Involutivity of integrals of sine-Gordon, modified KdV and potential KdV maps

Dinh T. Tran, P.H. van der Kamp, G.R.W. Quispel
Department of Mathematics, La Trobe University, Victoria, 3086, Australia
E-mail:dinhtran82@yahoo.com

February 10, 2011

Abstract

Closed form expressions in terms of multi-sums of products have been given in [13, 16] of integrals of sine-Gordon, modified Korteweg-de Vries and potential Korteweg-de Vries maps obtained as so-called $(p, -1)$-traveling wave reductions of the corresponding partial difference equations. We prove the involutivity of these integrals with respect to recently found symplectic structures for those maps. The proof is based on explicit formulae for the Poisson brackets between multi-sums of products.

1 Introduction

Integrable systems boast a long and venerable history. The history dates back to the 17th century with the work of Newton on the two body problem. The notion of integrability was first introduced by Liouville in the 19th century in the context of finite dimensional continuous Hamiltonian systems. Since then, it has been expanded to classes of nonlinear (partial) differential equations, see for example [4, 5]. More recently, there has been a shift of interest into systems with discrete time, e.g. integrable ordinary difference equations (or maps) and integrable partial difference (or lattice) equations. Some of the first examples of discrete integrable systems appeared in [6, 11]. And a classification of integrable lattice equations defined on a elementary square of the lattice has recently been obtained [1], based on the notion of multi-dimensional consistency. For maps there is the notion of complete or Liouville-Arnold integrability [2, 8, 17], analogues to the same notion for continuous systems. Briefly speaking, a mapping is said to be completely integrable if it has a sufficient number of functionally independent integrals that are in involution, that is, they Poison commute.

In this paper we study the involutivity of integrals of a certain class of integrable maps related to the fully discrete sine-Gordon, modified Korteweg-de Vries (mKdV) and potential Korteweg-de Vries (pKdV) equations. These maps arise as travelling wave reductions from the corresponding lattice equations. Such maps typically come in an infinite family of increasing dimension, and for this reason it is not feasible to calculate Poisson brackets one by one and show that they all vanish. One way to circumvent this problem is to use the so-called Yang-Baxter structure, and that is the approach taken in [3, 9]. This approach was used to prove the involutivity of integrals for the so-called $(p, -p)$-reduction of the lattice pKdV equation. We refer to [10, 15] for the background on (p, q)-travelling wave reductions. In this paper we study $(p, -1)$-reductions and we take a different approach. Starting from recently found symplectic structures [7, 12], and recently obtained closed-form expressions in terms of multi-sums of products for integrals of our family of sine-Gordon, mKdV and pKdV maps [16, 13], we proceed to prove involutivity of the integrals directly, using explicit formulae for the Poisson brackets between multi-sums of products. These formulae will be proven by induction on the number of variables, that is, on the dimension of the maps.

Recall, cf. [2, 7, 17], that a $2n$-dimensional discrete map $L : x \mapsto x^{'}$ is said to be completely integrable if:

- there is a $2n \times 2n$ anti-symmetric non-degenerate matrix Ω satisfying the Jacobi identity
$$\sum_l \left(\Omega_{li} \frac{\partial}{\partial x_l} \Omega_{jk} + \Omega_{lj} \frac{\partial}{\partial x_l} \Omega_{ki} + \Omega_{lk} \frac{\partial}{\partial x_l} \Omega_{ij} \right) = 0,$$
such that $dL(x)\Omega(x)dL^T(x) = \Omega(x^{'})$, where dL is the Jacobian of the map, $dL_{ij} := \frac{\partial x_i'}{\partial x_j}$.
- there exist n functionally independent integrals $I_1, I_2, \ldots, I_n$ satisfying $\{I_r, I_s\}_x = 0$ for all $1 \leq r, s \leq n$, where the Poisson bracket is defined by
$$\{f, g\}_x = \nabla_x(f).\Omega.(\nabla_x(g))^T, \tag{1}$$
with $\nabla_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{2n}} \right)$. Note that we will encounter several (related) Poisson brackets which are distinguished by the label x denoting the coordinates in which the bracket is expressed. Also, ∇_x will always have the right number of components.

The families of ordinary difference sine-Gordon, mKdV and pKdV equations are given as follows, [16, 13]

$$\text{sine-Gordon} : \alpha_1(v_n v_{n+p+1} - v_{n+1}v_{n+p}) + \alpha_2 v_n v_{n+1} v_{n+p} v_{n+p+1} - \alpha_3 = 0, \tag{2}$$
$$\text{modified KdV} : \beta_1(v_n v_{n+p} - v_{n+1}v_{n+p+1}) + \beta_2 v_n v_{n+1} - \beta_3 v_{n+p} v_{n+p+1} = 0, \tag{3}$$
$$\text{potential KdV} : (v_n - v_{n+p+1})(v_{n+1} - v_{n+p}) - \gamma = 0. \tag{4}$$

These equations are obtained from the $(p, -1)$-traveling wave reductions of the corresponding partial difference equations of the form
$$f(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+1,m+1}) = 0, \tag{5}$$
where we have taken $v_n = u_{l,m}$ with $n = l + mp$, introducing the periodicity $u_{l,m} = u_{l+p,m-1}$, cf. [10, 15].

The corresponding $d = p + 1$ dimensional maps derived from equations (2), (3), (4) are $\mathbb{R}^d \to \mathbb{R}^d$,
$$(v_1, v_2, \ldots, v_d) \mapsto (v_2, v_3, \ldots, v_{d+1}), \tag{6}$$
where
$$v_{d+1} = v_1^{-1} \frac{\alpha_1 v_2 v_d + \alpha_3}{\alpha_2 v_2 v_d + \alpha_1}, \quad v_{d+1} = v_1 \frac{\beta_1 v_d + \beta_2 v_2}{\beta_1 v_2 + \beta_3 v_d}, \quad v_{d+1} = v_1 - \frac{\gamma}{v_2 - v_d},$$
respectively. The integrals of sine-Gordon and mKdV maps can be expressed in terms of multi-sums of products, which we call Theta:
$$\Theta_{r,\epsilon}^{a,b}(f_a, f_{a+1}, \ldots, f_b) := \sum_{a \leq i_1 < i_2 < \cdots < i_r \leq b} \prod_{j=1}^{r} (f_{i_j})^{(-1)^{j+\epsilon}}, \tag{7}$$
with $f_i = v_i v_{i+1}$. In [16] it was shown that $\lfloor d/2 \rfloor$ integrals of the sine-Gordon map are given by
$$I_r^{\text{sG}} = \alpha_1 \left(\frac{v_d}{v_1} \Theta_{2r,1}^{1,d-1} + \frac{v_1}{v_d} \Theta_{2r,0}^{1,d-1} \right) + \alpha_2 \Theta_{2r+1,1}^{1,d-1} + \alpha_3 \Theta_{2r+1,0}^{1,d-1}, \quad 0 \leq 2r < d - 1 \tag{8}$$

and $\lfloor (d-1)/2 \rfloor$ integrals of the mKdV map are given by

$$I_r^{\text{mKdV}} = \beta_1 \left(v_1 v_d \Theta_{2r-1,0}^{1,d-1} + \frac{1}{v_1 v_d} \Theta_{2r-1,1}^{1,d-1} \right) + \beta_2 \Theta_{2r,1}^{1,d-1} + \beta_3 \Theta_{2r,0}^{1,d-1}, \quad 0 < 2r < d. \tag{9}$$

