{B}_a^4$ equals six (the number of symmetric polynomials in two variables of degree less than four).

We write $\mathcal{G}_4[a]$ as a function of X, Y and factorize

$$\mathcal{G}_4[a](X,Y) = (a-1)X^4 + 4X^2Y - 2Y^2$$

$$= -2(Y - (1+\alpha)X^2)(Y - (1-\alpha)X^2)$$

with

$$\alpha \stackrel{\text{def}}{=} \sqrt{\frac{a+1}{2}}.$$

We see that $\mathcal{G}_4[a]$ divides $\mathcal{G}_m[B_m(1\pm\alpha)]$ only if $B_m(1+\alpha)$ equals $B_m(1-\alpha)$. This condition is trivially satisfied when $\alpha = 0$. We call the point where this happens $(a = -1)$ the trivial root. It will be removed by division,

$$Q_m(\alpha) \stackrel{\text{def}}{=} \frac{B_m(1+\alpha) - B_m(1-\alpha)}{\alpha}. \tag{2}$$

By the results of Appendix A the roots of $Q_m(\alpha)$ give the eigenvalues a of all almost integrable $\mathcal{B}_a^4(K)$.

Lemma 1. *$Q_m(\alpha)$ has a nontrivial root if $5 < m \neq 1 \bmod 3$.*

Proof. Lemma 5 proves that $Q_m(\alpha)$ is a polynomial in a of degree $\lfloor m/4 - 1/2 \rfloor$, where $\lfloor x \rfloor$ denotes the largest $z \in \mathbb{Z}$ such that $z \leq x$. This means that $Q_m(\alpha)$ has a root for all $m > 5$. Lemma 6 proves that $\alpha = -1$ is a root only if $m = 1 \bmod 3$. This is not surprising, the family $\mathcal{B}^4_{-1}$ is integrable and has symmetries at all orders 1 mod 3, cf. [4]. □

Theorem 1. *There are infinitely many systems which are almost integrable of finite depth.*

Proof. By Lemma 1 we have infinitely many candidates, given by $\mathcal{B}^4_a(K)$ with a the nontrivial roots of $Q_m(\alpha)$. In principle all but a finite number of these roots could equal an eigenvalue of an integrable equation. This is not the case. By Corollary 7 and the results of [4] we know all integrable $\mathcal{B}^4_a$. Their eigenvalues are $a = 0, -1, -3$ and their symmetries appear at orders $m = 4 \bmod 8$, $1 \bmod 3$, $0 \bmod 4$, respectively. Now take for example roots of $Q_m(a)$ for $m = 6 \bmod 12$. The corresponding equations have at least one symmetry and they are not integrable. □

To end the section we remark that the Bakirov equation mentioned in the introduction is $\mathcal{B}^4_5(v^2)$ (its symmetry is $\mathcal{B}^6_{11}(5v_2v_0 + 4v_1^2)$) and that finding all almost integrable $\mathcal{B}^5_a(K)$ can be done with the same method.

4. The counterexample to Fokas' conjecture

Theorem 2. *The 2-component equation*

$$\begin{aligned} u_t &= au_7 + 2vv_2 + cv_1^2 \\ v_t &= v_7 \end{aligned}$$

where $a = -6 - 14\rho + 14\rho^2$, $c = 3 - \rho^2$ and $\rho^3 = \rho + 1$, is almost integrable of depth two.

Proof. We first prove the existence of an 7-th order equation that possesses symmetries at order 11 and 29. Second, we show how to calculate its coefficients and the symmetries. Third, we prove that the equation does not possess another higher order symmetry.

Lemma 2. *Let*

$$A_n(r) \stackrel{\text{def}}{=} \frac{1 + r^n}{(1+r)^n}.$$

We have

- *$(\eta_1 + \eta_2)$ divides $\mathcal{G}_n[a](\eta)$ for all odd $n \in \mathbb{N}$*
- *$(\eta_1 - r\eta_2)(r\eta_1 - \eta_2)$ divides $\mathcal{G}_n[A_n(r)](\eta)$ for all $1 < n \in \mathbb{N}$ and $-1 \neq r \in \mathbb{C}$.*

Proof. By substitution of $\eta_1 = -\eta_2$ in $\mathcal{G}_n[a](\eta)$ and substitution of $\eta_1 = r\eta_2$, $\eta_2 = r\eta_1$ in $\mathcal{G}_n[A_n(r)](\eta)$. □

