Symmetry condition in terms of Lie brackets.

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Abstract

A passive orthonomic system of PDEs defines a submanifold in the corresponding jet manifold, coordinated by so called parametric derivatives. We restrict the total differential operators and the prolongation of an evolutionary vector field v to this submanifold. We show that the vanishing of their commutators is equivalent to v being a generalized symmetry of the system.

1 The standard symmetry condition

In the majority of cases where exact solutions of differential equations can be found, the underlying property is a (continuous) symmetry of the equation [11, 7]. And, in the theory of integrable equations, the recognition and classification methods based on the existence of symmetries have been particular successful [5, 10, 13, 3].

A symmetry-group transforms one solution of an equation to another solution of the same equation. Although this idea goes back to Sophus Lie, we refer to [7] for a good introduction to the subject, numerous examples, applications and references. And we quote: 'The great power of Lie group theory lies in the crucial observation that one can replace the complicated, nonlinear conditions for the invariance of the solution set of an equation under the group transformations by an equivalent linear condition of infinitesimal invariance under the corresponding infinitesimal generators of the group action' [7]. In this paper we provide a characterization of symmetries that is different from the standard one, generalizing a similar characterization in the special setting of ordinary differential equations [11, eq. (3.35)], and evolution equations [7, Prop. 5.19] to the setting of passive orthonomic systems.

The natural framework in which symmetries of differential equations are studied is the so called jet-manifold M. Coordinates on M consist of p independent variables x_i, q dependent variables u^α and the derivatives of the dependent variables, which are denoted using multi-index notation, e.g.

$$u^2_{1,0,3} = \frac{\partial^4 w}{\partial r \partial t^3}$$

when $x = (r, s, t)$ and $u = (v, w)$. A typical point $z \in M$ is $z = (x_i, u^\alpha, u^\alpha_K)$. The ring of smooth functions on M will be denoted $\mathcal{A}$. To indicate functional dependence of $f \in \mathcal{A}$ we simply write $f(z)$. Thus the system $\Delta(z) = 0,\ \Delta \in \mathcal{A}^n$ is a system of n PDEs.

The action of a Lie group is defined on the space of dependent and independent variables, and then prolonged to an action on the jet manifold. Likewise the infinitesimal generator of the symmetry group is obtained by prolongation from an infinitesimal vector field on the base manifold. It turns out that any symmetry has an evolutionary representative [7, Prop. 5.5]. In terms of the total differential operators

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\alpha,K} u^\alpha_{Ki} \frac{\partial}{\partial u^\alpha_K},\ i = 1, \ldots, p, \tag{1}$$

the prolongation pr_Q of an evolutionary vector field $\nu_Q = \sum_\alpha Q^\alpha \partial/\partial u^\alpha$ is

$$\mathrm{pr}_Q = \sum_{\alpha,K} D_K Q^\alpha \frac{\partial}{\partial u^\alpha_K}. \tag{2}$$

A simple computation shows that these derivations on $\mathcal{A}$ commute among each other, we have $[D_i, D_j] = 0$, $i, j = 1, \ldots, p$ and

$$[D_i, \mathrm{pr}_Q] = 0, \quad i = 1, \ldots, p,\ Q \in \mathcal{A}. \tag{3}$$

In fact, up to a linear combination of translational fields $\partial/\partial x_i$, evolutionary vector fields are uniquely determined by property (3), cf. [7, Lemma 5.12].

The condition of infinitesimal invariance, *the standard symmetry condition*, is [7, Theorem 2.31]

$$\mathrm{pr}_Q \Delta \equiv 0 \bmod \Delta \tag{4}$$

in which case ν_Q, or the tuple $Q \in \mathcal{A}^n$ itself, is called a (generalized) symmetry of the system $\Delta = 0$. The tuple Q is a trivial symmetry if $Q \equiv 0 \bmod \Delta$, which defines an equivalence relation on the space of symmetries. In section 2 we show this is well defined for *passive orthonomic systems*. We restrict Q to be a function $Q \in \mathcal{B}$ on the sub-manifold of the jet-manifold defined by our system of PDEs. Although this is a more than standard procedure, restricting the derivations to act on this sub-manifold is not standard at all, except possibly in the settings of ODEs and evolution equations. This is, at least from a philosophical point of view, not fully satisfying.

In section 3, for any passive orthonomic system of partial differential equations, we define intrinsic total differential operators $\mathfrak{D}_i$ and an intrinsic prolongation $\mathfrak{pr}_Q$, which are derivations on the sub-space $\mathcal{B}$. Subsequently we show that the vanishing of the Lie brackets

$$[\mathfrak{D}_i, \mathfrak{pr}_Q] = 0, \quad i = 1, \ldots, p,\ Q \in \mathcal{B}$$

is equivalent to Q being a symmetry.