In [13] it was shown that $\lfloor (d-1)/2 \rfloor$ integrals of the pKdV map are given by

$$I_r^{\text{pKdV}} = \Psi_{r-1}^{2,d-2} + (v_d - v_2)\Psi_{r-1}^{2,d-3} + (v_{d-1} - v_1)\Psi_{r-1}^{3,d-2} + \Psi_{r-2}^{3,d-3} + ((v_{d-1} - v_1)(v_d - v_2) - \gamma)\, \Psi_r^{2,d-2}, \tag{10}$$

where $0 \leq r < \lfloor (d-1)/2 \rfloor$ and

$$\Psi_r^{a,b}(c_a, c_{a+1}, \ldots, c_{b+1}) = \left(\sum_{a \leq i_1, i_1+1 < i_2, i_2+1, \ldots, < i_r \leq b} \prod_{j=1}^{r} \frac{1}{c_{i_j} c_{i_j+1}} \right) \prod_{i=a}^{b+1} c_i, \tag{11}$$

with $c_i = v_{i-1} - v_{i+1}$. In this paper we will prove that the integrals (8),(9) and (10) are in involution with respect to accompanying symplectic structures.

The paper is organized as follows. In section 2, we prove the involutivity of integrals of the sine-Gordon maps. Firstly, we consider the odd-dimensional maps. We introduce a transformation to reduce the dimension of the map by one and we present a symplectic structure of the reduced map. Then we use properties of Theta with respect to the Poisson bracket associated to this symplectic structure. These properties are proven in Appendix A. To prove the involutivity of the integrals, we write the Poisson bracket $\{I_r, I_s\}$ as a polynomial in $\alpha_1, \alpha_2, \alpha_3$ and prove that all the coefficients of this polynomial vanish. Secondly, we consider the even-dimensional map. We provide a symplectic structure for it, and show that it relates to the symplectic structure for the odd dimensional map. Therefore, many properties of Theta with respect to the new Poisson bracket can be obtained directly from the ones with respect to the old Poisson bracket. The proof of involutivity is similar to the first case.

In section 3, we present relationships between symplectic structures of the sine-Gordon and mKdV maps. We use these relationships to derive analogous properties of Theta with respect to the Poisson bracket of the mKdV maps. Involutivity of the integrals of the mKdV follows from these properties.

In section 4, we prove that the integrals of the pKdV map are in involution (with respect to the approriate symplectic structures). We again distinguish even and odd dimensional maps and present a relationship of symplectic structures between the two cases. For the even-dimensional map, the properties of multi-sums of products, Ψ, with respect to the symplectic structure are proved by induction in Appendix B. For the other case, the properties of Ψ with respect to its symplectic structure are derived from the previous case. The involutiviy of integrals (10) is proved by using these properties.

In section 5 we discuss results, obtained in [**?**], on the functional independence of the sets of integrals (8, 9, 10), and conclude the integrability of the difference equations (2, 3, 4) for any value of the order d.

2 Involutivity of sine-Gordon integrals

In this section, we distinguish two cases: the odd-dimensional and even-dimensional sine-Gordon maps. In [16] it is shown that for the even-dimensional map, we have enough integrals for integrability. For the odd-dimensional map, we need to reduce the dimension of the map by one. We expand the Poisson bracket between two integrals $\{I_r, I_s\}$ as a quadratic polynomial in the parameters $\alpha_1, \alpha_2, \alpha_3$ and prove the involutivity of integrals (8) by showing its coefficients vanish.

2.1 The case $d = 2n+1$

Using a reduction $f_i = v_i v_{i+1}$, we obtain a $2n$-dimensional map

$$\mathrm{sG} : (f_1, f_2, \ldots, f_{2n}) \mapsto \left(f_2, f_3, \ldots, f_{2n}, \frac{f_2 f_4 \ldots f_{2n}(\alpha_1 f_2 f_4 \ldots f_{2n} + \alpha_3 f_3 f_5 \ldots f_{2n-1})}{f_1 f_3 \ldots f_{2n-1}(\alpha_2 f_2 f_4 \ldots f_{2n} + \alpha_1 f_3 f_5 \ldots f_{2n-1})} \right). \tag{12}$$

This map has n integrals given by

$$I_r^{\mathrm{sG}} = \alpha_1 \left(\frac{f_2 f_4 \ldots f_{2n}}{f_1 f_3 \ldots f_{2n-1}} \Theta_{2r,1}^{1,2n} + \frac{f_1 f_3 \ldots f_{2n-1}}{f_2 f_4 \ldots f_{2n}} \Theta_{2r,0}^{1,2n} \right) + \alpha_2 \Theta_{2r+1,1}^{1,2n} + \alpha_3 \Theta_{2r+1,0}^{1,2n}, \tag{13}$$

where the argument of Θ is f_i and $0 \leq r \leq n-1$.

