The staircase method

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Abstract

We show, in full generality, that the staircase method provides integrals for mappings, or correspondences, obtained as traveling wave reductions of integrable partial difference equations. We apply the staircase method to a variety of equations, including the Korteweg-De Vries equation, the five-point Bruschi-Calogero-Droghei equation, the QD-algorithm, and the Boussinesq system. We show that, in all these cases, if the staircase method provides r integrals for a n-dimensional mapping, with $2r < n$, then one can introduce $q \leq 2r$ variables, which reduce the dimension of the mapping from n to q. These dimension reducing variables are obtained as joint invariants of k-symmetries of the mappings. Our results support the idea that the staircase method provides sufficiently many integrals for the periodic reductions of integrable lattice equations to be completely integrable. We also prove linear growth of the multi-valuedness of iterates of high-dimensional correspondences obtained as reductions of the QD-algorithm.

1 Introduction

The field of integrable partial difference equations emerged in the late nineteen seventies, early eighties [1, 9, 12, 19, 25]. An important, and well-studied, class of partial difference equations is the class of (scalar) equations that are defined on the elementary squares of a lattice. The integrable equations in this class, such as the lattice potential Korteweg-De Vries (pKdV) equation

$$(u_{l,m} - u_{l+1,m+1})(u_{l+1,m} - u_{l,m+1}) = \alpha. \tag{1}$$

are part of a multi-dimensional family of mutually consistent partial difference equations [16, 20]. A classification with respect to multi-dimensional consistency has been achieved recently [2, 3].

For lattice equations on the square, an initial value problem can be posed on a so called *staircase*: a connected path which is nondecreasing, or nonincreasing. In [23] initial values are given at lattice points $u_{l,l}$ and $u_{l+1,l}$ which satisfy the periodicity $u_{l,m} = u_{l+p,m+p}$. By doing so, the partial difference equation reduces to a multi-dimensional mapping. The authors of [23] used the linear spectral problem (Lax pair) of the lattice pKdV equation to derive a set of polynomial invariants for this mapping. They constructed a so called monodromy matrix, which is an ordered product of Lax matrices along the staircase over a one-period distance. This method, nowadays known as *the staircase method*, is an important tool in proving complete integrability in the sense of Liouville-Arnold, that is, a $2n$-dimensional mapping is said to be completely integrable if it admits n functionally independent integrals in involution with respect to a symplectic form [6, 30].

In [8, 17] the authors established the involutivity of the integrals for the mappings they introduced in [23]. Similar results have been obtained for mappings derived from the lattice Gel'fand-Dikii hierarchy [18] and for reductions of the time-discrete versions of the Bogoyavlensky equations [22]. In [24] more general staircases were given, corresponding to the **s**-*periodicity condition:*[1]

$$u_{l,m} = u_{l+s_1,m+s_2}, \tag{2}$$

where $s_1, -s_2 \in \mathbb{N}$ are relatively prime integers. The authors also suggested considering general $s_1, -s_2 \in \mathbb{N}^+$, see the third concluding remark in that paper. In recent work [13, 26] we have provided a unified picture for **s**-periodic reductions, with nonzero $\mathbf{s} = (s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}$. In [26] it was shown how, under periodicity condition (2), any lattice equation $f(u_{l,m}, \cdots) = 0$ reduces to a system of r ordinary difference equations $f(v_n^p, \cdots) = 0$, $p = 0, \ldots, r-1$, where r is the greatest common divisor of s_1 and s_2. Also it was proved that the monodromy matrix, denoted $\mathcal{L}$, is one of the Lax matrices for the reduction, that is, there is a matrix $\mathcal{M}$ such that for solutions of the system the following holds,

$$\mathcal{L}_n \mathcal{M}_n = \mathcal{M}_n \mathcal{L}_{n+1}. \tag{3}$$

In [13] a geometric description of **s**-reduction has been given. It was shown that for all **s** there exists a well-posed, or *nearly well-posed*, **s**-periodic initial value problem, for any given scalar lattice equation on some arbitrary stencil of lattice points. We expect something similar to hold for systems of lattice equations. Combining the two results; given the existence of a periodic solution, we can multiply equation (3) by $\mathcal{M}_n^{-1}$, and conclude that the trace of $\mathcal{L}$ is an invariant of the mapping $n \mapsto n+1$. In section 2 we provide a direct proof, in the spirit of the original work [24], that the staircase method applies to any given system of lattice equations, if a Lax-pair and a well-posed periodic initial value problem are known.

The trace of $\mathcal{L}$ depends on a spectral parameter, arising from the Lax representation of the PΔE. By expanding in this parameter we obtain a number of

[1] Our notation differs from the one used in [24], where $s_1 = z_2$ and $s_2 = -z_1$.

integrals. In relation to establishing the complete integrability of a mapping (or correspondence) obtained by periodic reduction a first question to ask is: *does the staircase method yield sufficiently many functionally independent integrals?*

For the reductions we perform in section 2.1, of the Bruschi-Calogero-Droghei equation [5]

$$(u_{l,m} - u_{l,m-1})(u_{l,m} - u_{l-1,m-1}) = (u_{l,m} - u_{l,m+1})(u_{l,m} - u_{l+1,m+1}),$$

the number of integrals is exactly half the dimension of the mapping. For the one-parameter families of reductions we perform in section 2.2, of the QD-system [21]

$$e_{l,m+1} + q_{l+1,m+1} = q_{l+1,m} + e_{l+1,m}, \quad e_{l,m+1}q_{l,m+1} = q_{l+1,m}e_{l,m}, \tag{4}$$

we have verified that the number of integrals is one more than half the dimension of the mapping.

In other cases, there are fewer integrals than half the dimension of the mapping obtained by periodic reduction. For example, performing periodic reductions of the lattice pKdV equation (1) we find $(2n+1)$-dimensional mappings and $(2n+2)$-dimensional mappings, for which the staircase method provides only n integrals, see section 3.1. It turns out that when the dimension of the mapping is $2n+1$ it can be dimensionally reduced by 1, whereas when the dimension is $2n+2$ it can be reduced by 3. All n integrals survive the dimensional reduction and we can conclude that the dimensionally reduced mappings posses sufficiently many integrals for complete integrability.

To distinguish the two kinds of reductions we say $\mathbf{s}$-*reduction* for a periodic reduction of a lattice equation with period $\mathbf{s} \in \mathbb{Z} \times \mathbb{Z}$, and we say *d-reduction* for a dimensional reduction which reduces the dimension of a mapping by $d \in \mathbb{N}$.

In section 3.2 we show how to pose $\mathbf{s}$-periodic initial value problems for the Boussinesq system [18, 27]

$$\begin{aligned} w_{l+1,m} + v_{l,m} &= u_{l,m}u_{l+1,m}, \\ w_{l,m+1} + v_{l,m} &= u_{l,m}u_{l,m+1}, \\ w_{l,m} + v_{l+1,m+1} &= u_{l,m}u_{l+1,m+1} + \frac{\gamma}{u_{l+1,m} - u_{l,m+1}}. \end{aligned}$$

Performing $\mathbf{s}$-reduction with $\mathbf{s} = (n-1, 1)$ we get a $2n$ dimensional mapping. For these mapping we verified, taking $n \leq 17$, that the staircase method provides $n-1$ integrals unless 3 divides n in which case it provides only $n-3$ functionally independent integrals. We show, for all n, that the mapping can be 6-reduced when 3 divides n, and that the mapping can be 2-reduced otherwise.

These examples suggest that if the staircase method provides r integrals for an n-dimensional mapping, with $2r < n$, then the mapping can be d-reduced, with $d \geq n - 2r$. The d-reduction is performed by introducing $n - d$ new variables. These reducing variables can be obtained as the joint invariants of symmetries, or k-symmetries [10], of the mapping, which in turn are obtained from point-symmetries of the partial difference equation, see section 3.

For certain $\mathbf{s}$-reductions the periodic solutions are generated by multi-valued mappings, or correspondences, see [13]. The staircase method applies equally well in such cases. Here, another important question is: what is the multi-valuedness of the iterates of the correspondence? In general, the number of image points of the nth iterate of an m-valued correspondence would be m^n. However, it has been shown that for completely integrable correspondences the number of images under the iterates grows polynomially, rather than exponentially [29]. In section 4 we provide two examples: a reduction of the pKdV equation (1), and a one parameter families of reductions of the QD-system (4). In these examples the multi-valuedness of the nth iterate is $n+1$ and $2n$, respectively.

2 The staircase method

Let $\mathbf{u}$ be a multi-component field on the lattice and let $\mathbf{f}[\mathbf{u}]$ be a multi-component function of $\mathbf{u}$ and finitely many shifts of $\mathbf{u}$. We call a (nontrivial) lattice equation $\mathbf{f}[\mathbf{u}]=0$ *integrable* if it arises as the compatibility condition of two linear equations $\tilde{\psi}=L\psi$ and $\hat{\psi}=M\psi$. Here the $\tilde{}$ denotes the horizontal shift $l\mapsto l+1$ and $\hat{}$ denotes the vertical shift $m\mapsto m+1$. Thus we have

$$\hat{L}M \equiv \tilde{M}L \bmod \mathbf{f}, \tag{5}$$

which is called the Lax-equation, or zero-curvature condition.

We say that a lattice equation $\mathbf{f}[\mathbf{u}]=0$ admits a *well-posed* initial value problem if from a set of generic initial points a solution can be constructed in a unique way. An initial value problem is called *nearly-well-posed* if from the set of generic initial points solutions can be constructed, which can take only finitely many values at each lattice point.

Theorem 1 *Suppose an integrable equation $\mathbf{f}[\mathbf{u}]=0$ allows a $\mathbf{s}$-periodic initial value problem which is well-posed, or nearly-well-posed. Then, with $\mathcal{L}$ being the inversely ordered product of Lax matrices over a connected path (staircase) of length $\mathbf{s}$, the trace of $\mathcal{L}^i$ is invariant under any shift on the lattice, $\forall i\in\mathbb{N}$.*

Proof: Let a,b be two points on the lattice. Let $\mathcal{L}_{a,b}$ be the inversely ordered product of Lax matrices along any connected path, or staircase, from a to b. We have to show that $\mathcal{L}_{a,b}$ does not depend on the path from a to b. This follows from the fact that every square can be passed in two ways: from (5) it follows that $\hat{L}M=\tilde{M}L=\mathcal{L}_{u,\hat{\tilde{u}}}$ is well-defined for solutions of $\mathbf{f}[\mathbf{u}]=0$. We have

$$\mathcal{L}_{\tilde{a},\tilde{b}}=\mathcal{L}_{b,\tilde{b}}\mathcal{L}_{a,b}\mathcal{L}_{\tilde{a},a}=\mathcal{L}_{b,\tilde{b}}\mathcal{L}_{a,b}\mathcal{L}^{-1}_{a,\tilde{a}}$$

Now let b be a lattice point on a i period distance from a, so that the value of the solution at a and b coincide. Then $\mathcal{L}_{a,b}=\mathcal{L}^i$. Also, $\mathcal{L}_{b,\tilde{b}}=\mathcal{L}_{\tilde{a},a}$. If we denote $I=\mathrm{Tr}(\mathcal{L}^i)$, it is clear that we have $\tilde{I}=I$ and, similarly $\hat{I}=I$. □

The mapping, or correspondence, which generates the $\mathbf{s}$-periodic solution, is defined by updating a set of initial values through a shift on the lattice. Therefore, an invariant for it is given by the trace of (any power of) $\mathcal{L}$. If the Lax-matrices depend on a spectral parameter, say k, one can expand the trace $\mathrm{Tr}(\mathcal{L}^i)$ in powers of k. Each coefficient then provides an integral for the mapping, or for the correspondence. However, these integrals are not all functionally independent.

