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Integrability of auto-Bäcklund transformations and solutions of a torqued ABS equation

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Abstract

An auto-Bäcklund transformation for the quad equation $Q1_1$ is considered as a discrete equation, called $H2^a$, which is a so called torqued version of H2. The equations $H2^a$ and $Q1_1$ compose a consistent cube, from which an auto-Bäcklund transformation and a Lax pair for $H2^a$ are obtained. More generally it is shown that auto-Bäcklund transformations admit auto-Bäcklund transformations. Using the auto-Bäcklund transformation for $H2^a$ we derive a seed solution and a one-soliton solution. From this solution it is seen that $H2^a$ is a semi-autonomous lattice equation, as the spacing parameter q depends on m but it disappears from the plane wave factor.

Keywords: auto-Bäcklund transformation, consistency, Lax pair, soliton solution, torqued ABS equation, semi-autonomous

1. Introduction

The subtle concept of integrability touches on global existence and regularity of solutions, exact solvability, as well as compatibility and consistency (see [1]). In the past two decades, the study of discrete integrable systems has achieved a truly significant development, which mainly relies on the effective use of the property of multidimensional consistency (MDC). In the two-dimensional case, MDC means the equation is consistent around the cube (CAC) and this implies it can be embedded consistently into lattices of dimension 3 and higher [2–4]. In 2003, Adler, Bobenko and Suris (ABS) classified scalar quadrilateral equations that are CAC (with extra restrictions: affine linear, D4 symmetry and tetrahedron property) [5]. The complete list contains 9 equations.

In this paper, our discussion will focus on two of them, namely

$$\begin{aligned} &Q1_\delta(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}; p, q) \\ &\quad = p(u-\hat{u})(\tilde{u}-\hat{\tilde{u}}) - q(u-\tilde{u})(\hat{u}-\hat{\tilde{u}}) \\ &\quad + \delta pq(p-q) = 0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} H2(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}; p, q) = {}&(u-\hat{\tilde{u}})(\tilde{u}-\hat{u}) \\ &+ (q-p)(u+\tilde{u}+\hat{u}+\hat{\tilde{u}}) \\ &+ q^2 - p^2 = 0. \end{aligned} \tag{1.2}$$

Here $u = u(n, m)$ is a function on $\mathbb{Z}^2$, p and q are spacing parameters in the n and m direction respectively, δ is an arbitrary constant which we set equal to 1 in the sequel, and conventionally, tilde and hat denote shifts, i.e.

$$\begin{aligned} u = u(n, m), \quad \tilde{u} &= u(n+1, m), \\ \hat{u} = u(n, m+1), \quad \hat{\tilde{u}} &= u(n+1, m+1). \end{aligned} \tag{1.3}$$

H2 is a new equation due to the ABS classification, while $Q1_\delta$ extends the well known cross-ratio equation, or lattice Schwarzian Korteweg–de Vries equation $Q1_{\delta=0}$. Note that spacing parameters p and q can depend on n and m respectively, which leads to nonautonomous equations.

For a quadrilateral equation that is CAC the equation itself defines its own (natural) auto-Bäcklund transformation (auto-BT), see [5]. For example, the system

$$Q1_\delta(u, \tilde{u}, \overline{u}, \overline{\tilde{u}}; p, r) = 0, \quad Q1_\delta(u, \hat{u}, \overline{u}, \overline{\hat{u}}; q, r) = 0,$$

where r acts as a wave number, composes an auto-BT between $Q1_\delta(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}; p, q) = 0$ and $Q1_\delta(\overline{u}, \overline{\tilde{u}}, \overline{\hat{u}}, \overline{\hat{\tilde{u}}}; p, q) = 0$.

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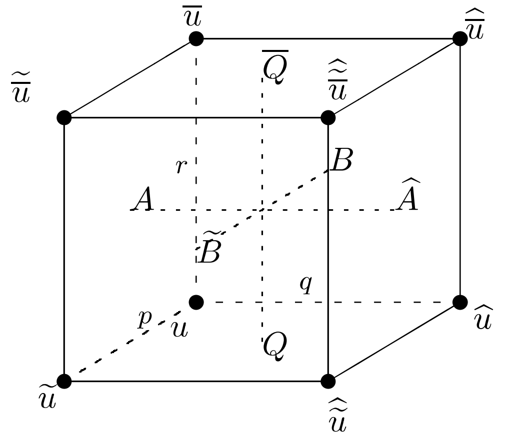

Figure 1. Consistent cube with equations A, B and Q on its faces.