A symplectic structure for the map (12) is given by $\Omega_{2n}^{\mathrm{sG}}$, where

$$\Omega_p^{\mathrm{sG}} = \begin{pmatrix} 0 & f_1 f_2 & f_1 f_3 & f_1 f_4 & \cdots & f_1 f_{p-1} & f_1 f_p \\ -f_2 f_1 & 0 & f_2 f_3 & f_2 f_4 & \cdots & f_2 f_{p-2} & f_2 f_p \\ -f_3 f_1 & -f_3 f_2 & 0 & f_3 f_4 & \cdots & f_3 f_{p-1} & f_3 f_p \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ -f_p f_1 & -f_p f_2 & -f_p f_3 & -f_p f_4 & \cdots & -f_p f_{p-1} & 0 \end{pmatrix}. \tag{14}$$

cf. [7, 12]. One can verify that $d\mathrm{sG} \cdot \Omega_{2n}^{\mathrm{sG}} \cdot d\mathrm{sG}^T = \Omega_{2n}^{\mathrm{sG}} \circ \mathrm{sG}$. Let g and h be functions differentiable with respect to the f_i's. The symplectic structure Ω_p^{sG} defines the following Poisson bracket

$$\begin{aligned} \{g, h\}_f &= \nabla_f(g) \cdot \Omega_p^{\mathrm{sG}} \cdot (\nabla_f(h))^T \\ &= \sum_{i<j} f_i f_j \left(\frac{\partial g}{\partial f_i} \frac{\partial h}{\partial f_j} - \frac{\partial g}{\partial f_j} \frac{\partial h}{\partial f_i} \right). \end{aligned} \tag{15}$$

We will prove that integrals (13) are in involution with respect to the symplectic structure $\Omega_{2n}^{\mathrm{sG}}$, i.e $\{I_r^{\mathrm{sG}}, I_s^{\mathrm{sG}}\}_f = 0$, for all $0 \leq r, s \leq n-1$. The proof is based on the following explicit expressions for the Poisson bracket between Theta multi-sums, which are proved in Appendix A.

Lemma 1. *Let $1 \leq r, s \leq p$ and $\epsilon \in \{0, 1\}$. We have*

$$\{\Theta_{r,\epsilon}^{1,p}, \Theta_{s,\epsilon}^{1,p}\}_f = \begin{cases} 0 & r, s \text{ are both odd or both even}, \\ \displaystyle\sum_{i \geq 0} (-1)^i \Theta_{r+i,\epsilon}^{1,p} \Theta_{s-i,\epsilon}^{1,p} & r \text{ even, } s \text{ odd and } r > s, \\ \displaystyle\sum_{i \geq 1} (-1)^{i-1} \Theta_{r-i,\epsilon}^{1,p} \Theta_{s+i,\epsilon}^{1,p} & r \text{ even, } s \text{ odd and } r < s. \end{cases} \tag{16}$$

Note that the right hand side of (16) is a finite sum.

The next proposition provides the Poisson bracket between two Theta multi-sums with different values of ϵ.

Lemma 2. *Let $1 \leq r, s \leq p$.*

1. *If $r \equiv s \pmod 2$, we have*

$$\{\Theta_{r,0}^{1,p}, \Theta_{s,1}^{1,p}\}_f = \begin{cases} \displaystyle\sum_{i \geq 0} (-1)^i \Theta_{r-1-2\lfloor i/2 \rfloor, i}^{1,p} \Theta_{s+1+2\lfloor i/2 \rfloor, i+1}^{1,p} & r \leq s, \\ \displaystyle\sum_{i \geq 0} (-1)^i \Theta_{s-1-2\lfloor i/2 \rfloor, i}^{1,p} \Theta_{r+1+2\lfloor i/2 \rfloor, i+1}^{1,p} & r > s. \end{cases} \tag{17}$$

2. If $r \not\equiv s \pmod 2$, we have

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_f = \begin{cases} \sum_{i\geq 0}(-1)^i \Theta^{1,p}_{s+i,i+1}\Theta^{0,p}_{r-i,i} & r \text{ odd, } s \text{ even,} \\ \sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i,i} & r \text{ even, } s \text{ odd.} \end{cases} \tag{18}$$

Using Lemma 1 and Lemma 2, we have the following corollary.

Corollary 3. *Let r and s be both even or both odd and let $\epsilon \in \{0, 1\}$. Then*

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_f + \{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,0}\}_f = 0, \tag{19}$$

$$\{\Theta^{1,p}_{r-1,\epsilon}, \Theta^{1,p}_{s,\epsilon}\}_f + \{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon}\}_f = \begin{cases} 0, & r, s \text{ even,} \\ \Theta^{1,p}_{r-1,\epsilon}\Theta^{1,p}_{s,\epsilon} - \Theta^{1,p}_{s-1,\epsilon}\Theta^{1,p}_{r,\epsilon}, & r, s \text{ odd,} \end{cases} \tag{20}$$

$$\{\Theta^{1,p}_{r-1,\epsilon\pm1}, \Theta^{1,p}_{s,\epsilon}\}_f + \{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon\pm1}\}_f = \begin{cases} 0, & r, s \text{ even,} \\ \Theta^{1,p}_{s-1,\epsilon\pm1}\Theta^{1,p}_{r,\epsilon} - \Theta^{1,p}_{s,\epsilon}\Theta^{1,p}_{r-1,\epsilon\pm1}, & r, s \text{ odd.} \end{cases} \tag{21}$$

Theorem 4. *Let $0 \leq r, s \leq n-1$. Let $I^{\mathrm{sG}}_r, I^{\mathrm{sG}}_s$ be given by formula (13). Then*

$$\{I^{\mathrm{sG}}_r, I^{\mathrm{sG}}_s\}_f = 0.$$

Proof. First of all, we denote

$$F = \frac{f_1 f_3 \cdots f_{2n-1}}{f_2 f_4 \cdots f_{2n}}.$$

For any $g(f_1, f_2, \ldots, f_{2n})$ we find $\{F^{\pm1}, g\}_f = \pm F^{\pm1} E_f g$, where

$$E_f = \sum_{i\geq 1} f_i \frac{\partial}{\partial f_i}, \tag{22}$$

which scales any homogeneous expression by its total degree. Every term in the multi-sum has total degree 0 if r is even and degree $(-1)^{\epsilon+1}$ if r is odd, hence

$$\{F^{\pm1}, \Theta^{1,p}_{r,\epsilon}\}_f = \begin{cases} 0 & \text{if } r \text{ even,} \\ \mp(-1)^{\epsilon} F^{\pm1}\Theta^{1,p}_{r,\epsilon} & \text{if } r \text{ odd.} \end{cases} \tag{23}$$