By the Cayley-Hamilton theorem any matrix $\mathcal{L}$ satisfies its own characteristic equation $P(\lambda) = \mathrm{Det}(\lambda I - \mathcal{L})$, i.e. we have $P(\mathcal{L}) = 0$. Therefore, given that $\mathcal{L}$ is a $n \times n$ matrix, it suffices to consider traces of $\mathcal{L}^i$, with $i < n$. Even better, there are certain combinations of $\mathrm{Tr}(\mathcal{L}^i)$, which, generally, yield a nicer basis of functionally independent integrals. These are provided by the coefficients in $P(\lambda) = 0$. For example, if $n = 2$ we have

$$P(\lambda) = \lambda^2 - \lambda \mathrm{Tr}(\mathcal{L}) + (\mathrm{Tr}(\mathcal{L})^2 - \mathrm{Tr}(\mathcal{L}^2))/2. \tag{6}$$

Note that the coefficient of λ^0 coincides with the determinant of $\mathcal{L}$.

For general n, the coefficients can be obtained using Newton's identities

$$ne_n = \sum_{i=1}^{n} (-1)^{i-1} e_{n-i} p_i, \tag{7}$$

where the power sums p_k are given by $p_k = x_1^k + x_2^k + \cdots + x_q^k$ and the elementary symmetric polynomials e_k are given by

$$e_k = \sum_{1 \le i_1} \sum_{i_1 < i_2} \cdots \sum_{i_{k-1} < i_k} \sum_{i_k \le q} \prod_{j=1}^{k} x_{i_j},$$

and appear as coefficients in the (Vieta) expansion

$$\prod_{i=1}^{q} (\lambda - x_i) = \sum_{i=0}^{q} (-1)^i e_i \lambda^{q-i}. \tag{8}$$

If we denote the q eigenvalues of the matrix $\mathcal{L}$ by x_i, the characteristic polynomial equals the left hand side of equation (8). Using Newton's identities (7) the right hand side can be expanded recursively in terms of $p_k = \mathrm{Tr}(\mathcal{L}^k)$. Taking $k = 1, 2$ in (7) we find familiar coefficients $e_1 = p_1$, $e_2 = (p_1^2 - p_2)/2$, see equation (6). Taking $k = 3, 4$ Newton's identities yield $e_3 = (p_1^3 - 3p_1 p_2 + 2p_3)/6$, and $e_4 = (p_1^4 - 6p_1^2 p_2 + 3p_2^2 + 8p_1 p_3 - 6p_4)/24$.

For scalar equations that are defined on elementary squares, initial values are given on staircases. So the dimension of the initial value problem is $|s_1| + |s_2|$. A so-called standard staircase, cf. [25, 26, 13] gives rise to a particularly simple mapping. In fact, any mapping, defined by a shift on the lattice, is (equivalent to) a power of this basic one.

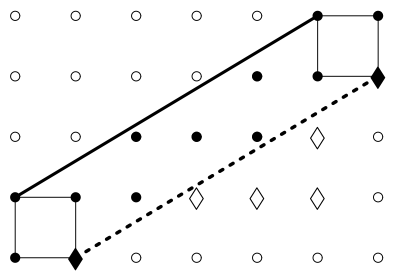

Figure 1: (5,3)-periodic initial value problem for equations defined on elementary squares

To illustrate this, we have presented the standard $(5,3)$-staircase in Figure 1. The standard staircase is the path through the points between the two lines, the black dots. The black dots are the points were initial values are given. The standard mapping will be the shift $u_{l,m} \mapsto u_{l+2,m+1}$. Note that by this shift almost all black dots are shifted to another black dot. The one black dot which is closest to the dotted line, is shifted to the black diamond, whose value can be calculated using the equation on the square. We can also see that the mapping defined by the shift $u_{l,m} \mapsto u_{l+1,m}$ is given as the third power of the standard one.

To evaluate the monodromy matrix $\mathcal{L}$ one would take the product of matrices along the same staircase on which the initial values are given. Since for this type of equations the matrices L, M depend on $(u, \tilde{u})$, $(u, \hat{u})$, respectively, the matrix $\mathcal{L}$ is then automatically expressed in terms of the initial values. We note that one can just as well take the product over any other one-period long path. For example, assuming that $\mathbf{s} \in \mathbb{N} \times \mathbb{N}$, one could consider the product

$$\mathcal{L}(l,m) = \prod_{j=0}^{\overset{\curvearrowleft}{s_2-1}} M_{l+s_1,m+j} \prod_{i=0}^{\overset{\curvearrowleft}{s_1-1}} L_{l+i,m}.$$

Then, one first has to calculate the points on the corresponding path to be able to evaluate $\mathcal{L}$. In the example given in Figure 1, we would need to calculate the points indicated by the white diamonds, which amounts to iterate the mapping 5 times.

For equations, or systems, that are not defined on elementary squares the initial value problem does, in general, not lie on a staircase. Depending on the type of stencil and on the particular periodicity condition, there could either be more, or less than $|s_1| + |s_2|$ initial values. In [13] it was shown how to write down, for a given scalar equation on an arbitrary stencil, a piece-wise linear expression (as a function of $\mathbf{s}$) for the dimension of an $\mathbf{s}$-periodic initial value problem.

Note that the monodromy matrix is still a product of $|s_1| + |s_2|$ matrices (if the product is taken over a staircase, which is the sensible thing to do). Also note that, in general, the Lax matrices depend on $\mathbf{u}$ and a number of shifts of $\mathbf{u}$. Therefore, for certain choices of $\mathbf{s}$, one needs to determine a number of points, by iterations of the mapping, or correspondence, in order to evaluate the monodromy matrix in terms of the initial values. It might be possible to avoid this by using the equation to change the $[\mathbf{u}]$-dependence of the Lax-matrices. However, that would have to be adjusted to the particular choice of $\mathbf{s}$.

2.1 The Bruschi-Calogero-Droghei equation

In [5] the five-point equation $E(u, \hat{u}, \underset{\hat{}}{u}, \tilde{\hat{u}}, \underset{\hat{\tilde{}}}{u}) = 0$ is given, with[2]

$$E = (u - \underset{\hat{}}{u})(u - \underset{\hat{\tilde{}}}{u}) - (u - \hat{u})(u - \tilde{\hat{u}}),$$

see [5, Equation (4a)] in which we have set $\alpha^{(\nu)} = 0$. The two recursive formulas [5, Equations (1a),(6)], with coefficients [5, Equations (5),(7)], yield the following Lax-pair

$$L(u, \tilde{u}, \underset{\hat{}}{u}, \underset{\hat{\tilde{}}}{u}) = \begin{pmatrix} k + \tilde{u} - u & (u - \underset{\hat{}}{u})(u - \underset{\hat{\tilde{}}}{u}) \\ 1 & 0 \end{pmatrix},$$

and

$$M(u, \hat{u}, \underset{\sim}{u}) = \begin{pmatrix} 1 & (u - \underset{\hat{}}{u}) \\ (\hat{u} - \underset{\sim}{u})^{-1} & 1 - (k + u - \underset{\sim}{u})(\hat{u} - \underset{\sim}{u})^{-1} \end{pmatrix},$$

where we have denoted the spectral parameter (x in [5]) by k. The method laid out in [13] tells us how to pose well-defined $\mathbf{s}$-periodic initial value problems for this equation. This can be done for all $\mathbf{s} = (s_1, s_2)$ such that $s_2(s_2 - 2s_1) \neq 0$. The dimension of the periodic solutions is given by the following piece-wise-linear function

$$2\max\{|s_2 - s_1|, |s_1|\}.$$

We apply the staircase method to a few reductions, in the different regions distinguished by this function, see [13, Figure 10].

(0,3)-reduction

Assigning initial values as in Figure 2 and updating them using the right-shift, we get a six-dimensional mapping

Figure 2:

x_3	x_4	$\circ$
x_1	x_2	$\circ$
x_5	x_6	$\circ$
x_3	x_4	$\circ$
x_1	x_2	$\circ$

$$\begin{aligned} x_1, x_2 &\mapsto x_2, \frac{x_2x_6 + x_3x_4 - x_3x_6 - x_4x_6}{x_2 - x_6}, \\ x_3, x_4 &\mapsto x_4, \frac{x_2x_5 + x_2x_6 - x_2x_4 - x_5x_6}{x_2 - x_4}, \\ x_5, x_6 &\mapsto x_6, \frac{x_1x_4 + x_2x_4 - x_1x_2 - x_4x_6}{x_4 - x_6}. \end{aligned}$$

Three integrals for this mapping can be obtained from the coefficients in the characteristic polynomial (6), where the monodromy matrix , which we take from x_2 upwards to x_2, is $\mathcal{L} = M(x_6, x_2, x_5)M(x_4, x_6, x_3)M(x_2, x_4, x_1)$. They are $(x_1 - x_4)(x_2 - x_5)(x_3 - x_6)$, $(x_1 - x_6)(x_2 - x_3)(x_4 - x_5)$, and $x_1(x_2 - x_4) + x_3(x_4 - x_6) + x_5(x_6 - x_2)$.

[2]For brevity we adopt the notation $u = u_{l,m}$, $\hat{u} = u_{l,m+1}$, $\underset{\sim}{u} = u_{l-1,m}$, et cetera.

(1,3)-reduction

Figure 3:

$$\begin{array}{ccc} \circ & x_1 & x_4 \\ \circ & x_2 & \circ \\ \circ & x_3 & \circ \\ x_1 & x_4 & \circ \end{array}$$

Assigning initial values as in Figure 3, and updating them using the down-shift, we get a four-dimensional mapping

$$(x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4, \frac{x_1 x_4 + x_2 x_3 - x_1 x_3 - x_3 x_4}{x_2 - x_3}). \tag{9}$$

We first calculate $\hat{x}_1$ by solving $E(x_2, x_1, x_3, x_4, \hat{x}_1) = 0$, and then we get $\hat{\hat{x}}_1$ from $E(x_1, \hat{x}_1, x_2, x_3, \hat{\hat{x}}_1) = 0$. Two integrals for the mapping can be obtained from the coefficients in (6) with $\mathcal{L} = M(x_2, x_1, \hat{\hat{x}}_1) M(x_3, x_2, \hat{x}_1) M(x_4, x_3, x_1) L(x_1, x_4, x_2, \hat{\hat{x}}_1)$. They are $J_1 = (x_1 - x_2)(x_3 - x_4)$, and $J_2 = (x_1 - x_3)(x_2 - x_4)/(x_2 - x_3)$.