Such a property has been employed in solving CAC equations, see e.g. [6–10].

Some CAC equations allow auto-BTs of other forms. For example, in [11] it was shown that the coupled system

$$A:\ (u-\tilde{u})(\widetilde{\overline{u}}-\overline{u})-p(u+\tilde{u}+\overline{u}+\widetilde{\overline{u}}+p+2r)=0, \tag{1.4a}$$

$$B:\ (u-\hat{u})(\widehat{\overline{u}}-\overline{u})-q(u+\hat{u}+\overline{u}+\widehat{\overline{u}}+q+2r)=0 \tag{1.4b}$$

provides an auto-BT between

$$Q:\ \mathrm{Q1}_1(u,\tilde{u},\hat{u},\widehat{\tilde{u}};p,q)=0 \tag{1.5}$$

and $\overline{Q}$: $\mathrm{Q1}_1(\overline{u},\widetilde{\overline{u}},\widehat{\overline{u}},\widehat{\widetilde{\overline{u}}};p,q)=0$, and, that H2 acts as a nonlinear superposition principle for the BT (1.4). One can think of the auto-BT as equations posed on the side faces of a consistent cube with Q and $\overline{Q}$ respectively on the bottom and the top face, as in figure 1. Here one interprets $\overline{u}=u(n,m,l+1)$, and r serves as a spacing parameter for the third direction l. The superposition principle can be understood as consistency of a 4D cube, see [12, 13].

In [14] the auto-BT (1.4) and its superposition principle have been derived from the natural auto-BT for H2, employing a transformation of the variables and the parameters. The equation

$$\mathrm{H2}^a(u,\tilde{u},\hat{u},\widehat{\tilde{u}};p,q)=\mathrm{H2}(u,\widehat{\tilde{u}},\hat{u},\tilde{u};p+q,q)$$
$$=(u-\tilde{u})(\widehat{\tilde{u}}-\hat{u})-p(u+\tilde{u}+\hat{u}+\widehat{\tilde{u}}+p+2q)=0 \tag{1.6}$$

was identified as a torqued version of the equation H2. The superscript ^{a} refers to the *a*dditive transformation of the spacing parameter. In [11], equation (1.6) appeared as part of an auto-BT for $\mathrm{Q1}_1$. The corresponding consistent cube is a special case of [15, equation (3.9)]. In [14], equation (1.6) was shown to be an integrable equation in its own right, with an asymmetric auto-BT given by $A=\mathrm{H2}^a=0$ and $B=\mathrm{H2}=0$. Here we provide an alternative auto-BT for equation (1.6) to the one that was provided in [14].

In section 2, we establish a simple but quite general result, namely that if a system of equations $A=B=0$ comprises an auto-BT, then both equations $A=0$ and $B=0$ admit an auto-BT themselves. In particular, the equation $\mathrm{H2}^a$ given by (1.6) is CAC, with $\mathrm{H2}^a$ and $\mathrm{Q1}_1$ providing its an auto-BT. We construct a Lax pair for $\mathrm{H2}^a$, which is asymmetric. In section 3, we employ the auto-BT for $\mathrm{H2}^a$ to derive a seed-solution and the corresponding one-soliton solution. In the seed-solution the spacing parameter q depends explicitly on m, which makes $\mathrm{H2}^a$ inherent semi-autonomous. Some conclusions are presented in section 4.

2. Auto-BTs for auto-BTs and a Lax pair for H2^a

To have a consistent cube with $\mathrm{H2}^a$ and $\mathrm{Q1}_1$ on the side faces, providing an auto-BT for $\mathrm{H2}^a$, we assign equations to six faces as follows:

$$Q:\ \mathrm{H2}^a(u,\tilde{u},\hat{u},\widehat{\tilde{u}};p,q)=0,\quad \overline{Q}:\ \mathrm{H2}^a(\overline{u},\widetilde{\overline{u}},\widehat{\overline{u}},\widehat{\widetilde{\overline{u}}};p,q)=0, \tag{2.1a}$$