This Lemma implies that every $\mathcal{G}$-function of the form $\mathcal{G}_7[A_7(r)]$ is divisible by some third degree polynomial. The quotient is a fourth degree symmetric polynomial for which we can determine whether it divides the higher $\mathcal{G}$-functions by using the method described in Section 3,

$$\frac{\mathcal{G}_7[A_7(r)](\eta_1,\eta_2)}{(\eta_1 - r\eta_2)(r\eta_1 - \eta_2)(\eta_1+\eta_2)} = -7(1+r)^4(Y - fX^2)(Y - hX^2)$$

with $f(r), h(r)$ the solutions of

$$(1+r)^4x^2 - (1+r)^2(2r^2+3r+2)x + (r^2+r+1)^2 = 0.$$

As in Section 3 there exists a symmetry at order m if $B_m(f)$ equals $B_m(h)$. The fact that $(B_{11}(f) - B_{11}(h))$ and $(B_{29}(f) - B_{29}(h))$ share the common (nontrivial) factor

$$\mathcal{P}(r) \stackrel{\text{def}}{=} (1+r)^6 + r(1+r)^4 - r^3$$

proves the existence claim.

Take $\mathcal{P}(\beta) = 0$. Then we have $(1+\beta)^2\rho - \beta = 0$ with $\rho^3 - \rho - 1 = 0$. The equation and its symmetries can be expressed in terms of ρ. To obtain the eigenvalue a of the equation we compute $A_7(\beta)$. Since the orders of the symmetries are odd, X is not necessarily a divisor of the quadratic part. To obtain the coefficient c we translate

$$\widehat{K} = (\eta_1 - \beta\eta_2)(\eta_1 - \frac{1}{\beta}\eta_2)v^2$$

back to differential language. We now observe that $A_7(\rho) = a$, i.e., ρ is a root of $\mathcal{G}_7[a]$. Therefore, to obtain the eigenvalues of the symmetries we can compute $A_{11}(\rho)$ and $A_{29}(\rho)$. Finally the quadratic parts are given by

$$\widehat{S}_i = \frac{\mathcal{G}_i[A_i(\rho)]\ \widehat{K}}{\mathcal{G}_7[a]}, \qquad i = 11, 29.$$

The symmetries are

$$u_t = (67 + 66\rho - 88\rho^2)u_{11} + \frac{11}{7}(-1+\rho)\big(4vv_6 + (24 - 6\rho - 6\rho^2)v_1v_5$$

$$+(66 - 16\rho - 22\rho^2)v_2v_4 + (45 - 11\rho - 16\rho^2)v_3^2\big)$$

$$v_t = v_{11}$$

and

$$\begin{aligned}
u_t = {} & (114841 + 139200\rho - 170520\rho^2)u_{29} + \frac{29}{7}(-15 + 10\rho + \rho^2)\,(240vv_{24} \\
& + (5008 - 1682\rho - 1392\rho^2)v_1v_{23} + (59262 - 20344\rho - 18180\rho^2)v_2v_{22} \\
& + (449892 - 149154\rho - 143536\rho^2)v_3v_{21} + (2438248 - 778062\rho \\
& - 801660\rho^2)v_4v_{20} + (10030734 - 3088458\rho - 3383928\rho^2)v_5v_{19} \\
& + (32544982 - 9700890\rho - 11221688\rho^2)v_6v_{18} + (85419408 - 24747138\rho \\
& - 29993464\rho^2)v_7v_{17} + (184573840 - 52205892\rho - 65767368\rho^2)v_8v_{16} \\
& + (332407424 - 92241166\rho - 119786848\rho^2)v_9v_{15} + (503230226 \\
& - 137718440\rho - 182798468\rho^2)v_{10}v_{14} + (644020664 - 174769884\rho \\
& -235057120\rho^2)v_{11}v_{13} + (349486397 - 94573740\rho - 127758838\rho^2)v_{12}^2) \\
v_t = {} & v_{29}
\end{aligned}$$

Now we prove that the equation is almost integrable of depth at most 2. Let δ be a root of $\mathcal{G}_7[a]$ not equal to $-1, \beta, 1/\beta, \rho, 1/\rho$. If the equation has a symmetry on order k, we have

$$A_k(\delta) = A_k(\rho) \tag{3}$$

which can be written in the form

$$U_k \stackrel{\text{def}}{=} a^k + b^k - c^k - d^k = 0.$$

with

$$a = 1 + \rho,\ b = (1 + \rho)\delta,\ c = 1 + \delta,\ d = (1 + \delta)\rho.$$

Now the problem is to find all k for which $U_k = 0$. The following lemma was proved in [3] and used to prove that the Bakirov equation is almost integrable of depth 1.