(2,3)-reduction

$$\begin{array}{ccc} \circ & \circ & x_1 \\ \circ & \circ & x_3 \\ \circ & x_2 & \circ \\ x_1 & x_4 & \circ \end{array}$$

Figure 4:

We assigning initial values as in Figure 4 and updating them using the diagonal shift $u \mapsto \hat{\tilde{u}}$. The points $\tilde{x}_2, \tilde{x}_3$ are determined by $E(x_3, x_1, \tilde{x}_2, x_4, x_2) = 0$, and $E(x_4, x_2, \tilde{x}_3, \tilde{x}_2, x_3) = 0$, successively. We find the same four-dimensional mapping as in the previous case. The monodromy matrix $M(x_4, x_2, x_1) M(\tilde{x}_3, x_4, x_3) L(x_3, \tilde{x}_3, \tilde{x}_2, x_2) M(\tilde{x}_2, x_3, x_2) \cdot L(x_2, \tilde{x}_2, x_4, x_1)$ yields the same integrals J_1, J_2.

(2,-1)-reduction

Assigning initial values as in Figure 5, and updating them using the right-shift, we get a six-dimensional mapping

$$\begin{array}{ccc} x_5 & x_6 & \circ \\ x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 \\ \circ & \circ & x_1 \end{array}$$

Figure 5:

$$\begin{aligned} x_i &\mapsto x_{i+1}, \quad 1 \leq i \leq 5 \\ x_6 &\mapsto \frac{x_1 x_4 + x_2 x_4 - x_1 x_2 - x_4 x_6}{x_4 - x_6} \end{aligned}$$

The trace of the monodromy matrix $M^{-1}(x_4, x_6, x_3) L(x_5, x_6, x_3, x_2) L(x_4, x_5, x_2, x_1)$ yields two integrals, $(x_1 - x_6)(x_2 - x_4)(x_3 - x_5)$ and $x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 + x_5 x_6 - x_1 x_4 - x_2 x_4 - x_2 x_5 - x_3 x_5 - x_3 x_6$. A third integral, $(x_1 - x_4)(x_2 - x_4)((x_2 - x_5) x_3^2 + (x_3 - x_6) x_5^2 + (x_2 + x_3) x_5 x_6 - x_2 x_3 (x_5 + x_6))$, is obtained by taking the determinant of the monodromy matrix. In the previous cases, the determinant does not provide any other functional independent integrals.

Dimensional reduction

In the above cases the number of functionally independent integrals is exactly half the dimension of the mapping, sufficiently many for integrability. We note that the equation $E = 0$ admits the Lie-point symmetries $u \mapsto u + \epsilon$ and $u \mapsto \lambda u$. This yields two symmetries for the mappings, which can be used to reduce the dimension of the mappings by 2. The integrals given only admit admit the first (translation) symmetry. However, certain homogeneous combinations of them also admit the second (scaling) symmetry. Applying 2-reduction therefore produces 2-, respectively 4-dimensional mappings with 1, respectively 2 integrals. For example, in reducing variables

$$z_1 = \frac{x_1 - x_2}{x_2 - x_3}, \quad z_2 = \frac{x_2 - x_3}{x_3 - x_4},$$

the mapping (9) reads

$$z_1, z_2 \mapsto z_2, \frac{1}{z_1}$$

which has integral

$$\frac{J_2^2}{J_1} = \frac{(z_1 + 1)^2 (z_2 + 1)^2}{z_1 z_2}.$$

Note that the 4th iterate equals the identity and hence the reduction provides an explicit solution for mapping (**??**). If the $(n-1)$st iterate of the mapping 9 is denoted $(x_n, x_{n+1}, x_{n+2}, x_{n+3})$, then

$$x_n = x_1 + \lfloor \frac{n+2}{4} \rfloor (x_2 - x_1) + \lfloor \frac{n+1}{4} \rfloor (x_3 - x_2) + \lfloor \frac{n}{4} \rfloor (x_4 - x_1) + \lfloor \frac{n-1}{4} \frac{(x_2 - x_1)(x_4 - x_3)}{x_3 - x_2} \rfloor,$$

which is obtained similarly as the solution to the (3,1)-reduction of lattice pKdV given in the appendix of [13].

2.2 The QD-algorithm

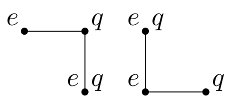

Figure 6: QD-type system

The quotient-difference (QD) algorithm, see for example [21, equations 7,8],

$$\hat{e} + \hat{\tilde{q}} = \tilde{q} + \tilde{e}, \quad \hat{e}\hat{q} = e\tilde{q}, \tag{10}$$

is used to construct continued fractions whose convergents form ordered sequences in a normal Padé table [7], and to find the zeros of a polynomial [11]. It is an integrable two-component equation defined on the stencils depicted in Figure 6.

A Lax-pair for the QD-algorithm can be obtained using the following relations between the higher adjacent orthogonal polynomials [4], cf. [21, equations (3,4)]

$$x\widehat{P} = \widetilde{P} + qP, \quad \widetilde{P} = \widehat{\widetilde{P}} + e\widehat{P}.$$

With $\Psi = (P, \widehat{P})$ we have $\widetilde{\Psi} = L(e,q)\Psi$ and $\widehat{\Psi} = k^{-1}M(e,q,\widehat{q})\Psi$, where

$$L = \begin{pmatrix} -q & k \\ -q & k-e \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & k \\ -q & k-e+\widehat{q} \end{pmatrix}.$$

This (small) Lax-pair differs from the (big) Lax-pair obtained in [21, equation (9)]. A big Lax-pair incorporates a particular choice of initial values and we would like to consider general periodic initial value problems.

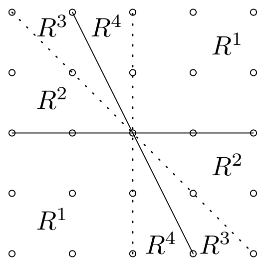

Figure 7: Distinct regions for the QD-stencil

In [13], it has been shown that for all $s = (s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $s_2(s_2 - 2s_1) \neq 0$, there exists a well-posed **s**-periodic initial value problem, with dimension

$$2\max(|s_1 + s_2|, |s_1|).$$

This function tells us there are two different regions, where the dimension is given by a different linear function for the dimension (up to a sign). However, to pose the initial value problems one has to distinguish four different regions, as depicted in Figure 7, see [13].

We give four examples of families of periodic reductions, where the dimension depends on an arbitrary variable $n \in \mathbb{N}$, one family in each of the different regions.

$(0, n)$-reduction

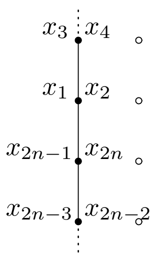

Figure 8: A vertical staircase.

We take initial values $(e,q)_{0,i} = (x_{2i-1}, x_{2i})$, where $x_k = x_m$ when $k \equiv m \mod 2n$, see Figure 8. Note, whenever one is dealing with a system of equations, there is in general more than one x_i associated with each lattice point.

The $2n$-dimensional volume-preserving mapping is, with $i = 1, 2, \ldots, 2n$,

$$x_{2i-1} \mapsto x_{2i+1} + \frac{x_{2i+3}x_{2i+4}}{x_{2i+1}} - \frac{x_{2i+1}x_{2i+2}}{x_{2i-1}},$$
$$x_{2i} \mapsto \frac{x_{2i+1}x_{2i+2}}{x_{2i-1}}.$$

The monodromy matrix is

$$\mathcal{L} = M(x_{2n-1}, x_{2n}, x_2)M(x_{2n-3}, x_{2n-2}, x_{2n}) \cdots \\ \cdots M(x_3, x_4, x_6)M(x_1, x_2, x_4)$$

We verified up to $n = 9$ that the coefficients in the k-expansions of its trace and determinant yield $n + 1$ functionally independent integrals. The lowest non-trivial mapping, taking $n = 2$, is

$$x_1, x_2, x_3, x_4 \mapsto x_3 + \frac{x_1 x_2}{x_3} - \frac{x_3 x_4}{x_1}, \frac{x_3 x_4}{x_1}, x_1 + \frac{x_3 x_4}{x_1} - \frac{x_1 x_2}{x_3}, \frac{x_1 x_2}{x_3},$$

which admits the three functionally independent integrals $x_1 + x_3, x_2 x_4, x_1 x_2 + x_3 x_4 - x_1 x_3$.

$(3n, -2n)$-reduction

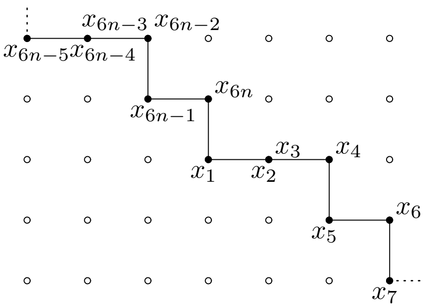

Figure 9: Initial values on part of a standard $(3n, -2n)$-staircase.

The mapping is the horse-jump $(l, m) \mapsto (l + 2, m - 1)$. We obtain the $6n$-dimensional mapping, with $i = 0, 1, \ldots, n - 1$,

$$\begin{aligned}
x_{6i+1} &\mapsto x_{6i+5} \\
x_{6i+2} &\mapsto \frac{x_{6i+7} x_{6i+9}}{x_{6i+6}} \\
x_{6i+3} &\mapsto x_{6i+6} \\
x_{6i+4} &\mapsto x_{6i+8} + x_{6i+9}(1 - \frac{x_{6i+7}}{x_{6i+6}}) \\
x_{6i+5} &\mapsto x_{6i+8} \\
x_{6i+6} &\mapsto x_{6i+10},
\end{aligned}$$

which is measure-preserving with density $(\prod_{i=1}^{n} x_{6i-3})^{-1}$.