Now we expand $\{I^{\mathrm{sG}}_r, I^{\mathrm{sG}}_s\}_f$ in terms of polynomials in $\alpha_1, \alpha_2, \alpha_3$ as follows

$$\{I^{\mathrm{sG}}_r, I^{\mathrm{sG}}_s\}_f = \alpha_1^2 A_1 + \alpha_2^2 A_2 + \alpha_3^2 A_3 + \alpha_1\alpha_2 A_{12} + \alpha_1\alpha_3 A_{13} + \alpha_2\alpha_3 A_{23},$$

where

$$\begin{aligned}
A_1 &= \{F^{-1}\Theta^{1,2n}_{2r,1} + F\Theta^{1,2n}_{2r,0}, F^{-1}\Theta^{1,2n}_{2s,1} + F\Theta^{1,2n}_{2s,0}\}_f, \\
A_2 &= \{\Theta^{1,2n}_{2r+1,1}, \Theta^{1,2n}_{2s+1,1}\}_f, \\
A_3 &= \{\Theta^{1,2n}_{2r+1,0}, \Theta^{1,2n}_{2s+1,0}\}_f, \\
A_{12} &= \{F^{-1}\Theta^{1,2n}_{2r,1} + F\Theta^{1,2n}_{2r,0}, \Theta^{1,2n}_{2s+1,1}\}_f + \{\Theta^{1,2n}_{2r+1,1}, F^{-1}\Theta^{1,2n}_{2s,1} + F\Theta^{1,2n}_{2s,0}\}_f, \\
A_{13} &= \{F^{-1}\Theta^{1,2n}_{2r,1} + F\Theta^{1,2n}_{2r,0}, \Theta^{1,2n}_{2s+1,0}\}_f + \{\Theta^{1,2n}_{2r+1,0}, F^{-1}\Theta^{1,2n}_{2s,1} + F\Theta^{1,2n}_{2s,0}\}_f, \\
A_{23} &= \{\Theta^{1,2n}_{2r+1,1}, \Theta^{1,2n}_{2s+1,0}\}_f + \{\Theta^{1,2n}_{2r+1,0}, \Theta^{1,2n}_{2s+1,1}\}_f.
\end{aligned}$$

We prove that all these coefficients equal 0. Using Lemma 1 and Corollary 3, we have $A_2 = A_3 = A_{23} = 0$. We now expand A_1, A_{12} and A_{13}, we obtain

$$
\begin{aligned}
A_1 &= F^{-2}\{\Theta^{1,2n}_{2r,1}, \Theta^{1,2n}_{2s,1}\} + \{\Theta^{1,2n}_{2r,1}, \Theta^{1,2n}_{2s,0}\} + \{\Theta^{1,2n}_{2r,0}, \Theta^{1,2n}_{2s,1}\} + F^2\{\Theta^{1,2n}_{2r,0}, \Theta^{1,2n}_{2s,0}\} \\
&\quad + \Theta^{1,2n}_{2r,1}\Theta^{1,2n}_{2s,1}\{F^{-1}, F^{-1}\} + \Theta^{1,2n}_{2r,0}\Theta^{1,2n}_{2s,1}\{F, F^{-1}\} + \Theta^{1,2n}_{2r,1}\Theta^{1,2n}_{2s,0}\{F^{-1}, F\} + \Theta^{1,2n}_{2r,0}\Theta^{1,2n}_{2s,0}\{F, F\} \\
&\quad + F^{-1}\left(\Theta^{1,2n}_{2s,1}\{\Theta^{1,2n}_{2r,1}, F^{-1}\}_f + \Theta^{1,2n}_{2r,1}\{F^{-1}, \Theta^{1,2n}_{2s,1}\}_f + \Theta^{1,2n}_{2s,0}\{\Theta^{1,2n}_{2r,1}, F\}_f + \Theta^{1,2n}_{2r,0}\{F, \Theta^{1,2n}_{2s,1}\}_f\right) \\
&\quad + F\left(\Theta^{1,2n}_{2r,1}\{F^{-1}, \Theta^{1,2n}_{2s,0}\}_f + \Theta^{1,2n}_{2s,1}\{\Theta^{1,2n}_{2r,0}, F^{-1}\}_f + \Theta^{1,2n}_{2r,0}\{F, \Theta^{1,2n}_{2s,0}\}_f + \Theta^{1,2n}_{2s,0}\{\Theta^{1,2n}_{2r,0}, F\}_f\right) \\
&= 0,
\end{aligned}
$$

where the second and third terms cancel each other, due to (19), and all other terms vanish according to equations (16), and (23).

We also get

$$
\begin{aligned}
A_{12} &= F^{-1}\left(\{\Theta^{1,2n}_{2r,1}, \Theta^{1,2n}_{2s+1,1}\}_f + \{\Theta^{1,2n}_{2r+1,1}, \Theta^{1,2n}_{2s,1}\}\right)_f + F\left(\{\Theta^{1,2n}_{2r,0}, \Theta^{1,2n}_{2s+1,1}\}_f + \{\Theta^{1,2n}_{2r+1,1}, \Theta^{1,2n}_{2s,0}\}_f\right) \\
&\quad + \Theta^{1,2n}_{2r,1}\{F^{-1}, \Theta^{1,2n}_{2s+1,1}\}_f + \Theta^{1,2n}_{2r,0}\{F, \Theta^{1,2n}_{2s+1,1}\}_f + \Theta^{1,2n}_{2s,1}\{\Theta^{1,2n}_{2r+1,1}, F^{-1}\}_f + \Theta^{1,2n}_{2s,0}\{\Theta^{1,2n}_{2r+1,1}, F\}_f \\
&= F^{-1}\left(\Theta^{1,2n}_{2r,1}\Theta^{1,2n}_{2s+1,1} - \Theta^{1,2n}_{2r+1,1}\Theta^{1,2n}_{2s,1}\right) + F\left(\Theta^{1,2n}_{2s,0}\Theta^{1,2n}_{2r+1,1} - \Theta^{1,2n}_{2s+1,1}\Theta^{1,2n}_{2r,0}\right) \\
&\quad - F^{-1}\Theta^{1,2n}_{2r,1}\Theta^{1,2n}_{2s+1,1} + F\Theta^{1,2n}_{2r,0}\Theta^{1,2n}_{2s+1,1} + F^{-1}\Theta^{1,2n}_{2s,1}\Theta^{1,2n}_{2r+1,1} - F\Theta^{1,2n}_{2s,0}\Theta^{1,2n}_{2r+1,1} \\
&= 0,
\end{aligned}
$$

where we have used (20), (21), and (23). Similarly we get $A_{13} = 0$. Hence, we have $\{I^{\mathrm{sG}}_r, I^{\mathrm{sG}}_s\}_f = 0$. $\square$

2.2 The case $d = 2n$

In this section, we consider a $2n$-dimensional map

$$
\widetilde{\mathrm{sG}} : (v_1, v_2, \cdots, v_{2n}) \mapsto (v_2, v_3, \ldots, v_{2n}, v_1^{-1}\frac{\alpha_1 v_2 v_{2n} + \alpha_3}{\alpha_2 v_2 v_{2n} + \alpha_1}). \tag{24}
$$