Lemma 3 (Skolem). *Suppose p is an odd prime. Let $a, b, c, d \in \mathbb{Z}_p$ and suppose they are not zero modulo p. By the little theorem of Fermat we can find unique $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{Z}_p$ such that*

$$a^{p-1} = 1 + p\tilde{a},\ b^{p-1} = 1 + p\tilde{b},\ c^{p-1} = 1 + p\tilde{c},\ d^{p-1} = 1 + p\tilde{d}.$$

Define

$$V_k \stackrel{\text{def}}{=} \tilde{a}a^k + \tilde{b}b^k - \tilde{c}c^k - \tilde{d}d^k.$$

The following holds:

- *If $U_k \not\equiv 0 \bmod p$ then $U_{k+r(p-1)} \neq 0$ for all $r \in \mathbb{Z}$.*
- *If $U_k = 0$ and $V_k \not\equiv 0 \bmod p$ then $U_{k+r(p-1)} = 0$ implies $r = 0$.*

From now on everything can be done p-adically. Due to the lemma of Skolem we only have to check a finite number $(p-1)$ of orders if all the involved roots are in $\mathbb{Z}_p$. Moreover all the calculations can be done modulo p^2. For our purpose the prime $p = 101$ suffices.

We choose $\rho \equiv 20+76p$, a particular root of x^3-x-1 and calculate $a \equiv 62+37p$. We take $\delta = 52+76p$, another root of $\mathcal{G}_7[a]$ that is neither a root of $\mathcal{P}(r)$ nor equal to $1/\rho$. This gives

$$\begin{array}{llll} a \equiv 21+76p, & b \equiv 82+3p, & c \equiv 53+76p, & d \equiv 50+3p, \\ \tilde{a} \equiv 54, & \tilde{b} \equiv 97, & \tilde{c} \equiv 99, & \tilde{d} \equiv 16. \end{array}$$

For $0 \leq k < p-1$ we have $U_k \equiv 0 \bmod p$ only if $k = 0, 1, 7, 11, 29$. Moreover for $k = 0, 1, 7, 11, 29$ we have $V_k \not\equiv 0 \bmod p$. Lemma 3 then states that $k = 0, 1, 7, 11, 29$ are the only values for which condition (3) can be satisfied. $\square$

Corollary 1. *There is still some freedom in the choice of $\widehat{K}$. One can multiply $(\eta_1-\beta\eta_2)(\eta_1-\eta_2/\beta)v^2$ with any symmetric polynomial in η of degree less than 5 not divisible by $Y - fX^2$ or $Y - hX^2$. Although the symmetries evidently change, the proof does not. Also the root ρ can have three different values. Therefore, instead of an example of an equation with 2 symmetries, we have found three 9-parameter families.*

5. Conclusions and further developments

We have presented a counterexample to the conjecture of Fokas. This result puts a burden of proof on anyone claiming (almost) integrability (with respect to generalized symmetries): it is necessary to produce an explicit proof for the existence of finitely or infinitely many symmetries. The symbolic method in combination with number theory is very well suited to this purpose.

With the method employed in Section 3 and 4 we can determine all $\mathcal{B}$-equations at order $4, 5, 6$ and 7 that possesses a symmetry at some fixed order. Another method, using resultants, that covers also the cases where the order of the equations is higher than 7 is introduced in [23].

The method of Skolem depends on the existence of a suitable prime. It is not always easy to find such a prime (try $n = 5, m = 19$). Some refinements were made to make more primes suitable, cf. [23]. Extensive computer calculations, presented in [23], have shown the following.

Theorem 3. *Take $3 < n < 11, n < m < n + 151$ and $m \neq 11, 29$ when $n = 7$. Then, all n-th order non-integrable $\mathcal{B}$-equations with a symmetry of order m are almost integrable of depth 1.*

Thus, the depth of any non-integrable $\mathcal{B}$-equation might be at most 1, except of course for the counterexamples we presented, whose depth is 2. In this respect one could conjecture the following.

Conjecture 2. *Let $r, s \in \mathbb{C}$ and be given such that both r, s are neither roots of unity nor zero and $\overline{r} \neq s, s^{-1}$. Moreover, assume that they are not both zeros of*

$$(x^3 - x - 1)(x^3 + x^2 - 1)(x^6 + 3x^5 + 5x^4 + 5x^3 + 5x^2 + 3x + 1).$$

Then the Diophantine equation

$$(1 + r^n)(1 + s)^n - (1 + s^n)(1 + r)^n = 0$$

has at most two solutions $n > 1$.