We obtain integrals by expanding the trace and determinant of the monodromy matrix

$$\begin{aligned}
\overset{\curvearrowleft}{\prod_{i=0}^{n-1}} M^{-1}(x_{6i+7}, x_{6i+5} + x_{6i+6} - x_{6i+7}, x_{6i+6}) L(x_{6i+5}, x_{6i+2} + x_{6i+4} - x_{6i+5}) \cdot \\
M^{-1}(x_{6i+5}, x_{6i+2} + x_{6i+4} - x_{6i+5}, x_{6i+4}) L(x_{6i+2}, x_{6i+3}) \cdot \\
L(x_{6i+1}, x_{6i-1} + x_{6i} - x_{6i+1})
\end{aligned}$$

in powers of the spectral parameter k. We verified that, up to $n = 3$, $3n + 1$ of them are functionally independent. For $n = 1$ the mapping reads

$$x_1, x_2, x_3, x_4, x_5, x_6 \mapsto x_5, \frac{x_1 x_3}{x_6}, x_6, x_2 + x_3(1 - \frac{x_1}{x_6}), x_2, x_4,$$

which admits the following four functionally independent integrals

$$x_6 + x_4 + x_2 + x_3, x_3(x_4 - x_5)(x_1 - x_6), x_3(x_6 + x_4 - x_1) + x_6(x_4 + x_2 - x_5), x_1 x_2 x_3 x_5.$$

$(2n, -3n)$-reduction

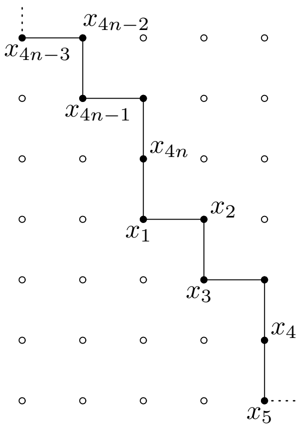

Figure 10: part of the (2n,-3n)-staircase

We choose initial values on the standard staircase as in Figure 10,

$$e_{2i,-3i} = x_{4i+1}, \quad q_{2i+1,-3i} = x_{4i+2}$$
$$e_{2i+1,-3i-1} = x_{4i+3}, \quad q_{2i+2,-3i-2} = x_{4i+4},$$

with $x_n = x_{n \bmod 4n}$. The standard mapping is obtained by shifting $(l, m) \mapsto (l+1, m-1)$. We obtain a $4n$-dimensional mapping, with $i = 0, 1, \ldots, n-1$,

$$x_{4i+1}, x_{4i+2} \mapsto x_{4i+3}, \frac{x_{4i+5}x_{4i+6}}{x_{4i+4}} + x_{4i+4} - x_{4i+3}$$
$$x_{4i+3}, x_{4i+4} \mapsto \frac{x_{4i+5}x_{4i+6}}{x_{4i+4}}, x_{4i+6},$$

which is measure-preserving with density $\prod_{i=1}^{n} x_{4i}$.

The monodromy matrix is

$$\overset{\curvearrowleft}{\prod_{i=0}^{n-1}} M^{-1}(x_{4i+5}, p_i, x_{4i+4}) M^{-1}(e_i, x_{4i+4}, q_i) L(x_{4i+3}, r_i) \cdot$$
$$M^{-1}(x_{4i+3}, r_i, x_{4i+2}) L(x_{4i+1}, p_{i-1}),$$

where $e_i = x_{4i+5}x_{4i+6}/x_{4i+4}$, $q_i = e_i + x_{4i+4} - x_{4i+3}$, $r_i = x_{4i+1} + x_{4i+2} - x_{4i+3}$, $p_i = z_i + x_{4i+4} - x_{4i+5}$, and $z_i = x_{4i+3}r_i/x_{4i+4}$. We verified up to $n = 3$ that its trace and determinant yield $2n + 1$ functionally independent integrals. For $n = 1$ the measure preserving 4-dimensional mapping reads

$$x_1, x_2, x_3, x_4 \mapsto x_3, \frac{x_1x_2}{x_4} + x_4 - x_3, \frac{x_1x_2}{x_4}, x_2, \tag{11}$$

which admits the following three functionally independent integrals,

$$x_2 + x_4 - x_3, \; \frac{x_1x_3}{x_4}, \; \frac{(x_1 - x_4)(x_2 - x_3)(x_3 - x_4)}{x_4}.$$

$(1, -1-n)$-reduction

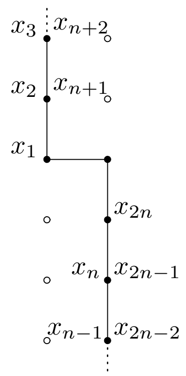

Figure 11:

We choose initial values on a standard staircase as in Figure 11, with $j = 1, 2, \ldots, n$, and $i \in \mathbb{Z}$,

$$e_{i,-i(n+1)+j-1} = x_j, \;\; q_{i,-i(n+1)+j} = x_{n+j}.$$

The standard mapping is obtained by shifting $(l, m) \mapsto (l, m+1)$. We obtain the $2n$-dimensional mapping,

$$\begin{aligned} x_i &\mapsto x_{i+1}, \quad 1 \le i \le n-1, \;\; n+1 \le i \le 2n-1 \\ x_n &\mapsto x_1 + \frac{x_2 x_{n+1}}{x_1} - x_{2n} \\ x_{2n} &\mapsto \frac{x_2 x_{n+1}}{x_1} \end{aligned}$$

which is measure-preserving with density x_1.

The monodromy matrix is

$$M^{-1}(x_1, (x_n + x_{2n-1} - x_{2n})\frac{x_{2n}}{x_1}, x_{n+1}) \left(\prod_{i=2}^{n} M^{-1}(x_i, x_{n+i-1}, x_{n+i}) \right)$$
$$M^{-1}(x_1 + \frac{x_2 x_{n+1}}{x_1} - x_{2n}, x_{2n}, \frac{x_2 x_{n+1}}{x_1}) L(x_1, (x_n + x_{2n-1} - x_{2n})\frac{x_{2n}}{x_1}),$$

whose trace and determinant yield $n+1$ functionally independent integrals, which we verified up to $n = 8$. For $n = 2$ we find mapping (11) again, under the change of variables

$$x_1, x_2, x_3, x_4 \mapsto x_4, x_2, x_1, x_3.$$

Dimensional reduction

As the QD-system admits 1 (scaling) symmetry all mappings can be 1-reduced. The reduced mapping has one integral less, as only homogeneous combinations (with scaling eigenvalue 0) are invariant under scaling. For example, the mapping (11) with new variables $y_i = x_i/x_4$, $i = 1, 2, 3$, reduces to the 3-dimensional mapping

$$y_1, y_2, y_3 \mapsto \frac{y_3}{y_2}, y_1 + \frac{1 - y_3}{y_2}, y_1$$

which has invariants

$$\frac{y_1 y_3}{y_2 - y_3 + 1}, \qquad \frac{(y_2 - y_3)(y_1 - 1)(y_3 - 1)}{(y_2 - y_3 + 1)^2}.$$

3 Dimensional reduction of mappings

In all cases we have encountered, the staircase method provided sufficiently many integrals, i.e. at least half the dimension of the mapping. In the examples given in this section this is not the case. However, after dimensional reduction the number of integrals does suffice. We will see that the number of dimensions to be reduced varies with the period $\mathbf{s}$. This can be obtained exploiting k-symmetries of the PΔE. Recall, a k-symmetry of a mapping is a symmetry of the k-th iterate of that mapping [10].

For the KdV equation (1) we have verified up to $n = 17$ that the trace of the monodromy matrix for the $(n-1,1)$-reduction, which is an n-dimensional mapping, provides $\lfloor (n-1)/2 \rfloor$ functionally independent integrals. So, for the odd dimensional mappings we need one reduction, but for the even dimensional mappings we need two. In fact, as we shall see, in the even case there exist three. How to explain this? As we will see below, the KdV-equation has a Lie-point symmetry that does not depend on the lattice variables. This symmetry gives rise to one reduction for both the odd and the even dimensional mappings. Also, there are two symmetries that do depend on the lattice variables. These yield two 2-symmetries of the mapping if and only if its dimension is even, giving us two more reductions.

Another example we provide is the Boussinesq system, see equation (7). Performing $(n-1,1)$-reductions, with $1 < n < 10$, we find $n-1$ integrals for a $2n$-dimensional mapping, except if 3 divides n when the number of integrals is $n-3$. This means that we need to d-reduce by $d = 2$ dimensions, or, if 3 divides n, by $d = 6$ dimensions. We show this can be done using symmetries, respectively 3-symmetries of the mappings.

3.1 The lattice potential Korteweg-De Vries equation, $(n-1,1)$-reduction

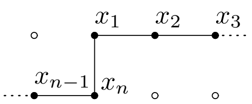

Figure 12:

The matrices

$$\begin{pmatrix} u & -k - u\tilde{u} \\ 1 & -\tilde{u} \end{pmatrix}, \quad \begin{pmatrix} u & -\alpha - k - u\hat{u} \\ 1 & -\hat{u} \end{pmatrix},$$

form a Lax pair for the lattice pKdV equation

$$(u - \hat{\tilde{u}})(\tilde{u} - \hat{u}) = \alpha. \tag{12}$$

We consider the initial value problem $u_{(n-1)k+i-1,k} = x_i$, with $i = 1, \ldots, n$ and $k \in \mathbb{Z}$, see Figure 12. We find, for all k, $u_{(n-1)k+n,k} = x_1 + \alpha/(x_n - x_2)$. The right-shift $\tilde{}$ induces the n dimensional mapping

$$x_1, x_2, \ldots, x_n \mapsto x_2, x_3, \ldots, x_n, x_1 + \frac{\alpha}{x_n - x_2},$$

which is volume-preserving when n is odd and anti-volume-preserving when n is even. Equation (12) has the following symmetry $u \mapsto u + \epsilon$. This yields a symmetry for the mapping, whose infinitesimal generator is

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}. \tag{13}$$

A set of functionally independent invariants is given by $y_i = x_i - x_{i+1}$, $i = 1..n-1$ and in these reduced variables the mapping reads

$$y_1, y_2, \ldots, y_{n-1} \mapsto y_2, y_3, \ldots, y_{n-1}, -\sum_{i=1}^{n-1} y_i + \frac{\alpha}{\sum_{i=2}^{n-1} y_i}. \tag{14}$$

In addition, equation (1) also has the following two symmetries

$$u_{l,m} \mapsto u_{l,m} - (-1)^{l+m}\epsilon, \qquad u_{l,m} \mapsto u_{l,m}\epsilon^{(-1)^{l+m}}.$$

Suppose now that n is even. Then the above symmetries of the PΔE (12) give rise to 2-symmetries of the mapping (14), with generators

$$\sum_{i=1}^{n} (-1)^i \frac{\partial}{\partial x_i}, \qquad \sum_{i=1}^{n} (-1)^i x_i \frac{\partial}{\partial x_i}. \tag{15}$$

It can be verified that a set of $n-3$ functionally independent joint invariants of the above three vector fields (13,15) is given by $q_i = (x_i - x_{i+2})(x_{i+1} - x_{i+3})$, $i = 1..n-3$. We will take the q_i as reduced variables and perform the reduction. We define, with $k, m \in \mathbb{N}$,

$$F_k^m := \frac{x_{1+k} - x_{2m+3+k}}{x_{2m+1+k} - x_{2m+3+k}}.$$

The F_k^m satisfy the recurrence relation

$$(F_k^m - 1)q_{2m+k} = q_{2m+k-1}F_k^{m-1}, \tag{16}$$

with initial condition $F_k^0 = 1$. Therefore the F_k^m can be expressed in the q_i, with $i \leq n-3$, when $m < n/2 - 1$ and $k < n/2 - 2m + 3$. We have the following equalities

$$\begin{aligned} x_i &= x_{n-1} + \frac{q_{n-3}}{x_{n-2} - x_n} F_{i-1}^{(n-3-i)/2} && \text{when } i \text{ odd, and} \\ x_i &= x_{n-2} + \frac{q_{n-4}}{q_{n-3}}(x_{n-2} - x_n) F_{i-1}^{(n-4-i)/2} && \text{when } i \text{ even,} \end{aligned}$$

which is an inverse reduction. In terms of reduced variables the mapping is

$$\begin{aligned} q_i &\mapsto q_{i+1}, \quad i < n-3 \\ q_{n-3} &\mapsto -q_{n-3}F_0^{(n-4)/2} + \alpha / F_1^{(n-4)/2}. \end{aligned} \tag{17}$$

An explicit expression for F_k^m in terms of q_i is

$$F_k^m = \sum_{i=1}^{m+1} \prod_{j=2i}^{2m+1} q_{j-1+k}^{(-1)^j}, \tag{18}$$

since this expression solves the recurrence (16) with $F_k^0 = 1$. The mapping (17) is anti-measure preserving with density $\prod_{i=1}^{(n-4)/2} q_{2i}$.