This map has n integrals given by

$$
I^{\widetilde{\mathrm{sG}}}_r = \alpha_1\left(\frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2r,1} + \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2r,0}\right) + \alpha_2\Theta^{1,2n-1}_{2r+1,1} + \alpha_3\Theta^{1,2n-1}_{2r+1,0}, \tag{25}
$$

where $0 \le r \le n-1$ and $f_i = v_i v_{i+1}$ in the argument of Theta (7). The sine-Gordon map (24) has a symplectic structure $\Omega^{\widetilde{\mathrm{sG}}}_{2n}$, where

$$
\Omega^{\widetilde{\mathrm{sG}}}_p = \begin{pmatrix}
0 & v_1v_2 & 0 & v_1v_4 & \ldots & 0 & v_1v_p \\
-v_2v_1 & 0 & v_2v_3 & 0 & \ldots & v_2v_{p-1} & 0 \\
0 & -v_3v_2 & 0 & v_3v_4 & \ldots & 0 & v_3v_p \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-v_pv_1 & 0 & -v_pv_3 & 0 & \ldots & -v_pv_{p-1} & 0
\end{pmatrix}, \tag{26}
$$

cf. [7, 12]. The Poisson bracket $\frac{1}{2}\nabla_v(g)\Omega^{\widetilde{\mathrm{sG}}}_p(\nabla_v(h))^T$ is denoted $\{g, h\}_v$. Before we prove that the integrals (25) are in involution with respect to this bracket, we first establish the following Poisson brackets between Theta multi-sums:

$$
\{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s,\delta}\}_v = \{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s,\delta}\}_f|_{f_i = v_iv_{i+1}}, \tag{27}
$$

where the right-hand-side is given by Lemmas 1 and 2. Equation (27) follows as a corollary from the next Lemma. Consider the map $\mathbb{R}^p \to \mathbb{R}^{p-1}$,

$$G_p : (v_1, v_2, \ldots, v_p) \mapsto (v_1 v_2, v_2 v_3, \ldots, v_{p-1} v_p).$$

Lemma 5. *With g, h differentiable functions on $\mathbb{R}^{p-1}$ we have*

$$\{g \circ G_p, h \circ G_p\}_v = \{g, h\}_{f=G_p(v)},$$

i.e. G_p is a Poisson map.

Proof. The $(p-1) \times p$ Jacobian of the map G_p is

$$dG_p = \begin{pmatrix} v_2 & v_1 & 0 & \cdots & 0 \\ 0 & v_3 & v_2 & \cdots & 0 \\ \vdots & & & & 0 \\ 0 & 0 & \cdots & v_p & v_{p-1} \end{pmatrix}.$$

By direct calculation, we have

$$dG_p \cdot \Omega_p^{\widetilde{\mathrm{sG}}} \cdot (dG_p)^T = 2\Omega_{p-1}^{\mathrm{sG}}|_{f=G_p(v)}. \tag{28}$$

Applying ∇ to $(g \circ G_p)(v) = g(f)|_{f=G_p(v)}$ (and omitting some arguments) we find

$$\nabla_v(g \circ G_p) = \nabla_f(g) dG_p|_{f=G_p(v)}.$$

Hence, we have

$$\begin{aligned} \{g \circ G_p, h \circ G_p\}_v &= \frac{1}{2}\nabla_v(g \circ G_p)\Omega_p^{\widetilde{\mathrm{sG}}}\nabla_v(g \circ G_p)^T \\ &= \frac{1}{2}\nabla_f(g) dG_p|_{f=G_p(v)}\Omega_p^{\widetilde{\mathrm{sG}}}(\nabla_f(h) dG_p)^T|_{f=G_p(v)} \\ &= \nabla_f(g)\Omega_p^{\mathrm{sG}}(\nabla_f(h))^T|_{f=G_p(v)} \\ &= \{g, h\}_{f=G_p(v)} \end{aligned}$$

□

Now we will prove the involutivity of the integrals (25) of the sine-Gordon map (24).

Theorem 6. *Let $I_r^{\widetilde{\mathrm{sG}}}$ and $I_s^{\widetilde{\mathrm{sG}}}$, with $0 \le r, s \le n-1$, be given by the formula (25). Then we have*

$$\{I_r^{\widetilde{\mathrm{sG}}}, I_s^{\widetilde{\mathrm{sG}}}\}_v = 0.$$

Proof. With $V = v_1/v_{2n}$ we have

$$\{V^{\pm 1}, \Theta_{r,\epsilon}^{1,p}\}_v = V^{\pm 1} E_v \Theta_{r,\epsilon}^{1,p} = \begin{cases} 0 & \text{if } r \text{ even}, \\ \mp(-1)^\epsilon V^{\pm}\Theta_{r,\epsilon}^{1,p} & \text{if } r \text{ odd}. \end{cases}$$

The Poisson bracket between 2 integrals is expanded as

$$\{I_r^{\widetilde{\mathrm{sG}}}, I_s^{\widetilde{\mathrm{sG}}}\}_v = \alpha_1^2 B_1 + \alpha_2^2 B_2 + \alpha_3^2 B_3 + \alpha_1\alpha_2 B_{12} + \alpha_1\alpha_3 B_{13} + \alpha_2\alpha_3 B_{23},$$

where the coefficients B_I are similar to the A_I given in section 2.1, replacing F by V and $2n$ by $2n-1$. The rules for simplification are also similar. Therefore, $\{I_r^{\widetilde{\mathrm{sG}}}, I_s^{\widetilde{\mathrm{sG}}}\}_v = 0$. □

3 Involutivity of mKdV integrals

We consider the d-dimensional mKdV map

$$(v_1, v_2, \ldots, v_d) \mapsto \left(v_2, v_3, \ldots, v_d, v_1 \frac{\beta_1 v_d + \beta_2 v_2}{\beta_1 v_2 + \beta_3 v_d} \right). \tag{29}$$

As shown in [16], this map has $\lfloor (d-1)/2 \rfloor$ integrals given by the formula (9) with $0 < 2r < d$. If $d = 2n+1$, the map (29) reduces to a $2n$-dimensional map with exactly n integrals via a reduction $z_i = v_{i+1}/v_i$. For the other case, where $d = 2n+2$, the map (29) reduces to a $2n$-dimensional map with exactly n integrals via the reduction $z_i = v_{i+2}/v_i$. We will show that the integrals of these reduced maps are in involution. In each case, we present a relationship between the relevant symplectic structures and the symplectic structures of the sine-Gordon map in the even case (14). This relation can be used to derive properties of Theta with new symplectic structures.