$$A:\ \mathrm{Q1}_1(u,\tilde{u},\overline{u},\widetilde{\overline{u}};p,r)=0,\quad \widehat{A}:\ \mathrm{Q1}_1(\hat{u},\widehat{\tilde{u}},\widehat{\overline{u}},\widehat{\widetilde{\overline{u}}};p,r)=0, \tag{2.1b}$$

$$B:\ \mathrm{H2}^a(u,\overline{u},\hat{u},\widehat{\overline{u}};r,q)=0,\quad \widetilde{B}:\ \mathrm{H2}^a(\tilde{u},\widetilde{\overline{u}},\widehat{\tilde{u}},\widehat{\widetilde{\overline{u}}};r,q)=0. \tag{2.1c}$$

Then, given initial values u, $\tilde{u}$, $\hat{u}$, $\overline{u}$, by direct calculation, one can find that the value $\widehat{\widetilde{\overline{u}}}$ is uniquely determined. Thus, the cube in figure 1 with (2.1) is a consistent cube.

By means of such a consistency, the side equations A and B, i.e.

$$A:\ p(u-\overline{u})(\tilde{u}-\widetilde{\overline{u}})-r(u-\tilde{u})(\overline{u}-\widetilde{\overline{u}})+pr(p-r)=0, \tag{2.2a}$$

$$B:\ (u-\overline{u})(\widehat{\overline{u}}-\hat{u})-r(u+\overline{u}+\hat{u}+\widehat{\overline{u}}+r+2q)=0, \tag{2.2b}$$

compose an auto-BT for the $\mathrm{H2}^a$ equation (1.6). Here r acts as the Bäcklund parameter.

We note that the order of the variables in the equations (2.1) is quite particular. Since equation (1.6) is not D4 symmetric, i.e. we have

$$\mathrm{H2}^a(u,\overline{u},\hat{u},\widehat{\overline{u}};r,q)\neq\mathrm{H2}^a(u,\hat{u},\overline{u},\widehat{\overline{u}};q,r),$$

one has to be careful. The above result is explained by the following general result, see [16, section 2.1] where the same idea was used to reduce the number of triplets of equations to consider for the classification of consistent cubes.

Lemma 2.1. *Let*

$$A(u,\tilde{u},\overline{u},\widetilde{\overline{u}};p,r)=0,\quad B(u,\hat{u},\overline{u},\widehat{\overline{u}};q,r)=0 \tag{2.3}$$

be an auto-BT for

$$Q(u,\tilde{u},\hat{u},\widehat{\tilde{u}};p,q)=0. \tag{2.4}$$

Then we have (i)

$$Q(u,\tilde{u},\bar{u},\tilde{\bar{u}};p,r)=0,\quad B(u,\bar{u},\hat{u},\hat{\bar{u}};r,q)=0 \tag{2.5}$$

is an auto-BT for

$$A(u,\tilde{u},\hat{u},\hat{\tilde{u}};p,q)=0; \tag{2.6}$$

and (ii)

$$Q(u,\bar{u},\tilde{u},\tilde{\bar{u}};r,p)=0,\quad A(u,\bar{u},\hat{u},\hat{\bar{u}};r,q)=0 \tag{2.7}$$

is an auto-BT for

$$B(u,\tilde{u},\hat{u},\hat{\tilde{u}};p,q)=0. \tag{2.8}$$

Proof. If $A=B=0$ is an auto-BT of $Q=0$, then they compose a consistent cube as in figure 1. We prove the result by relabeling the fields at the vertices, see [13, lemma 2.1]. For (i) we interchange $\hat{u}\leftrightarrow\bar{u}$ and $q\leftrightarrow r$, and for (ii) we perform the cyclic shifts $\hat{u}\to\tilde{u}\to\bar{u}\to\hat{u}$ and $q\to p\to r\to q$. □

Applying (i) to the consistent cube with (1.4*a*) and (1.5) we obtain (2.1*a*). Applying (ii) yields the same, as $Q1_1$ has D4 symmetry.