At $n = 4$ the mapping is $q_1 \mapsto \alpha - q_1$, which admits one integral, $q_1(\alpha - q_1)$. The second iterate of this mapping equals the identity. What happened? Well, the joint invariant q_1 turns out to be a 2-integral of the mapping. Let us define another set of functions

$$H_k^m := -x_{1+k}x_{2m+2+k} - \sum_{i=1}^{2m+1} (-1)^i x_{i+k}x_{i+1+k}.$$

They can be expressed in terms of the q_i, with $i \leq n-3$, when $m < n/2$ and $k < n/2 + 4 - 2m$, using the recurrence

$$H_k^m = H_k^{m-1} + q_{2m-1+k}F_k^{m-1}, \tag{19}$$

with initial condition $H_k^0 = 0$. For all n the n-dimensional mapping admits the 2-integral $H_0^{(n-2)/2}$, we have $\delta_n(H_0^{(n-2)/2}) = \alpha - H_0^{(n-2)/2}$. An explicit expression for H_k^m in terms of the q_i is

$$H_k^m = \sum_{i=1}^{m} \sum_{j=1}^{m+1-i} \prod_{l=0}^{2i-2} q_{2j+l-1+k}^{(-1)^l},$$

since this solves the initial value problem (19). At $n = 6$ we have the mapping

$$(q_1, q_2, q_3) \mapsto (q_2, q_3, -q_3(q_1 + q_2)/q_2 + \alpha q_3/(q_2 + q_3))$$

for which the staircase method provides the two integrals

$$(q_1(q_2 + q_3)/q_2 + q_2 + q_3 - \alpha)q_1q_3,\ (q_1 + q_3 + q_1q_3/q_2)(q_1q_3/q_2 + q_1 + q_3 - \alpha),$$

of which the latter can be expressed in terms of the 2-integral $H_0^2 = q_1 + q_3 + q_1q_3/q_2$. One can choose the 2-integral as a variable. In terms of q_i, $i < n-3$, and $p = H_0^{(n-2)/2}$ we get for $n = 6$

$$(q_1, q_2, p) \mapsto (q_2, q_2(p - q_1)/(q_1 + q_2), \alpha - p)$$

which has integrals

$$\frac{q_1q_2}{q_1 + q_2}(p - q_1)(p + q_2 - \alpha), p(\alpha - p).$$

In general, with $n > 2$ even, the reduced mapping can be written as

$$(q_1, q_2, \ldots, q_{n-4}, p) \mapsto (q_2, q_3, \ldots, q_{n-4}, \frac{p - H_0^{(n-4)/2}}{F_0^{(n-4)/2}}, \alpha - p),$$

where we used (19) to solve $p = H_0^{(n-2)/2}$ for q_{n-3}.

We note that all $\lfloor (n-1)/2 \rfloor$ functionally independent integrals we have calculated for $n \leq 17$ dimensional mappings survive these reductions. Taking $n = 2m + 1$ odd, the reduced mapping is $2m$ dimensional and has m integrals. With $n = 2m + 2$, the reduced mapping is $2m - 1$ dimensional and has m integrals.

3.2 The Boussinesq system

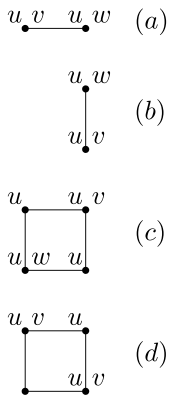

Figure 13:

The Boussinesq system [18, 27]

$$\tilde{w} = u\tilde{u} - v \tag{20a}$$

$$\hat{w} = u\hat{u} - v \tag{20b}$$

$$w = u\hat{\tilde{u}} - \hat{\tilde{v}} + \frac{\gamma}{\tilde{u} - \hat{u}}. \tag{20c}$$

is defined on the square as depicted in Figure 13(a,b,c). In Figure 13(d) we have depicted the consequence of equations (20a,20b),

$$\hat{u}\hat{\tilde{u}} - \hat{v} = \tilde{u}\hat{\tilde{u}} - \tilde{v}. \tag{20d}$$

From the Boussinesq system one can eliminate the variables v and w using the identity

$$\widehat{(\widetilde{\widehat{v}} + w)} - \widetilde{(\widetilde{\widehat{v}} + w)} = \widetilde{\widehat{(\widehat{v} - \widetilde{v})}} + (\widehat{w} - \widetilde{w})$$

to get a 9-point scalar equation on a 2×2 square, called the Boussinesq equation, cf. [18, equation 1.3].

We denote $\mathbf{u} = (u, v, w)$. A Lax-pair for the system (20) is given by [27],

$$L_{\mathbf{u}} = \begin{pmatrix} -\widetilde{u} & 1 & 0 \\ -\widetilde{v} & 0 & 1 \\ \widetilde{u}w - \widetilde{v}u - k & -w & u \end{pmatrix}, \quad M_{\mathbf{u}} = \begin{pmatrix} -\widehat{u} & 1 & 0 \\ -\widehat{v} & 0 & 1 \\ \widehat{u}w - \widehat{v}u + \gamma - k & -w & u \end{pmatrix}.$$

Invariants for traveling wave reductions of the system can be obtained by expanding traces of powers of the monodromy matrix. Since $\mathcal{L}$ is a 3×3 matrix with constant determinant a full set of functional independent integrals can be obtained from k-expansions of $\mathrm{Tr}(\mathcal{L})$ and $\mathrm{Tr}^2(\mathcal{L}) - \mathrm{Tr}(\mathcal{L}^2)$, see section 2.

The following proposition tells us how to pose initial value problems for the Boussinesq system. The proof uses a different technique than the one used in [13], which is possible due to the fact that here initial values can be given on staircases. However, the staircases are not necessarily standard staircases (which they would in the framework of [13], at least for equations defined on the square, such as the Boussinesq system). We call a staircase **ascending**, if it goes from the lower left to the upper right, that is, if it is a sequence of neighboring lattice sites with l and m nondecreasing. And we call a staircase **descending** if it goes from the upper left to the lower right, that is, if it is a sequence of neighboring lattice sites with l and $-m$ nondecreasing.

Proposition 2 *The following initial values problems for the Boussinesq system are well-posed.*

- *At every point on an ascending staircase take the components u, v of the vector $\mathbf{u}$ as initial values.*
- *On a descending staircase take u, v, w at the lower left corners, v at the upper right corners, and u, v at the other points as initial values.*

Proof: The proof consist of two parts. Firstly, we show that the values at all points of the staircase can be obtained from the initial values. Secondly we show that any of the four vectors $\mathbf{u}, \tilde{\mathbf{u}}, \hat{\mathbf{u}}, \hat{\tilde{\mathbf{u}}}$, can be determined from the other three.

i) For ascending staircases the first part is easy. Going along the staircase from the lower-left to the upper-right at each horizontal step the component w is obtained using equation (20a), whereas at the vertical steps equation (20b) can be used. For a descending staircase we can do a similar thing, except at the upper-right corners. Equation (20d) can be solved for $u_{l+1,m+1}$ and used to get the u-components at the upper-right corners. Once u has been calculated w can be calculated in two ways, using either (20a) or (20b), leading to the same result.

ii) The values of $\hat{\mathbf{u}}$ can be obtained as follows. First calculate $\hat{u}$ from (20c). Then $\hat{w}$ can be obtained from (20b) and $\hat{v}$ from the up-shifted consequence of (20a). We can obtain $\tilde{\mathbf{u}}$ in a similar way. This follows from the fact that interchanging the left-shift with the up-shift and $\gamma \mapsto -\gamma$ is a (discrete) symmetry of the system. Finally, to obtain $\hat{\tilde{\mathbf{u}}}$ one uses the consequence (20d) to calculate $\hat{\tilde{u}}$, after which $\hat{\tilde{w}}$ is found using a shifted version of either (20a) or (20b), and $\hat{\tilde{v}}$ is calculated using (20c). Also, due to the discrete symmetry which interchanges the up-shift with the down-shift and the left-shift with the right-shift, together with $u \leftrightarrow w$ and $\gamma \mapsto -\gamma$, it follows that $\mathbf{u}$ can be obtained from given values at the other sites.

$\square$

It follows that, with $s = (s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $s_1 s_2 \neq 0$, the dimension of a $\mathbf{s}$-periodic reduction is $2(|\ s_1\ | + |\ s_2\ |)$.

(n-1,1)-reduction

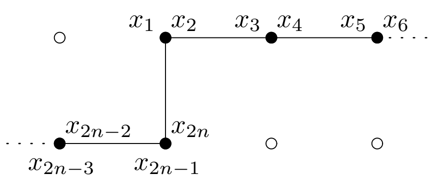

Figure 14: $2n$ initial values

We take $s_1 = n - 1$ positive (but $n \neq 3$), $s_2 = 1$, and we consider the following initial value problem, with $i = 0, 1, \dots, n-1$ and $l \in \mathbb{Z}$.

$$u_{(n-1)l+i,l} = x_{2i+1},$$
$$v_{(n-1)l+i,l} = x_{2i+2},$$

see Figure 14.

We calculate, $w_{(n-1)l,l-1} = x_{2n-3}x_{2n-1} - x_{2n-2}$, $w_{(n-1)l-1,l} = x_1 x_3 - x_2$, $u_{(n-1)l+1,l-1} = x_1 + \gamma P$, $v_{(n-1)l+1,lk-1} = x_2 + x_3 \gamma P$, where

$$P = \frac{1}{x_{2n-1}(x_{2n-3} - x_3) - x_{2n-2} + x_4}.$$

The right-shift induces the $2n$-dimensional mapping $\phi_n : x \mapsto x'$,

$$\begin{aligned} x_i &\mapsto x_{i+2}, \quad 1 \leq i \leq 2n-2, \\ x_{2n-1} &\mapsto x_1 + \gamma P, x_{2n} \qquad \mapsto x_2 + \gamma x_3 P. \end{aligned} \tag{21}$$

The Jacobian matrix of ϕ_n is given by

$$J = \begin{pmatrix} 0 & I_{n-2} \\ I_2 & \gamma P^2 H \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix and H is the $2 \times (n-2)$ matrix

$$\begin{pmatrix} x_{2n-1} & -1 & 0 & \cdots & 0 & -x_{2n-1} & 1 & (x_3 - x_{2n} - 3) & 0 \\ 1/P + x_3 x_{2n-1} & -x_3 & 0 & \cdots & 0 & -x_3 x_{2n-1} & x_3 & x_3(x_3 - x_{2n} - 3) & 0 \end{pmatrix}.$$

The mapping ϕ has two symmetries, generated by

$$v_1 = \sum_{i=1}^{n} \frac{\partial}{\partial x_{2i}}, \qquad v_2 = \sum_{i=1}^{n} \frac{\partial}{\partial x_{2i-1}} + x_{2i-1} \frac{\partial}{\partial x_{2i}}.$$