3.1 The case $d = 2n + 1$

Using the reduction $z_i = v_{i+1}/v_i$, we obtain the map

$$\text{mKdV} : (z_1, z_2, \ldots, z_{2n}) \mapsto \left(z_2, z_3, \ldots, z_{2n}, \frac{1}{z_1 z_2 \ldots z_{2n}} \cdot \frac{\beta_1 z_2 z_3 \ldots z_{2n} + \beta_2}{\beta_1 + \beta_3 z_2 z_3 \ldots z_{2n}} \right). \tag{30}$$

The integrals of this map are given by

$$I_r^{\text{mKdV}} = \beta_1 \left(z_1 z_2 \ldots z_{2n} \Theta^{1,2n}_{2r-1,0} + \frac{1}{z_1 z_2 \ldots z_{2n}} \Theta^{1,2n}_{2r-1,1} \right) + \beta_2 \Theta^{1,2n}_{2r,1} + \beta_3 \Theta^{1,2n}_{2r,0}, \tag{31}$$

where arguments for Theta are $f_i = z_1^2 z_2^2 \ldots z_{i-1}^2 z_i$. Here we have used an 'inverse reduction', $v_i = v_1 z_1 z_2 \cdots z_{i-1}$ to express $f_i = v_i v_{i+1}$ in terms of the z_j and we omitted the v_1 dependence as both the integral and the map do not depend on it.

We obtain a symplectic structure $\Omega_{2n}^{\text{mKdV}}$ for the map (30), where

$$\Omega_p^{\text{mKdV}} := \begin{pmatrix} 0 & z_1 z_2 & -z_1 z_3 & z_1 z_4 & \ldots & (-1)^p z_1 z_p \\ -z_2 z_1 & 0 & z_2 z_3 & z_2 z_4 & \ldots & (-1)^{p-1} z_2 z_p \\ z_3 z_1 & -z_3 z_2 & 0 & z_3 z_4 & \ldots & (-1)^{p-2} z_3 z_p \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ (-1)^{p-1} z_p z_1 & (-1)^p z_p z_2 & (-1)^{p-1} z_p z_3 & (-1)^p z_p z_4 & \ldots & 0 \end{pmatrix}, \tag{32}$$

cf. [7, 12]. This gives us a Poisson bracket $\{g, h\}_z = \nabla_z(g) \Omega_{2n}^{\text{mKdV}} (\nabla_z(h))^T$. As before we can express the z-Poisson brackets between Theta multi-sums in terms of the corresponding f-Poisson brackets. Consider the map

$$M_p : (z_1, z_2, \ldots, z_p) \mapsto (z_1, z_1^2 z_2, \ldots, z_1^2 z_2^2 \cdots z_{p-1}^2 z_p).$$

We have the following result.

Lemma 7. *With g, h differentiable functions on $\mathbb{R}^p$ we have*

$$\{g \circ M_p, h \circ M_p\}_z = \{g, h\}_{f = M_p(z)},$$

i.e. M_p is a Poisson map.

Proof. The $p \times p$ Jacobian of the map M_p is

$$dM_p = \begin{cases} 0 & i < j, \\ \prod_{k=1}^{i-1} z_k^2 & i = j, \\ 2z_i z_j^{-1} \prod_{k=1}^{i-1} z_k^2 & i > j. \end{cases}$$

and a calculation shows

$$dM_p.\Omega_p^{\mathrm{mKdV}}.dM_p^T = \Omega_p^{\mathrm{sG}} \mid_{f=M_p(z)} . \tag{33}$$

The argument is finished along the lines of the proof for Lemma 5. $\square$

We are ready now to prove the following theorem

Theorem 8. *Let I_r^{mKdV} and I_s^{mKdV} be given by the formula (31) with $1 \le r, s \le n$. Then we have*

$$\{I_r^{\mathrm{mKdV}}, I_s^{\mathrm{mKdV}}\}_z = 0. \tag{34}$$

Proof. With $Z = (z_1 z_2 \dots z_{2n})^{-1}$ we have $F^{\pm 1} \circ M_{2n} = Z^{\pm 1}$. Thus, Lemma 7 implies

$$\{Z^{\pm 1}, \Theta_{r,\epsilon}^{1,2n}\}_z = \begin{cases} 0 & \text{if } r \text{ even}, \\ \mp(-1)^\epsilon Z^{\pm 1} \Theta_{r,\epsilon}^{1,2n} & \text{if } r \text{ odd}. \end{cases} \tag{35}$$

Writing the left hand side of equation (34) as

$$\{I_r^{\mathrm{mKdV}}, I_s^{\mathrm{mKdV}}\}_z = \beta_1^2 P_1 + \beta_2^2 P_2 + \beta_3^2 P_3 + \beta_1\beta_2 P_{12} + \beta_1\beta_3 P_{13} + \beta_2\beta_3 P_{23}, \tag{36}$$

yields coefficients P_I similar to the A_I given in section 2.1, replacing F by Z, $2r$ by $2r-1$, and $2s$ by $2s-1$. Now that we know the brackets between Z, Z^{-1}, and the $\Theta_{2s,1}^{1,2n}$, we can expand the coefficient and show they vanish.

As before, the coefficients P_2, P_3, and P_{23} are the easy ones. For P_1 we get, using equation (35) and Lemma 7 in conjunction with equations (19) and (16),

$$\begin{aligned} P_1 &= Z \left(\Theta_{2s-1,0}^{1,2n} \{\Theta_{2r-1,0}^{1,2n}, Z\}_z + \Theta_{2r-1,0}^{1,2n} \{Z, \Theta_{2s-1,0}^{1,2n}\}_z + \Theta_{2s-1,1}^{1,2n} \{\Theta_{2r-1,0}^{1,2n}, Z^{-1}\}_z + \Theta_{2r-1,1}^{1,2n} \{Z^{-1}, \Theta_{2s-1,0}^{1,2n}\}_z \right) \\ &\quad + Z^{-1} \left(\Theta_{2s-1,1}^{1,2n} \{\Theta_{2r-1,1}^{1,2n}, Z\}_z + \Theta_{2r-1,1}^{1,2n} \{Z^{-1}, \Theta_{2s-1,1}^{1,2n}\}_z + \Theta_{2r-1,0}^{1,2n} \{Z, \Theta_{2s-1,1}^{1,2n}\}_z + \Theta_{2s-1,0}^{1,2n} \{\Theta_{2r-1,1}^{1,2n}, Z\}_z \right) \\ &\quad + \left(\{\Theta_{2r-1,0}^{1,2n}, \Theta_{2s-1,1}^{1,2n}\}_z + \{\Theta_{2r-1,1}^{1,2n}, \Theta_{2s-1,0}^{1,2n}\}_z \right) \\ &= Z^2 \left(-\Theta_{2s-1,0}^{1,2n} \Theta_{2r-1,0}^{1,2n} + \Theta_{2r-1,0}^{1,2n} (-1)^1 \Theta_{2s-1,0}^{1,2n} \right) + Z^{-2} \left(-\Theta_{2s-1,1}^{1,2n} \Theta_{2r-1,1}^{1,2n} + \Theta_{2r-1,1}^{1,2n} \Theta_{2s-1,1}^{1,2n} \right) \\ &\quad + \left(\{\Theta_{2r-1,0}^{1,2n}, \Theta_{2s-1,1}^{1,2n}\}_z + \{\Theta_{2r-1,1}^{1,2n}, \Theta_{2s-1,0}^{1,2n}\}_z \right) \\ &\quad + \Theta_{2s-1,1}^{1,2n} \Theta_{2r-1,0}^{1,2n} - \Theta_{2r-1,0}^{1,2n} \Theta_{2s-1,1}^{1,2n} + \Theta_{2s-1,0}^{1,2n} \Theta_{2r-1,1}^{1,2n} - \Theta_{2r-1,1}^{1,2n} \Theta_{2s-1,0}^{1,2n} \\ &= 0, \end{aligned}$$