3D consistency can be used to construct Lax pairs for quadrilateral equations (see [3, 5, 17]). To achieve a Lax pair for $H2^a$, we rewrite (2.2*a*) as

$$\tilde{\bar{u}}=\frac{u(p\tilde{u}-r\bar{u})+(p-r)(pr-\tilde{u}\bar{u})}{(p-r)u+r\tilde{u}-p\bar{u}}, \tag{2.9a}$$

$$\hat{\bar{u}}=-r+\hat{u}-\frac{2r(q+\hat{u}+u)}{r-u+\bar{u}}. \tag{2.9b}$$

Then, introducing $\bar{u}=G/F$ and $\varphi=(G,F)^{\mathrm{T}}$, from (2.9*a*) we have

$$\tilde{\varphi}=L\varphi,\quad \hat{\varphi}=M\varphi, \tag{2.10}$$

where

$$L=\gamma\begin{pmatrix}-ur-(p-r)\tilde{u} & pu\tilde{u}+(p-r)pr\\ -p & (p-r)u+r\tilde{u}\end{pmatrix},$$

$$M=\gamma'\begin{pmatrix}\hat{u}-r & (-r+\hat{u})(r-u)-2r(q+u+\hat{u})\\ 1 & r-u\end{pmatrix},$$

with $\gamma=\frac{1}{\sqrt{p^2-(u-\tilde{u})^2}}$, $\gamma'=\frac{1}{\sqrt{q+u+\hat{u}}}$. The linear system (2.10) is compatible for solutions of (1.6) in the sense that $H2^a$ is a divisor of $(\widehat{L}M)^2-(\widetilde{M}L)^2$, where the square can be taken either as matrix multiplication, or as component-wise multiplication.

3. Seed and one-soliton solution

In this section, we use the auto-BT (2.2*a*) to construct solutions for (1.6). First, we need to have a simple solution as a 'seed'. To find such a solution, we take $\bar{u}=u$ in the BT (2.2*a*), i.e.

$$(u-\tilde{u})^2=p(p-r),\quad u+\hat{u}=-q-\frac{r}{2}. \tag{3.1}$$

This so-called *fixed point approach* has proved to be effective in finding seed solutions [6, 8].

Proposition 3.1. *Parametrizing*

$$p=\frac{\alpha}{a},\quad \alpha=-\frac{ac}{a^2-1},\quad q=(-1)^m\beta-\frac{c}{2}, \tag{3.2}$$

and setting the seed BT parameter equal to r = c, *the equations (3.1) allow the solution*

$$u_0=(-1)^m(\alpha n+\beta m+c_0), \tag{3.3}$$

where c_0 *is a constant.*

Proof. By direct calculation, with the given parameterizations the equations (3.1) read

$$(u-\tilde{u})^2=\alpha^2,\ u+\hat{u}=(-1)^{m+1}\beta.$$

□

It can be verified directly that (3.3) also provides a solution to (1.6). Next, we derive the one-soliton solution for (1.6), from the auto-BT (2.2*a*) with $u=u_0$ as a seed solution.

Proposition 3.2. *The equation (1.6), with lattice parameters (3.2) admits the one-soliton solution*

$$u_1=(-1)^m\left(\alpha n+\beta m+c_0+\frac{ck}{1-k^2}\frac{1-\rho_{n,m}}{1+\rho_{n,m}}\right), \tag{3.4}$$

where

$$\rho_{n,m}=\rho_{0,0}\left(\frac{a+k}{a-k}\right)^n\prod_{i=0}^{m-1}\frac{(-1)^i-k}{(-1)^i+k} \tag{3.5}$$

with constant $\rho_{0,0}$, *is the plane wave factor.*

Proof. Let

$$u_1=u_0+(-1)^m(\kappa+\nu), \tag{3.6}$$

where $\kappa=kr$. With (3.2) and parametrizing the first BT parameter by

$$r=\frac{c}{1-k^2}, \tag{3.7}$$

then substitution of $u=u_0$ and $\bar{u}=u_1$ into the auto-BT (2.2*a*) yields

$$\tilde{\nu}=\frac{\nu E_+}{\nu+E_-},\quad \hat{\nu}=\frac{\nu F_+(m)}{\nu+F_-(m)}, \tag{3.8}$$

where

$$E_\pm=-r(a\pm k),\quad F_\pm(m)=r((-1)^m\mp k). \tag{3.9}$$

The difference system (3.8) can be linearized using $\nu = \frac{f}{g}$ and $\Phi = (f,\, g)^{\mathrm{T}}$, which leads to

$$\Phi(n+1,\, m) = M\Phi(n,\, m), \quad \Phi(n,\, m+1) = N(m)\Phi(n,\, m), \tag{3.10}$$

where

$$M = \begin{pmatrix} E_+ & 0 \\ 1 & E_- \end{pmatrix}, \quad N(m) = \begin{pmatrix} F_+ & 0 \\ 1 & F_- \end{pmatrix}. \tag{3.11}$$