As one can easily check we have $Jv = v'$ when $v = (0,1,0,1,\ldots,0,1)$ or $v = (1, x_1, 1, x_3, \ldots, 1, x_{2n-1})$. They correspond to the symmetries

$$(u,v,w) \mapsto (u, v+\epsilon, w-\epsilon), \qquad (u,v,w) \mapsto (u+\epsilon, v+\epsilon u, w+\epsilon u) \tag{22}$$

of the original system (3). Taking joint invariants of the symmetries as reduced variables, $y_i = x_{2i-1} - x_{2i+1},\ y_{n+i-1} = x_{2i} - x_{2i+2} + x_{2i-1}(x_{2i+1} - x_{2i-1})$, with $i = 1, \ldots, n-1$, we get a $2(n-1)$ dimensional volume preserving mapping

$$\begin{aligned} y_i &\mapsto y_{i+1}, \quad 1 \le i < n-1,\ n-1 < i < 2n-2, \\ y_{n-1} &\mapsto -\sum_{i=1}^{n-1} y_i - \gamma Q, \qquad Q = \sum_{i=1}^{n-3} (y_{n+i} + y_{i+1} \sum_{j=i+1}^{n-1} y_j), \\ y_{2n-2} &\mapsto -\sum_{i=0}^{n-2} (y_{n+i} + y_{i+1} \sum_{j=i+1}^{n-1} y_j) - \gamma Q \sum_{i=2}^{n-1} y_i. \end{aligned}$$

Setting $L_{\mathbf{u}} = L(u, w, \tilde{u}, \tilde{v})$ and $M_{\mathbf{u}} = M(u, w, \hat{u}, \hat{v})$ the monodromy matrix is

$$\mathcal{L} = M(x_{2n-1}, w_n, x_1, x_2) L(x_{2n-3}, w_{n-1}, x_{2n-1}, x_{2n}) \cdots L(x_1, w_1, x_3, x_4),$$

where $w_1 = x_1 x_{2n-1} - x_{2n}$ and $w_{i+1} = x_{2i-1} x_{2i+1} - x_{2i}$, $i = 1, 2, \ldots, n-1$. We obtain functionally independent integrals from k-expansions of the coefficients in

$$\begin{aligned} Det(\lambda I - \mathcal{L}) &= \lambda^3 - \lambda^2 \mathrm{Tr}(\mathcal{L}) + \lambda \frac{\mathrm{Tr}(\mathcal{L})^2 - \mathrm{Tr}(\mathcal{L}^2)}{2} \\ &\quad - \frac{\mathrm{Tr}(\mathcal{L})^3 - 3\mathrm{Tr}(\mathcal{L})\mathrm{Tr}(\mathcal{L}^2) + 2\mathrm{Tr}(\mathcal{L}^3)}{6}. \end{aligned} \tag{23}$$

cf. section 2, in particular equation (8).

When $n = 2$ the reduced mapping is, in terms of $X = -y_1, Y = -y_2$, cf. [18, equation 5.31]

$$(X, Y) \mapsto (-X + \gamma(Y - X^2), -Y + X^2)$$

which carries the invariant $X^2Y - Y^2 + \gamma X$. In [18] the case $s_1 = s_2$ was studied, in particular they established the involutivity of the integrals. The authors state they obtain a sufficient number of functionally independent integrals, but give no general proof nor indicate for which values of $s_1 = s_2$ they verified their statement.

n	2	3	4	5	6	7	8	9
#	1	0	3	4	3	6	7	6

Figure 15: Number of functionally independent integrals of ϕ.

The number of functionally independent integrals we have obtained is given in table 15. We have verified that all integrals survive the reduction. Hence, except if 3 divides n, the staircase method provides enough integrals for integrability. Next we will show that if 3 divides n, but $n \neq 3$, two more reductions are possible.

The Boussinesq system (3) has some additional symmetries, which depend on the lattice variables

$$\begin{aligned}
&(u,v,w)_{l,m} \mapsto (u, v + \epsilon\zeta^{l+m+1}, w - \epsilon\zeta^{l+m})_{l,m},\\
&(u,v,w)_{l,m} \mapsto (u + \epsilon\zeta^{l+m}, v + \epsilon\zeta^{l+m+1}u, w + \epsilon\zeta^{l+m-1}u)_{l,m},
\end{aligned} \tag{24}$$

where ζ is a primitive third root of unity, that is, $\zeta^2 + \zeta + 1 = 0$. The generators of the corresponding transformations acting on the initial values (14) are

$$v_3 = \sum_{i=1}^{n} \zeta^i \frac{\partial}{\partial x_{2i}}, \quad v_4 = \sum_{i=1}^{n} \zeta^{i-1} \frac{\partial}{\partial x_{2i-1}} + \zeta^i x_{2i-1} \frac{\partial}{\partial x_{2i}},$$

and, taking the conjugate root ζ^2,

$$v_5 = \sum_{i=1}^{n} \zeta^{2i} \frac{\partial}{\partial x_{2i}}, \quad v_6 = \sum_{i=1}^{n} \zeta^{2i+1} \frac{\partial}{\partial x_{2i-1}} + \zeta^{2i} x_{2i-1} \frac{\partial}{\partial x_{2i}}.$$

Since we want real vector fields we take the following linear combinations, with $i = 1, 2, 3$,

$$\begin{aligned}
w_i &= (v_1 + \zeta^{2i} v_3 + \zeta^i v_5)/3 = \sum_{j=0}^{n/3-1} \frac{\partial}{\partial x_{6j+2i}}\\
w_{3+i} &= (v_2 + \zeta^{2i} v_4 + \zeta^i v_6)/3 = \sum_{j=0}^{n/3-1} x_{6j+2i-1} \frac{\partial}{\partial x_{6j+2i}} + \frac{\partial}{\partial x_{6j+2i+1}},
\end{aligned}$$

where $x_{2n+1} = x_1$ and 3 divides n. These vector fields are 3-symmetries of the mapping ϕ_n. Let J^3 be the Jacobian matrix of δ_n^3. The vector fields w_i, $i \neq 1, 4$ can be obtained as follows $w_i = J^3 \delta_n w_{i-1}$. According to [10, proposition 1] it suffices to verify that w_1 and w_4 are 3-symmetries. We note that $v_i = J^3 \delta_n v_i$ for $i = 1, \ldots, 6$. The following polynomials form a complete set of joint invariants of the vector fields w_i, $i = 1, .., n-6$

$$\begin{aligned}
z_i &= x_{2i-1} - x_{2i+5}, \quad i = 1, .., n-3 \\
z_{n-3+i} &= x_{2i+1}(x_{2i-1} - x_{2i+5}) - x_{2i} + x_{2i+6}, \quad i = 1, 2, 3 \\
z_{n+i} &= (x_{2i-1} - x_{2i+5})(x_{2i+6} - x_{2i+12}) \\
&\quad - (x_{2i} - x_{2i+6})(x_{2i+5} - x_{2i+11}), \quad i = 1, .., n-6
\end{aligned}$$

Another joint invariant is given by P, which therefore can be expressed in terms of the z_i. However, we haven't found a general formula for $P(z)$. In terms of the z-variables ϕ_6 is

$$\begin{aligned}
z_i &\mapsto z_{i+1}, \quad i \neq 3, 6 \\
z_3, z_6 &\mapsto -z_1 + \frac{\gamma}{z_5 - z_2 z_3}, -z_4 + z_1 z_2 - \gamma \frac{z_2}{z_5 - z_2 z_3},
\end{aligned}$$

and ϕ_n, with $n = 3m > 6$,

$$\begin{aligned}
z_i &\mapsto z_{i+1}, \quad i \neq n-3, n, 2n-6 \\
z_{n-3}, z_n, z_{2n-6} &\mapsto -\sum_{i=1}^{n/3-1} z_{3i-2} - \gamma P, -\frac{z_{n+1} + z_4(z_1 z_2 - z_{n-2})}{z_1}, A - \gamma BP,
\end{aligned}$$

where $A = (x_{2n-11} - x_{2n-5})(x_{2n-4} - x_2) - (x_{2n-10} - x_{2n-4})(x_{2n-5} - x_1)$, and $B = x_3(x_{2n-11} - x_{2n-5}) - x_{2n-10} + x_{2n-4}$ are also joint invariants of the six vector fields w_i. All the integrals we found survive the above reduction. As we can see in Table 15 there are enough integrals for the map to be integrable. Apart from the integrals obtained by the staircase method we have found two functions which are 3-integrals, that is, integrals of δ_n^3. These are

$$\begin{aligned}
i_1 &= \sum_{j=1}^{n/3} (x_{6j+1} - x_{6j-5})(x_{6j-2} - x_{6j-3} x_{6j-1}) + x_{6j-1}(x_{6j+2} - x_{6j-4}), \\
i_2 &= \sum_{j=1}^{n/3} x_{6j-1}(x_{6j-5} x_{6j-3} - x_{6j-4} - x_{6j+1} x_{6j+3} + x_{6j+2}) + x_{6j}(x_{6j+3} - x_{6j-3}).
\end{aligned}$$

We have the following action of ϕ_n on the 3-integrals

$$(i_1, i_2) \mapsto (i_2 - i_1 + \gamma, -i_1 + \gamma), \tag{25}$$

whose third power is the identity. Note that by applying the map to one of the 3-integrals gives us the other one but no third functionally independent 3-integral can be obtained in this way. The 3-integrals admit the vector fields w_i as symmetries and two of the $n-3$ functionally independent integrals found by the staircase method can be written in terms of them. For $n = 6$ we have

$$i_1 = -z_1 z_2 z_3 + z_1 z_5 + z_3 z_4, \quad i_2 = z_2 z_6 + z_3 z_4.$$

and two of the three functionally independent integrals found by the staircase method are given by

$$i_2^2 + i_1(i_1 - i_2 - \gamma), i_1(\gamma - i_2)(\gamma - i_1 + i_2)$$

which are both integrals of the 2-dimensional map (25). An extra advantage of working with expansion (23) instead of Traces of powers of $\mathcal{L}$ is that the third functionally independent integral factorizes nicely as

$$z_4 z_5 z_6 (z_1 z_3 + z_6)(z_4 z_5 - z_4 z_2 z_3 - z_2 z_1 z_5 + z_2^2 z_1 z_3 + \alpha z_2).$$

(n-1,-1)-reduction

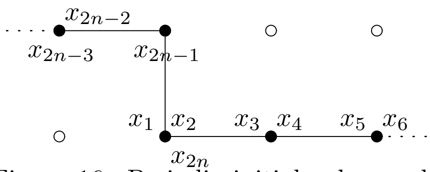

Figure 16: Periodic initial value problem for the Boussinesq system

We take $s_1 = n-1$ positive ($n \neq 2$), $s_2 = -1$, and we consider the following initial value problem, with $i = 0, 1, \ldots, n-2$, $k \in \mathbb{Z}$,

$$\begin{aligned} u_{(n-1)k+i,-k} &= x_{2i+1}, \\ v_{(n-1)k+i,-k} &= x_{2i+2}, \\ v_{(n-1)(k+1),-k} &= x_{2n-1}, \\ w_{(n-1)k,-k} &= x_{2n}, \end{aligned}$$

see Figure 16.