The p-adic method of Skolem is used to calculate the depth of almost integrable equations. For recognition and classification of integrable equations we use another method from number theory. This method is based on the Theorem of Lech–Mahler and was, as the p-adic method, introduced first in connection with generalized symmetries in [3].

Theorem 4 (Lech-Mahler)**.** *Let $a_1, a_2, \ldots, a_n, A_1, A_2, \ldots, A_n$ be nonzero complex numbers. Suppose that none of the ratios A_i/A_j with $i \neq j$ is a root of unity. Then the equation*

$$a_1 A_1^k + a_2 A_2^k + \ldots + a_n A_n^k = 0$$

in the unknown integer k has finitely many solutions.

In [3] equations of the form $\mathcal{B}_a^n(v^2)$ with $a \neq 1, n > 1$ where considered. Using the Lech–Mahler Theorem it was proven that such an equation is not integrable when $n \geq 6$ or when $n = 4, 5$ under some condition on the zeros of $\mathcal{G}_n[a]$ [3, Theorem 2.2]. Moreover it was conjectured that this condition was violated in a finite number of cases [3, Conjecture 2.3]. Using an algorithm of C.J. Smyth [5], that solves polynomial equations for roots of unity, this conjecture became a theorem [4, theorem 2.1] and the following list was proven to be exhaustive ($a \in \mathbb{C}, a \neq 0$):

$$\begin{cases} u_t = & au_2 + v^2 \\ v_t = & v_2 \end{cases}, \qquad \begin{cases} u_t = & au_3 + v^2 \\ v_t = & v_3 \end{cases},$$

$$\begin{cases} u_t = & -u_4 + v^2 \\ v_t = & v_4 \end{cases}, \qquad \begin{cases} u_t = & -3u_4 + v^2 \\ v_t = & v_4 \end{cases},$$

$$\begin{cases} u_t = & -\frac{1}{4}u_5 + v^2 \\ v_t = & v_5 \end{cases}, \qquad \begin{cases} u_t = & \frac{-13\pm 5\sqrt{5}}{2}u_5 + v^2 \\ v_t = & v_5 \end{cases},$$

$$\begin{cases} u_t = & u_5 + v^2 \\ v_t = & v_5 \end{cases}, \qquad \begin{cases} u_t = & u_7 + v^2 \\ v_t = & v_7 \end{cases}.$$

A corollary, cf. [4, corollary 2.1], says that each of the above equations with arbitrary quadratic part (in derivatives of v) is integrable as well. It was remarked that the list is not necessarily complete in this more general class $\mathcal{B}$.

The complete classification of integrable $\mathcal{B}$-equations is presented in [21]. In that article a more direct method than the one in [4] is presented by which the eigenvalues of all integrable $\mathcal{B}$-systems at order 4 and 5 can be obtained. This method makes it possible to classify integrable $\mathcal{B}$-systems of higher order, a task that has been carried out till order 23. It was noticed that more structure was to be found in the collection of roots of the $\mathcal{G}$-functions than in the collection of eigenvalues of the integrable $\mathcal{B}$-equations. Unraveling this structure seemed an impracticable task until the day the roots were plotted in the complex plane. A beautiful perfectly regular pattern did arise! To describe this pattern it is convenient to use so called bi-unit coordinates. If $r \in \mathbb{C} \setminus \mathbb{R}$ is the intersection of the lines $\psi\mathbb{R}$ and $\phi\mathbb{R} - 1$ where $|\psi| = |\phi| = 1,\ \psi, \phi \neq \pm 1$ the bi-unit coordinate expression for r is

$$r(\psi, \phi) = \psi^2 \frac{(\phi + 1)(\phi - 1)}{(\psi + \phi)(\psi - \phi)}.$$

Moreover, roots of unity play a special role. The set of all n-th roots of unity ζ where $\zeta \neq \pm 1$ will be denoted by Φ_n. The following Theorem is proven in [21].

Theorem 5. *Let $n > 3$. The eigenvalues of all integrable n-th order $\mathcal{B}$-equations, that are not in a hierarchy of order smaller than 4, are $A_n(r)$ with*

1. $r \in r(\Phi_{2n}, \Phi_{2n})$ *such that* $|r| \neq 1$, *or*
2. $r \in \Phi_{n-1}$, *or*
3. $r \in \Phi_{2n}$ *such that* $r^n = -1$.

In [21] the recognition problem is solved as well. An effective method, using resultants is given to determine whether a given $\mathcal{B}$-equation is integrable and whether the equation is contained in a lower hierarchy.