where we have used (35) and and (19). Expanding P_{12} yields

$$\begin{aligned} P_{12} &= Z \left(\{\Theta_{2r-1,0}^{1,2n}, \Theta_{2s,1}^{1,2n}\}_z + \{\Theta_{2r,1}^{1,2n}, \Theta_{2s-1,0}^{1,2n}\}_z \right) + Z^{-1} \left(\{\Theta_{2r-1,1}^{1,2n}, \Theta_{2s,1}^{1,2n}\}_z + \{\Theta_{2r,1}^{1,2n}, \Theta_{2s-1,1}^{1,2n}\}_z \right) \\ &\quad + \Theta_{2r-1,0}^{1,2n} \{Z, \Theta_{2s,1}^{1,2n}\}_z + \Theta_{2s-1,0}^{1,2n} \{\Theta_{2r,1}^{1,2n}, Z\}_z + \Theta_{2r-1,1}^{1,2n} \{Z^{-1}, \Theta_{2s,1}^{1,2n}\} + \Theta_{2s-1,1}^{1,2n} \{\Theta_{2r,1}^{1,2n}, Z^{-1}\}_z \\ &= 0, \end{aligned}$$

where the first and the second terms, as well as the third and the fourth terms cancel each other, and the last four terms are equal to zero. Similarly, we get $P_{13} = 0$. $\square$

3.2 The case $d = 2n+2$

Now using a reduction $w_i = v_{i+2}/v_i$, we obtain the map

$$\widetilde{\text{mKdV}}: \ (w_1, w_2, \dots, w_{2n}) \mapsto \left(w_2, w_3, \dots, w_{2n}, \frac{1}{w_1 w_3 \dots w_{2n-1}} \cdot \frac{\beta_1 w_2 w_4 \dots w_{2n} + \beta_2}{\beta_1 + \beta_3 w_2 w_4 \dots w_{2n}} \right). \tag{37}$$

Integrals of this map are given by

$$I_r^{\widetilde{\text{mKdV}}} = \alpha_1 \left(w_2 w_4 \dots w_{2n} \Theta^{1,2n+1}_{2r-1,0} + \frac{1}{w_2 w_4 \dots w_{2n}} \Theta^{1,2n+1}_{2r-1,1} \right) + \alpha_2 \Theta^{1,2n+1}_{2r,1} + \alpha_3 \Theta^{1,2n+1}_{2r,0}, \tag{38}$$

where $\Theta = \Theta[e_i]$ with $e_i = f_{i-1}$, with $f_0 = 1$ and $f_i = w_1 w_2 \dots w_i$ $(i > 0)$. Note, we have changed notation in order to relate the next Poisson bracket to the bracket $\{,\}_f$; the argument of $\Theta^{a,b}$ (7) is $(e_a, \cdots, e_b)$ with $e_i = v_i v_{i+1}$. In the 'inverse reduction', we have

$$v_n = \begin{cases} v_1 \prod_{j=1}^{i} w_{2j-1} & n = 2i+1, \\ v_2 \prod_{j=1}^{i-1} w_{2j} & n = 2i. \end{cases}$$

Therefore (similar to the case $d = 2n+1$) both the reduced map as well as the reduced integrals depend on the variables w_i. Using (57), we obtain

$$\begin{aligned} \Theta^{1,2n+1}_{s,\epsilon}[e_i] &= \Theta^{2,2n+1}_{s,\epsilon}[e_i] + \Theta^{2,2n+1}_{s-1,\epsilon+1}[e_i] \\ &= \Theta^{1,2n}_{s,\epsilon}[f_i] + \Theta^{1,2n}_{s-1,\epsilon+1}[f_i]. \end{aligned}$$

Let $K_p : (w_1, w_2, \dots, w_p) \mapsto (w_1, w_1 w_2, \dots, w_1 w_2 \cdots w_p)$ and $W = w_2 w_4 \dots w_{2n}$. Then, the integrals can be written

$$\begin{aligned} I_r^{\widetilde{\text{mKdV}}} = \alpha_1 & \left(W^{-1} \left(\Theta^{1,2n}_{2r-1,0} + \Theta^{1,2n}_{2r-2,1} \right) + W \left(\Theta^{1,2n}_{2r-1,1} + \Theta^{1,2n}_{2r-2,0} \right) \right) \\ & + \alpha_2 \left(\Theta^{1,2n}_{2r,1} + \Theta^{1,2n}_{2r-1,0} \right) + \alpha_3 \left(\Theta^{1,2n}_{2r,0} + \Theta^{1,2n}_{2r-1,1} \right). \end{aligned} \tag{39}$$

where $\Theta = \Theta[f_i]$ with $f = K_p(w)$.

The map (37) has a symplectic structure $\Omega_{2n}^{\widetilde{\text{mKdV}}}$, where

$$\Omega_p^{\widetilde{\text{mKdV}}} = \begin{pmatrix} 0 & w_1 w_2 & 0 & 0 & \dots & 0 \\ -w_2 w_1 & 0 & w_2 w_3 & 0 & \dots & 0 \\ 0 & -w_3 w_2 & 0 & w_3 w_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & w_{p-1} w_p \\ 0 & 0 & 0 & \dots & -w_p w_{p-1} & 0 \end{pmatrix}. \tag{40}$$

This gives us a Poisson bracket $\{g, h\}_w = \nabla_w(g) \Omega_{2n}^{\widetilde{\text{mKdV}}} (\nabla_w(h))^T$. Once again we can express the w-Poisson brackets between Theta multi-sums in terms of the corresponding f-Poisson brackets.