2 Passive orthonomic systems

Restricting to the sub-manifold is particularly easy when dealing with orthonomic systems, in which case this amounts to using the equations as substitution rules. However, in general the order of substituting and differentiating does matter, one encounters integrability conditions. For example, for the system $u_x = X, u_y = Y$ to be formally integrable we need $D_y X = D_x Y$. In general, a finite number of integrability conditions suffices to make the system formally integrable, in which case the system is called passive.

The idea of a passive orthonomic system is the main idea behind Riquier's existence theorems [12] and the corresponding algorithms for solving systems of PDEs due to Janet [2]. Riquier-Janet theory extended the works of Cauchy and Kovalevskaya, it takes a prominent place in computer algebra applied to PDE theory [8], and it has lead to important developments in polynomial elimination theory [1]. The passive orthonomic system was the predecessor of what is now called an involutive system of PDEs. For our purpose, the concept of involutivity does not play a role. We adopt a similar philosophy as in [4], and stick to the setting of passive orthonomic systems. In that paper an efficient algorithm is given by which any orthonomic system can be made passive by construction of a sufficient set integrability conditions free of redundancies [4].

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_q = \{1, 2, \ldots, q\}$. We denote the i-th component of $J \in \mathbb{N}^p$ by J_i and addition in $\mathbb{N}^p$ is denoted by concatenation. A set of basis vectors for $\mathbb{N}^p$ is given by $\{1, 2, \ldots, p\}$, where $i_j = 1$ when $i = j$ and $i_j = 0$ otherwise. Thus, with $J, K \in \mathbb{N}^p$, we have $(JK)_i = J_i + K_i$, and in particular $(Kj)_i$ equals K_i or, when $j = i$, $K_i + 1$. Also, when $L = JK$ we write $J = L/K$. Since total differential operators commute we can define a multi-differential operator D_K as

$$D_K = D_1^{K_1} D_2^{K_2} \cdots D_p^{K_p}, \tag{5}$$

and we have $D_K u^j = u_K^j$.

Choose n points $(i^\alpha, J^\alpha) \in \mathbb{N}_q \times \mathbb{N}^p$, with nonzero J^α, $\alpha = 1, \ldots, n$. A derivative u_K^j is called *principal* if there exist $L \in \mathbb{N}^p$ such that $(j, K) = (i^\alpha, J^\alpha L)$ for some α. The remaining ones are called *parametric*. The set of all (j, K) such that u_K^j is parametric is denoted $\mathcal{S}$, and the subspace of $\mathcal{A}$ consisting of smooth functions of the parametric derivatives is denoted $\mathcal{B}$.

We also choose a ranking $\leq$ on $\mathbb{N}_q \times \mathbb{N}^p$, that is, a total order relation which is positive:

$$\forall L, (j, K) \leq (j, KL),$$

and, compatible with differentiation:

$$(i, J) \leq (j, K) \Leftrightarrow (i, JL) \leq (j, KL),$$

cf. [6, 9].

We consider systems of n partial differential equations, with $\alpha = 1, \dots, n$,

$$u^{i^\alpha}_{J^\alpha} = P^\alpha, \qquad P^\alpha \in \mathcal{B}. \tag{6}$$

The system (6) will be written shortly $\Delta = 0$, where $\Delta^\alpha = u^{i^\alpha}_{J^\alpha} - P^\alpha$. We make the following assumptions:

i) the P^α only depend on u^j_K with $(j, K) < (i^\alpha, J^\alpha)$, and

ii) $(i^\alpha, J^\alpha K) = (i^\beta, J^\beta L) \Rightarrow D_K P^\alpha = D_L P^\beta$.

Such systems are called *passive orthonomic systems*. Their crucial property is that for any $Q \in \mathcal{A}$, there is a unique $\widehat{Q} \in \mathcal{B}$ such that $\widehat{Q} \equiv Q \mod \Delta$. This $\widehat{Q}$ can be obtained from the following reduction algorithm.

Algorithm 1 Input: *Expression* $Q \in \mathcal{A}$. Output: *Expression* $\widehat{Q} \in \mathcal{B}$.

- $\star$ *if no principal derivative appears in* Q *then return* Q
- $\star$ *let* u^j_K *be the* $\leq$*-highest principal derivative appearing in* Q*, and let* α, L *be such that* $j, K = i^\alpha, J^\alpha L$
- $\star$ *substitute* $u^j_K = D_L P^\alpha$ *in* Q *and call the result* R
- $\star$ *return* $\widehat{R}$

The algorithm terminates because the highest principal derivative of R, if it exists, is $\leq$-smaller than u^j_K, due to assumption i). And, a different choice of α wouldn't change the result because of assumption ii).

The following lemma states that differentiation is compatible with the above reduction $\mathcal{A} \to \mathcal{B}$, cf. [4, Theorem 4.8].