By 'integrating' (3.10) we have

$$\Phi(n,\, m) = \mathcal{M}(n)\Phi(0,\, m), \quad \Phi(n,\, m) = \mathcal{N}(m)\Phi(n,\, 0), \tag{3.12}$$

where

$$\mathcal{M}(n) = \begin{pmatrix} E_+^n & 0 \\ \dfrac{E_-^{\,n} - E_+^n}{2\kappa} & E_-^{\,n} \end{pmatrix},$$

$$\mathcal{N}(m) = \begin{pmatrix} \prod_{i=0}^{m-1} F_+(i) & 0 \\ \dfrac{1-(-1)^m}{2} \prod_{i=0}^{m-2} F_+(i) & \prod_{i=0}^{m-1} F_-(i) \end{pmatrix}.$$

Thus, we get a solution to (3.12):

$$\Phi(n,\, m) = \mathcal{M}(n)\mathcal{N}(m)\Phi(0,\, 0), \tag{3.13}$$

from which $\nu = f/g$ is obtained as

$$\nu = \frac{E_+^n \prod_{i=0}^{m-1} F_+(i) \cdot \nu_{0,0}}{E_-^{\,n} \prod_{i\,=\,0}^{m-1} F_-(i) + \frac{(E_-^n \prod_{i=0}^{m-1} F_-(i) - E_+^n \prod_{i=0}^{m-1} F_+(i))\nu_{0,0}}{2\kappa}}, \tag{3.14}$$

where $\nu_{0,0} = \frac{f_{0,0}}{g_{0,0}}$. Introducing the plane wave factor

$$\begin{aligned} \rho_{n,m} &= \rho_{0,0}\left(\frac{E_+}{E_-}\right)^n \prod_{i\,=\,0}^{m-1} \frac{F_+(i)}{F_-(i)} \\ &= \rho_{0,0}\left(\frac{a+k}{a-k}\right)^n \prod_{i=0}^{m-1} \frac{(-1)^i - k}{(-1)^i + k} \end{aligned} \tag{3.15}$$

with constant $\rho_{0,0}$, the above ν is written as

$$\nu = \frac{-2\kappa\rho_{n,m}}{1+\rho_{n,m}}, \tag{3.16}$$

where some constants are absorbed into $\rho_{0,0} = \frac{-\nu_{0,0}}{2\kappa + \nu_{0,0}}$. Substituting (3.16) into (3.6) yields the one-soliton solution (3.4), which solves (1.6) with (3.2) and (3.7). Note that in the plane wave factor (3.15) $n,\, m \in \mathbb{Z}$, and when $m \leqslant 0$ the product $\prod_{i=0}^{m-1}(\cdot)$ is considered as $\prod_{i=m-1}^{0}(\cdot)$. □

It is interesting that the solution has an oscillatory factor $(-1)^m$ in m-direction and in the plane wave factor $\rho_{n,m}$ the spacing parameter q for m-direction does not appear. Considering the parameterization (3.2) where p is constant while q depends on m, we can say that the H2^a equation (1.6) is semi-autonomous.

4. Conclusions

In this paper, we have shown that equations which constitute an auto-BT for a quad equation admit auto-BTs themselves. We have focussed on one such equation, the torqued H2 equation denoted H2^a (1.6), which forms an auto-BT for Q1$_1$. This equation is not part of the ABS list of CAC quad equations, as it is not symmetric with respect to $(n,\, p) \leftrightarrow (m,\, q)$. The integrability of this equation is guaranteed as it is part of a consistent cube, see [14]. The equations H2^a and Q1$_1$ comprise an auto-BT from which a Lax pair was obtained. Using this auto-BT we have derived a seed solution and a one-soliton solution. The parameterization of these solutions show that H2^a is a semi-autonomous equation. We hope to be able to construct higher order soliton solutions in a future paper.

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