Although $\mathcal{B}$-equations seems quite special, the implication for the general equation is immediate. Whenever a equation has a part that is a non integrable $\mathcal{B}$-equation, the equation is not integrable. Furthermore $\mathcal{B}$-equations can have quite complicated appearance. An equation in the classification list of second order 2-component equations presented in [19] appeared to be a $\mathcal{B}$-equation after a non-linear transformation [25]. Also the techniques that are employed and developed for the classification and recognition of both integrable and almost integrable $\mathcal{B}$-equations play a role in other classification programs, cf. [19], [22].

Appendix A.

Some technical results

Lemma 4. *For all $c \in \mathbb{C}$ and $1 < m \in \mathbb{N}$ there exist a unique $b \in \mathbb{C}$ such that $Y - cX^2$ divides $\mathcal{G}_m[b](X, Y)$.*

Proof. The polynomials $R_m(\eta) \stackrel{\text{def}}{=} \eta_1^m + \eta_2^m$ satisfy the recurrence relation

$$R_0 = 2, R_1 = X, R_{m+2} = XR_{m+1} - YR_m.$$

We substitute $Y = cX^2$ and obtain $R_m(X, Y) = B_m(c)X^m$ with

$$B_n(c) = \left(\frac{1+\sqrt{1-4c}}{2}\right)^n + \left(\frac{1-\sqrt{1-4c}}{2}\right)^n.$$

We see that $\mathcal{G}_m[b](X, Y) = bX^m - R_m(X, Y)$ is in the ideal of $Y - cX^2$ if $b = B_m(c)$. □

Lemma 5. *$Q_m(\alpha(a))$ is a polynomial in a of degree $\lfloor \frac{m-2}{4} \rfloor$.*

Proof. Using the binomial formula one can see that

$$\frac{(1+\alpha)^n - (1-\alpha)^n}{\alpha}.$$

has degree $\lfloor \frac{n-1}{2} \rfloor$. By Lemma 4 the highest power of the polynomial $B_m(x)$ is $\lfloor \frac{m}{2} \rfloor$. It follows that $Q_m(\alpha(a))$ is polynomial in a and has degree $\lfloor \frac{\lfloor \frac{m}{2} \rfloor - 1}{2} \rfloor$. □

Lemma 6. *$a = -1$ is a root of $Q_m(\alpha(a))$ only if $m = 1 \bmod 3$.*

Proof. Look at the multiplicity of α in $P_m(\alpha) \stackrel{\text{def}}{=} \alpha Q_m(\alpha)$ in the point where $a = -1$. One sees that $P_m'(0) = 0$ only if $m = 1 \bmod 3$. □

We look at the subclass of $\mathcal{B}$ where the eigenvalue is zero.

Lemma 7. *For all n the family $\mathcal{B}_0^n$ is integrable. There exists symmetries of order $n \bmod 2n$.*

Proof. There exists a symmetry $\mathcal{B}_0^m(S)$ if $\mathcal{G}_n[0]$ divides $\mathcal{G}_n[0]$. We have $\eta_1^n + \eta_2^n = 0$ implies $\eta_1^m + \eta_2^m = 0$ when $m = (2k+1)n$. □

Appendix B.

MAPLE code to compute the counterexample and its symmetries

```
G:=unapply(a*(x+y)^n-x^n-y^n,a,n):
A:=unapply(factor((1+r^n)/(1+r)^n),r,n):
alias(rho=RootOf(X^3-X-1,X)):
beta:=RootOf((X+1)^2*rho-X):
K:=x^2+y^2-factor(beta+1/beta)*x*y:
for i in [7,11,29] do
 a||i:=factor(A(rho,i)):
 K||i:=[a||i*u[i]+factor(TRANS(factor(G(a||i,i)/G(a7,7)*K))),v[i]]
od;
```

We used the program `TRANS` that translates polynomials in `x,y` to quadratic polynomials in derivatives of the function v.

```
TRANS:=proc(P)
local R,e,i,Q:
R:=0:Q:=expand(P):
if type(Q,`+`) then
 Q:=convert(Q,list) else
 Q:=[Q] fi:
for e in Q do
 e:=e*v[degree(e,x)]/x^degree(e,x)*v[degree(e,y)]/y^degree(e,y):
 R:=R+e od:
RETURN(R) end:
```

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P.H. van der Kamp
Vrije Universiteit
Faculty of Sciences
Department of Mathematics
De Boelelaan 1081A, Amsterdam
The Netherlands
e-mail: peter@cs.vu.nl

J.A. Sanders
Vrije Universiteit
Faculty of Sciences
Department of Mathematics
De Boelelaan 1081A, Amsterdam
The Netherlands
e-mail: jansa@cs.vu.nl