The right-shift induces a $2n$-dimensional measure preserving mapping with density $x_{2n-3} - x_1)$ ($n \neq 2$), $x \mapsto x'$:

$$\begin{aligned} x_i &\mapsto x_{i+2}, \quad i < 2n-3 \\ x_{2n-3} &\mapsto \frac{x_{2n-2} - x_2}{x_{2n-3} - x_1} \\ x_{2n-2} &\mapsto x_{2n-1} \\ x_{2n-1} &\mapsto -x_{2n} + \frac{(x_{2n-3} - x_1)(x_1(x_{2n-1} - x_4) - \gamma)}{x_{2n-2} - x_2 - x_3(x_{2n-3} - x_1)} \\ x_{2n} &\mapsto -x_2 + x_1 x_3. \end{aligned}$$

This mapping admits the symmetries

$$\frac{\partial}{\partial x_{2n-1}} - \frac{\partial}{\partial x_{2i}} + \sum_{i=1}^{n-1} \frac{\partial}{\partial x_{2n}}$$

and

$$x_1 \frac{\partial}{\partial x_{2n}} + \zeta(x_{2n-3}) \frac{\partial}{\partial x_{2n-1}} + \sum_{i=1}^{n-1} \frac{\partial}{\partial x_{2i-1}} + x_{2i-1} \frac{\partial}{\partial x_{2i}}$$

and hence can be reduced to a $(2n-2)$-dimensional mapping. The number of functionally independent integrals, obtained by k-expansion of the coefficients in (23), with

$$\mathcal{L} = M^{-1}(x_1, x_{2n}, \zeta_n(x_{2n-3}), x_{2n-1})L(x_{2n-3}, w_{n-1}, \zeta_n(x_{2n-3}), x_{2n-1}) \cdot \prod_{i=2}^{\overset{\frown}{n-1}} L(x_{2i-3}, w_{i-1}, x_{2i-1}, x_{2i}),$$

is $n-1$, except when 3 divides $(n+1)$, where the number is $n-3$, and all integrals admit the above vector fields as their symmetries. We have verified this up to $n = 7$. If $3 \mid (n-1)$ the symmetries (24) yield 3-symmetries of the mapping. As in the previous example we take linear combinations to get

$$w_1 = \sum_{j=0}^{(n-2)/3} \frac{\partial}{\partial x_{6j+2}},$$
$$w_4 = \sum_{j=0}^{(n-5)/3-1} \left(x_{6j+1} \frac{\partial}{\partial x_{6j+2}} + \frac{\partial}{\partial x_{6j+3}} \right) + x_{2n-3} \frac{\partial}{\partial x_{2n-2}},$$

together with $w_i = J^3 w'_{i-1}$, $i \neq 1, 4$. Taking $n = 5$ we obtain the following $(4 = 10 - 6)$-dimensional reduced mapping, in terms of $y_1 = x_7 - x_1, y_2 = x_6 + x_{10} - x_5 x_1, y_3 = x_8 - x_2 + x_3(-x_7 + x_1), y_4 = (-x_7 + x_1)(-x_9 + x_4 - x_5 x_3) + x_5(-x_8 + x_2)$:

$$(y_1, y_2, y_3, y_4) \mapsto (\frac{y_3}{y_1}, y_3, \frac{y_4}{y_1}, y_4 + \gamma - \frac{y_2 y_3}{y_1}).$$

This mapping is measure preserving with density y_1^2. It has two 3-integrals, $i_1 = y_2 y_3 / y_1$ and $i_2 = y_4$, which satisfy

$$(i_1, i_2) \mapsto (i_2, -i_1 - i_2 - \gamma),$$

whose third power is the identity. The two functionally independent invariants found by the staircase method can be expressed in terms of the 3-integrals as

$$i_1^2 + i_2^2 + i_1 i_2 + \gamma(i_1 + i_2), \quad i_1 i_2 (i_1 + i_2 + \gamma).$$

4 Multivaluedness of iterates of correspondences

There are certain lines in the (s_1, s_2)-plane where the periodic reduction yields a correspondence instead of a mapping, see [13]. Note, that here we impose the solution to be periodic, which is not implied by the periodicity of the initial condition. For the pKdV equation we find correspondences on lines given by $s_1 s_2 = 0$, whereas for the QD-system, see equation (10), they are $s_2(s_2 + 2s_1) = 0$. In both cases we get two-valued correspondences. In general the multi-valuedness of the n-th iterate of a two-valued correspondence would be 2^n. However, for integrable correspondences one expects the multi-valuedness to grow polynomially instead.

4.1 The potential Korteweg-De Vries equation, horizontal staircase

We take initial values for the pKdV equation (12) as follows: $u_{l,0} = x_k$ with $k \in \{1, 2, 3\}$, $k \equiv l \bmod 3$. Updating these by shifting them upwards, and imposing the image to be periodic, we have to solve the system

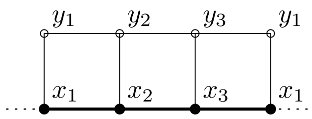

Figure 17: Initial values on a horizontal staircase, periodicity $u_{l,m} = u_{l+3,m}$.

$$
\begin{aligned}
(x_1 - y_2)(x_2 - y_1) &= \alpha \\
(x_2 - y_3)(x_3 - y_2) &= \alpha \\
(x_3 - y_1)(x_1 - y_3) &= \alpha
\end{aligned}
\tag{26}
$$

for the image points y_1, y_2, y_3. The three squares in Figure 17 represent the three equations (26).

In terms of reduced variables $q_1 = x_1 - x_2$, $q_2 = x_2 - x_3$ the volume preserving correspondence is $\varphi_\pm$:

$$
\begin{aligned}
q_1 &\mapsto q_1 \frac{\alpha(\alpha + q_2(q_1 + q_2)) + q_1^2 q_2(q_1 + q_2)/2 + (q_1/2 + q_2)R}{(\alpha + q_1 q_2)(\alpha - q_1(q_1 + q_2))} \\
q_2 &\mapsto q_2 \frac{\alpha(\alpha + q_1(q_1 + q_2)) + q_1 q_2^2(q_1 + q_2)/2 - (q_1 + q_2/2)R}{(\alpha + q_1 q_2)(\alpha - q_2(q_1 + q_2))},
\end{aligned}
$$

where

$$
R = \pm\sqrt{4\alpha^3 + (q_1 q_2 (q_1 + q_2))^2}.
$$

This correspondence admits the integral $q_1 q_2 (q_1 + q_2)$, which can be obtained by taking the trace of monodromy matrix.

Generically the multi-valuedness of the l-th iterate of a two-valued map would be 2^l. This is not the case here. Due to the $x \leftrightarrow y$ symmetry of the system (26) the correspondence φ equals its own inverse. The relations $\varphi_\pm^{-1} = \varphi_\mp$ imply that the l-th iterate of the correspondence is $(l + 1)$-valued, see Figure 18. All points on the same vertical line have the same value.

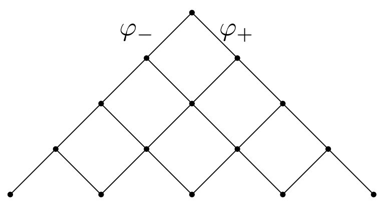

Figure 18: Polynomial growth of multivaluedness of iterates.

4.2 The QD-algorithm, horizontal staircase

We consider a horizontal staircase for the QD-system (10). Solving the non-local (or, implicit) scheme we find rational expressions for multivalued mappings.

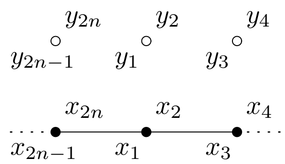

Figure 19: Initial values and their images under the up-shift

As initial conditions we take

$$e_{m,0} = x_{2m-1}, \quad q_{m,0} = x_{2m},$$

where

$$x_m = x_{m \mod 2n}, \tag{27}$$

see Figure 19. As mapping we take the up-shift, and the image of x_i will be denoted y_i.

We have the following $2n$ equations for the $2n$ unknowns y_i, with $i = 1, \ldots, n$,

$$y_{2i-1}y_{2i} = x_{2i-1}x_{2i+2}, \quad y_{2i-1} + y_{2i+2} = x_{2i+1} + x_{2i+2}. \tag{28}$$

We first solve for the odd variables, thereby obtaining a set of n equations for the even variables

$$y_{2i-1} = \frac{x_{2i-1}x_{2i+2}}{y_{2i}} = x_{2i+1} + x_{2i+2} - y_{2i+2}.$$

We write $y_{2i+2} = m_{2i+2}(y_{2i})$, where

$$m_k(z) = x_{k-1} + x_k - \frac{x_{k-3}x_k}{z}.$$

Now x_{2k} is a fixed point of the Möbius transformation

$$\begin{aligned} M_k &= m_{2k}m_{2(k-1)}m_{2(k-2)} \cdots m_{2(k-n+1)} \\ &= m_{2k} \cdots m_{2(k+3)}m_{2(k+2)}m_{2(k+1)}. \end{aligned}$$

One fixed point of M_k is given by x_{2k-1}, as $m_i(x_{i-3}) = x_{i-1}$. This gives us one way of updating our initial values, $\sigma : x \mapsto x'$,

$$\begin{array}{c} x_{2i-1} \mapsto x_{2i+2} \\ x_{2i} \mapsto x_{2i-1} \end{array}, \qquad i = 1, \ldots, n \tag{29}$$

where we assume periodicity (27). The other fixed point of M_{2k} is $z_k := x_{2(k+1)}Q^n_{2k}/Q^n_{2k-2}$, where[3]

$$Q^n_k(x) = \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i-1} x_{2j+2+k} \right) \left(\prod_{j=i}^{n-1} x_{2j+1+k} \right). \tag{30}$$

This follows from

[3]The function Q is closely related to the function F^m_k given by equation (18), we have $F^m_k = 1 + Q^{m-2}_{k+1}(q)/\prod_{j=1}^{k} q_{2j+m}$.