Lemma 2 *For any* $Q \in \mathcal{A}$ *we have*

$$\widehat{D_K Q} = \widehat{D_K \widehat{Q}}$$

Proof: Using a modified version of Algorithm 1 we can write $Q = \widehat{Q} + R(\Delta)$, where R is some differential function of Δ such that $R(0) = 0$. Clearly $\widehat{D_K R(\Delta)}$ vanishes. $\square$

3 The intrinsic symmetry condition

Definition 3 *We define an* intrinsic multi-differential operator $\mathfrak{D}_K : \mathcal{B} \to \mathcal{B}$ *by*

$$\mathfrak{D}_K P = \widehat{D_K P}, \qquad P \in \mathcal{B}$$

From this definition and Lemma 2 we obtain the following properties.

Proposition 4 *Intrinsic multi-differential operators are compatible with concatenation, $\mathfrak{D}_K\mathfrak{D}_L = \mathfrak{D}_{KL}$.*

Proof: We have $\mathfrak{D}_K\mathfrak{D}_L P = \widehat{D_K\widehat{D_L P}} = \widehat{D_K D_L P} = \widehat{D_{KL}P} = \mathfrak{D}_{KL}P$. $\square$

Corollary 5 *We have the analogue of equation (5), $\mathfrak{D}_K = \mathfrak{D}_1^{K_1}\mathfrak{D}_2^{K_2}\cdots\mathfrak{D}_p^{K_p}$.*

Corollary 6 *Intrinsic total differential operators commute, $[\mathfrak{D}_i, \mathfrak{D}_j] = 0$.*

We would like to have a more intrinsic characterization of $\mathfrak{D}_i$, that is, without reference to any principal derivative or total differential operator. For $L \in \mathbb{N}^p$ we denote $\mathcal{S}_L = \{(\alpha, K) : (\alpha, KL) \in \mathcal{S}\}$, which is a subset of $\mathcal{S}$. From equation (1) and Definition 3 it follows that

$$\mathfrak{D}_j = \frac{\partial}{\partial x_j} + \sum_{(k,L)\in\mathcal{S}_j} u^k_{Lj}\frac{\partial}{\partial u^k_L} + \sum_{(i^\alpha, J^\alpha M/j)\in\mathcal{S}\setminus\mathcal{S}_j} \mathfrak{D}_M P^\alpha \frac{\partial}{\partial u^{i^\alpha}_{J^\alpha M/j}}. \tag{7}$$

We note that when $(k, L) \in \mathcal{S}\setminus\mathcal{S}_j$ there exist $\alpha \in \mathbb{N}_n$, $M \in \mathbb{N}^p$ such that $(k, Lj) = (i^\alpha, J^\alpha M)$, and, for any $\beta \in \mathbb{N}_n$, $N \in \mathbb{N}^p$ such that $(k, Lj) = (i^\beta, J^\beta N)$ we have $i^\alpha = i^\beta$ and $\mathfrak{D}_M P^\alpha = \mathfrak{D}_N P^\beta$. Due to Corollary 5 equation (7) provides a recursive schema for intrinsic total differentiation.

Proposition 7 *The recursive schema (7) for $\mathfrak{D}_i$ is well defined.*

Proof: We show the schema terminates using transfinite induction. For any $Q \in \mathcal{A}$, to evaluate $\mathfrak{D}_j Q$, apart from some differentiation and multiplications, we need to evaluate a finite number of expressions $\mathfrak{D}_I P^\alpha$. We assume that $\mathfrak{D}_L P^\beta$ can be evaluated for all $(i^\beta, J^\beta L) < (i^\alpha, J^\alpha I)$. Suppose P^α depends on u^j_K. That implies $(j, K) < (i^\alpha, J^\alpha)$. Suppose there are $\beta \in \mathbb{N}_n$ and $L \in \mathbb{N}^p$ such that $(j, KI) = (i^\beta, J^\beta L)$. Then, to evaluate $\mathfrak{D}_I P^\alpha$ one may need to evaluate $\mathfrak{D}_L P^\beta$. By the induction hypothesis this can be done. $\square$

Definition 8 *We define intrinsic prolongation, denoted $\mathfrak{pr}_Q : \mathcal{B} \to \mathcal{B}$, of an evolutionary vector field ν_Q with $Q \in \mathcal{B}$ by*

$$\mathfrak{pr}_Q P = \widehat{pr_Q P}, \qquad P \in \mathcal{B}.$$

From equation (2) and Definition 8 we get the intrinsic formula

$$\mathfrak{pr}_Q = \sum_{j,K\in\mathcal{S}} \mathfrak{D}_K Q^j \frac{\partial}{\partial u^j_K}.$$

We now state and prove our main theorem.