Lemma 3

$$m_{2k}(z_{k-1}) = z_k$$

Proof: In terms of Q the statement is

$$x_{2k-3}Q^n_{2k-4} + x_{2k+2}Q^n_{2k} = (x_{2k-1} + x_{2k})Q^n_{2k-2} \tag{31}$$

From the definition (30) it follows that

$$x_{2n-1}Q_0^{n-1} + \prod_{j=2}^{n} x_{2j} \;=\; Q_0^n \;=\; x_4 Q_2^{n-1} + \prod_{j=2}^{n} x_{2j-1}. \tag{32}$$

Therefore we have

$$\begin{aligned}
&x_{2n-1}Q^n_{-2} + x_4 Q_2 \\
&= x_{2n-1}(x_2 Q_0^{n-1} + \prod_{j=1}^{n-1} x_{2j-1}) + x_4(x_1 Q_2^{n-1} + \prod_{j=3}^{n+1} x_{2j}) \\
&= x_2(x_{2n-1}Q_0^{n-1} + \prod_{j=2}^{n} x_{2j}) + x_1(x_4 Q_2^{n-1} + \prod_{j=2}^{n} x_{2j-1}) \\
&= (x_1 + x_2)Q_0^n,
\end{aligned}$$

which is equation (31) with $k = 1$. This implies that equation (31) holds for all k, as we may shift $x_i \mapsto x_{i+2(k-1)}$. □

The mapping that corresponds to the fixed point z_k will be denoted $\tau : x \mapsto x'$,

$$\begin{aligned} x_{2i-1} &\mapsto x_{2i-1}\frac{Q^n_{2i-2}}{Q^n_{2i}} \\ x_{2i} &\mapsto x_{2i+2}\frac{Q^n_{2i}}{Q^n_{2i-2}} \end{aligned}, \qquad i = 1, \ldots, n \tag{33}$$

where (27) is understood. Since a Möbius transformation has at most two fixed points we obtained a two-valued correspondence (σ, τ). Integrals for this correspondence are given by the coefficients of the k-expansions of the trace and determinant of the monodromy matrix $L(x_{2n}, x_{2n-1}) \cdots L(x_4, x_3)L(x_2, x_1)$. For all $n < 10$ we found $n + 1$ functionally independent integrals. The lowest non-trivial case is $n = 2$. Explicitly, both mappings σ :

$$(x_1, x_2, x_3, x_4) \mapsto (x_4, x_1, x_2, x_3)$$

and τ :

$$(x_1, x_2, x_3, x_4) \mapsto \left(x_1\frac{x_3 + x_4}{x_1 + x_2}, x_4\frac{x_1 + x_2}{x_3 + x_4}, x_3\frac{x_1 + x_2}{x_3 + x_4}, x_2\frac{x_3 + x_4}{x_1 + x_2} \right)$$

admit the three invariants $x_1 + x_2 + x_3 + x_4$, $x_1x_3 + x_2x_4$, and $x_1x_2x_3x_4$. In reducing coordinates $y_i = x_{i+1}/x_1$, $i = 1, 2, 3$, we have

$$\sigma : \;(y_1, y_2, y_3) \mapsto \left(\frac{1}{y_3}, \frac{y_1}{y_3}, \frac{y_2}{y_3}\right)$$

and

$$\tau: \ (y_1, y_2, y_3) \mapsto \left(y_3 \frac{(y_1+1)^2}{(y_3+y_2)^2}, y_2 \frac{(y_1+1)^2}{(y_3+y_2)^2}, y_1 \right),$$

which admit the invariants

$$\frac{y_2 + y_1 y_3}{(1+y_1+y_2+y_3)^2}, \quad \frac{y_1 y_2 y_3}{(1+y_1+y_2+y_3)^4}.$$

We are also interested in the growth of multi-valuedness. For integrable correspondences this should grow polynomially. As one can easily verify the above 4-dimensional mappings satisfy

$$\tau\sigma\tau = \sigma\sigma\sigma \quad \text{and} \quad \tau\sigma\sigma = \sigma\sigma\tau. \tag{34}$$

Due to these relations the l-th iterate of the correspondence is $2l$ valued, see figure 4.

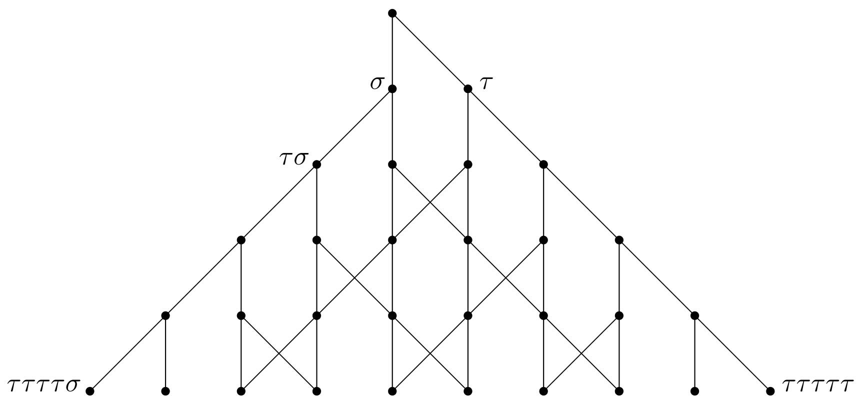

Figure 20: Graphical representation of Proposition 4

Proposition 4 *With compositions of mappings (34), the l-th iterate of the correspondence (σ, τ) is $2l$ valued.*

Proof: The relations (34) can be used to rewrite every word in two symbols σ and τ of length l into either a word that does not contain $\tau\sigma$ or a word that only contains $\tau\sigma$ at the end. There are $2l$ such words:

$$\tau\tau\tau\cdots\tau\tau\tau, \sigma\tau\tau\cdots\tau\tau\tau, \sigma\sigma\tau\cdots\tau\tau\tau, \ldots, \sigma\sigma\sigma\cdots\sigma\sigma\tau,$$
$$\tau\tau\tau\cdots\tau\tau\sigma, \sigma\tau\tau\cdots\tau\tau\sigma, \sigma\sigma\tau\cdots\tau\tau\sigma, \ldots, \sigma\sigma\sigma\cdots\sigma\sigma\sigma.$$

So there are at most $2l$ inequivalent words of length l. To show that there are exactly $2l$ inequivalent words of length l we proceed by induction. We suppose there are are exactly $2l$ inequivalent words of length l. For any three words u, v, w we have $uv = uw \Rightarrow v = w$. So if two different words of length l extend to an equivalent word of length $l+1$, this word can be written (we concatenate from the left) as $u = \tau v = \sigma w$. If a third word of length l would extend to u then we have $u = \tau z$ or $u = \sigma z$ which would imply $z = v$ or $z = w$ respectively. Therefore words of length l that extend to words of length $l+1$ coincide at most pairwise, giving an lower bound of $2l$ on the number of words of length $l+1$. However, since there are two words of length $l+1$ that are not equivalent to any other word, namely $\tau\tau\tau\cdots\tau\tau\tau$ and $\sigma\tau\tau\cdots\tau\tau\tau$, there are at least $2l+2$ inequivalent words of length $l+1$. $\square$

The same relations hold in the $2n$-dimensional case.

Proposition 5 *The $2n$-dimensional mappings (33),(29) satisfy the relations (34).*

Proof: The second of the relations (34) is easy as is $\sigma\sigma = x_k \mapsto x_{k+2}$ clearly commutes with τ.

For the first relation we solve the equations (28) for the x_i in terms of the y_k to find the inverse τ^{-1}:

$$x_{2i-1} \mapsto x_{2i-1}\frac{W^n_{2i}}{W^n_{2i-2}}x_{2i} \qquad \mapsto x_{2i-2}\frac{W^n_{2i-4}}{W^n_{2i-2}}, \qquad i = 1,\dots,n$$

where

$$W^n_k = \sum_{i=1}^{n-1}\left(\prod_{j=1}^{i-1} x_{2j-1+k}\right)\left(\prod_{j=i}^{n-1} x_{2j+2+k}\right). \tag{35}$$

One can verify that $\sigma(W^n_k) = Q^n_k$. This implies that $\sigma\tau^{-1}$ is given by

$$\begin{aligned} x_{2i-1} &\mapsto x_{2i+2}\frac{Q^n_{2i}}{Q^n_{2i-2}} \\ x_{2i} &\mapsto x_{2i-3}\frac{Q^n_{2i-4}}{Q^n_{2i-2}}, \qquad i = 1,\dots,n \end{aligned}$$

which equals its inverse $\tau\sigma^{-1}$. Multiplying $\tau\sigma^{-1} = \sigma\tau^{-1}$ from the left by $\tau\sigma$ and from the right by σ, using $\tau\sigma\sigma = \sigma\sigma\tau$, gives us $\tau\sigma\tau = \sigma\sigma\sigma$. $\square$

Remark: One can also directly prove that the τ^{-1} provided is the inverse of τ. This relies on the fact that $\tau(W^n_k) = Q^n_k$, which in terms of Q amounts to the identity

$$\prod_{i=1}^{n-1} Q^n_{2i} = \sum_{i=1}^{n}(\prod_{j=1}^{i-1} x_{2j-1})(\prod_{j=i}^{n-1} x_{2j+4})(\prod_{j=1}^{n-2} Q^n_{2j+2i}),$$

which can be proved using

$$Q_0^{n-m}Q_2^n = x_1 Q_2^{n-m-1}Q_0^n + Q^n_{2(n-m)}(\prod_{i=3}^{n-m+1} x_{2i})$$

which in turn relies on a generalization of the relations (32),

$$Q_0^{n-m} = Q_{2m}^{n-m} (\prod_{i=2}^{m+1} x_{2i}) + (Q_0^m \prod_{i=m}^{n-1} x_{2i+1}).$$

5 Concluding remarks

We showed that the staircase method provides integrals for mappings and correspondences derived as $\mathbf{s}$-periodic reductions of lattice equations and systems of lattice equations. We also showed that such mappings and correspondences can be dimensionally reduced systematically, using k-symmetries of the lattice equations. In all examples encountered the staircase method yields sufficiently many functionally independent integrals for the d-reduced mappings and correspondences to be completely integrable.

For posing initial value problems we used the method laid out in [13]. Although, for the Boussinesq system we presented an alternative approach. We have calculated integrals for mappings/correspondences with dimension smaller than say 20. Therefore we were able to check functional independence of the integrals using the symbolic software package Maple [15]. Closed form expressions for integrals of mappings with arbitrary dimensions have been given in [**?**, 28]. Here, the lattice equations are of Adler-Bobenko-Suris type [2, 3] and the integrals are expressed in terms of multisums of products. Their functional independence and involutivity is being investigated. It is an open problem whether such closed form expressions can be obtained in the examples given here, that is, for the Bruschi-Calogero-Droghei equation, the Quotient-Difference algorithm and the Boussinesq system. Another question is how to obtain symplectic structures for the mappings. This would enable one to conclude complete integrability.

We have obtained few results on mappings of arbitrary dimension. These include an explicit $d = 2 + (-1)^n$-reduction of the mapping related to the $(n-1,1)$-reduction of the lattice potential KdV equation, as well as an explicit expression for a 2-integral of the mapping in the case that n is even. We note that in the case $n = 2$ the 3-reduced mapping is a linear involution, which can be used to explicitly solve the $(3,1)$-mapping, see the appendix in [14]. We proved that the $(n-1,1)$-reduction of the Boussinesq system can be d-reduced with $d = 6$ if $3|n$ and $d = 2$ otherwise and we presented two 3-integrals for this mapping (expressed in terms of the original variables). For the $(n-1,-1)$-reduction of the Boussinesq system we have provided 2 symmetries (for all n) and 4 3-symmetries (when $3|(n-1)$, which show the mapping can be d-reduced with $d = 6$ if $3|(n-1)$ and $d = 2$ otherwise. Finally, the $(n,0)$-reduction the QD-system yields a $2n$-dimensional 2-valued correspondence. We have given an explicit expression for this correspondence and showed that its nth iterate is $2n$-valued. It would be interesting to investigate other ways of establishing such a result because one does not always have explicit expressions at hand.

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