Theorem 9 *A tuple $Q \in \mathcal{B}$ is a symmetry of equation (6) iff*

$$[\mathfrak{D}_j, \mathfrak{pr}_Q] = 0$$

for all j.

Proof:

$\Leftarrow$ We calculate $\mathrm{pr}_Q\Delta^\alpha$ modulo Δ

$$\begin{aligned} \widehat{\mathrm{pr}_Q\Delta^\alpha} &= \widehat{D_{J^\alpha}Q^{i^\alpha}} - \widehat{\mathrm{pr}_Q P^\alpha} \\ &= \mathfrak{D}_{J^\alpha}Q^{i^\alpha} - \mathfrak{pr}_Q P^\alpha \end{aligned} \tag{8}$$

Next we calculate the commutator $[\mathfrak{D}_j, \mathfrak{pr}_Q]$. Neglecting second order derivatives, we get

$$\mathfrak{D}_j\mathfrak{pr}_Q = \sum_{(k,L)\in\mathcal{S}} \mathfrak{D}_{Lj}Q^k\frac{\partial}{\partial u^k_L},$$

and

$$\mathfrak{pr}_Q\mathfrak{D}_j = \sum_{(k,L)\in\mathcal{S}_j} \mathfrak{D}_{Lj}Q^k\frac{\partial}{\partial u^k_L} + \sum_{(i^\alpha,J^\alpha M/j)\in\mathcal{S}\setminus\mathcal{S}_j} \mathfrak{pr}_Q\mathfrak{D}_M P^\alpha\frac{\partial}{\partial u^{i^\alpha}_{J^\alpha M/j}}.$$

Hence we get

$$[\mathfrak{D}_j, \mathfrak{pr}_Q] = \sum_{(i^\alpha,J^\alpha M/j)\in\mathcal{S}\setminus\mathcal{S}_j} \left(\mathfrak{D}_{J^\alpha M}Q^{i^\alpha} - \mathfrak{pr}_Q\mathfrak{D}_M P^\alpha\right)\frac{\partial}{\partial u^{i^\alpha}_{J^\alpha M/j}}.$$

Suppose that $J^\alpha_j \neq 0$. Then the action of the above vector field on $u^{i^\alpha}_{J^\alpha/j}$ yields the right hand side of equation (8). Since we have chosen $J^\alpha \neq 0 \in \mathbb{N}^p$ this proves our case.

$\Rightarrow$ Suppose $\mathfrak{D}_{J^\alpha}Q^{i^\alpha} = \mathfrak{pr}_Q P^\alpha$. Then

$$[\mathfrak{D}_j, \mathfrak{pr}_Q] = \sum_{(i^\alpha,J^\alpha M/j)\in\mathcal{S}\setminus\mathcal{S}_j} \left[\mathfrak{D}_M, \mathfrak{pr}_Q\right] P^\alpha\frac{\partial}{\partial u^{i^\alpha}_{J^\alpha M/j}}.$$

We will prove that $\left[\mathfrak{D}_M, \mathfrak{pr}_Q\right] P^\alpha = 0$ for all $\alpha \in \mathbb{N}_n$ and $M \in \mathbb{N}^p$. The statement is certainly true for $M = 0$. Assume that $\left[\mathfrak{D}_N, \mathfrak{pr}_Q\right] P^\beta = 0$ for all $(i^\beta, J^\beta N) < (i^\alpha, J^\alpha M)$. When $M \neq 0$ there exists j such that $M/j \in \mathbb{N}^p$. We write

$$\left[\mathfrak{D}_M, \mathfrak{pr}_Q\right] P^\alpha = \mathfrak{D}_j\left[\mathfrak{D}_{M/j}, \mathfrak{pr}_Q\right] P^\alpha + \left[\mathfrak{D}_j, \mathfrak{pr}_Q\right]\mathfrak{D}_{M/j}P^\alpha$$

The first term is zero by the induction hypothesis, so we concentrate on the second, which is

$$\sum_{(i^\beta,J^\beta N)\in S\setminus S_j} \left[\mathfrak{D}_N, \mathfrak{pr}_Q\right] P^\beta\frac{\partial}{\partial u^{i^\beta}_{J^\beta N/j}}\mathfrak{D}_{M/j}P^\alpha.$$

Suppose P^α depends on u^k_L. Then $(k, L) < (i^\alpha, J^\alpha)$. The function $\mathfrak{D}_{M/j}P^\alpha$ may depend on the derivative $u^k_{LM/j}$ (namely, if $u^k_{LM/j} \in \mathcal{S}$, otherwise it depends on smaller derivatives). But when $(i^\beta, J^\beta N/j) \leq (k, LM/j) < (i^\alpha, J^\alpha M/j)$ by the induction hypothesis $\left[\mathfrak{D}_N, \mathfrak{pr}_Q\right] P^\beta$ vanishes. $\square$

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