Lemma 9. *With g, h differential functions on $\mathbb{R}^p$ we have*

$$\{g \circ K_p, h \circ K_p\}_w = \{g, h\}_{f = K_p(w)},$$

i.e. K_p is a Poisson map.

Proof. This follows from

$$dK_p \Omega_p^{\widetilde{\mathrm{mKdV}}} dK_p^T = \Omega_p^{\mathrm{sG}} \mid_{f=K_p(w)} . \tag{41}$$

□

Because $F^{\pm 1} \circ K_{2n} = W^{\pm 1}$ this Lemma implies that $\{W, W^{-1}\}_w = 0$,

$$\{W^{\pm 1}, \Theta_{r,\epsilon}^{1,2n}\}_w = \begin{cases} 0 & \text{if } r \text{ even,} \\ \mp(-1)^\epsilon W^{\pm 1} \Theta_{r,\epsilon}^{1,2n} & \text{if } r \text{ odd,} \end{cases}$$

and we can also evaluate the brackets between $\Theta_{r,\epsilon}^{1,2n}$. Thus, the following theorem can be proven by mechanical expansion and evaluation of the bracket.

Theorem 10. *Let $I_r^{\widetilde{\mathrm{mKdV}}}$ and $I_s^{\widetilde{\mathrm{mKdV}}}$ be given by the formula (38). Then*

$$\{I_r^{\widetilde{\mathrm{mKdV}}}, I_s^{\widetilde{\mathrm{mKdV}}}\}_w = 0.$$

4 Involutivity of pKdV integrals

In this section, we prove the involutivity of the integrals of order-reduced pKdV maps. Similar to the sine-Gordon map, we consider two cases where the dimension d of the map (4) is even or odd. Here, in both cases there are not enough integrals for integrability, and therefore we perform reductions. We present symplectic structures for the reduced maps in both cases and give a relationship between these symplectic structures. For the case where d is even, properties of multi-sums of products, Ψ, with respect to its symplectic structure are proved in Appendix B. For the case where d is odd, the Poisson bracket between Ψ multi-sums are derived from those in the even case and the relationship between the two symplectic structures.

4.1 The case $d = 2n + 2$

We have a $(2n+2)$-dimensional map (6). The integrals I_r of this map are given by (10) with $0 \le r \le n-1$ which are not enough integrals for integrability in the sense of Liouville-Arnold. Therefore, we use a reduction $c_i = v_i - v_{i+2}$ to reduce the dimension of the map by 2. From equation (4), we obtain the following map:

$$\mathrm{pKdV}: \ (c_1, c_2, \ldots, c_{2n}) \mapsto (c_2, c_3, \ldots, c_{2n}, \frac{\gamma}{c_2 + c_4 + \ldots + c_{2n}} - c_1 - c_3 - \ldots - c_{2n-1}). \tag{42}$$

This map has exactly n integrals given by

$$\begin{aligned} I_r^{\mathrm{pKdV}} =& \Psi_{r-1}^{1,2n-1} - (c_2 + c_4 + \ldots + c_{2n}) \Psi_{r-1}^{1,2n-2} - (c_1 + c_3 + \ldots + c_{2n-1}) \Psi_{r-1}^{2,2n-1} \\ &+ \Psi_{r-2}^{2,2n-2} + ((c_1 + c_3 + \ldots + c_{2n-1})(c_2 + c_4 + \ldots + c_{2n}) - \gamma) \Psi_r^{1,2n-1}, \end{aligned} \tag{43}$$

with $r = 0, 1, \ldots, n-1$. The map is symplectic, we have $d\mathrm{pKdV} \cdot \Omega_{2n}^{\mathrm{pKdV}} \cdot d\mathrm{pKdV}^T = \Omega_{2n}^{\mathrm{pKdV}} \circ \mathrm{pKdV}$, where

$$\Omega_p^{\mathrm{pKdV}} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}, \tag{44}$$

which is given in [7, 12]. The corresponding Poisson bracket is denoted $\{g,h\}_c = \nabla_c(g)\Omega_{2n}^{\mathrm{pKdV}}\left(\nabla_c(h)\right)^T$. We prove that the integrals of the map pKdV are in involution with respect to this Poisson bracket. The proof is based on knowledge of the Poisson brackets between two Ψ multi-sums which is given as follows.

Lemma 11. *Let $p \geq 1$ and $0 \leq r, s \leq \lfloor (p+1)/2 \rfloor$. Then we have the following identities*

$$\{\Psi_r^{1,p}, \Psi_s^{1,p}\}_c = 0, \tag{45}$$

$$\{\Psi_r^{1,p}, \Psi_s^{1,p-1}\}_c + \{\Psi_r^{1,p-1}, \Psi_s^{1,p}\}_c = 0. \tag{46}$$

A corollary 17 derives from this Lemma is given in Appendix B.

Theorem 12. *For all $0 \leq r, s \leq n-1$, we have $\{I_r, I_s\}_c = 0$, where I_r, I_s are given by (43).*

Proof. To prove this theorem we need the following formulas. Let $g(c_1, c_2, \ldots, c_{2n})$ be a differentiable function on $\mathbb{R}^{2n}$. Denote

$$C_1 = c_1 + c_3 + \cdots + c_{2n-1}, \ C_2 = c_2 + c_4 + \cdots + c_{2n},$$

we have

$$\{g, C_1\}_c = -\frac{\partial g}{\partial c_{2n}}, \ \{g, C_2\}_c = \frac{\partial g}{\partial c_1}. \tag{47}$$

In addition, since we have

$$\Psi_r^{a,b} = c_{b+1}\Psi_r^{a,b} + \Psi_{r-1}^{a,b-1} \text{ and } \Psi_r^{a,b} = c_a\Psi_r^{a+1,b} + \Psi_{r-1}^{a+2,b},$$

we obtain

$$\frac{\partial \Psi_r^{a,b+1}}{\partial c_{b+2}} = \Psi_r^{a,b} \text{ and } \frac{\partial \Psi_r^{a,b+1}}{\partial c_a} = \Psi_r^{a+1,b}.$$

Now we write $\{I_r, I_s\}_c = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11}$, where

$$\begin{aligned}
A_1 &:= \{\Psi_{r-1}^{1,2n-1} - C_2\Psi_{r-1}^{1,2n-2}, \Psi_{s-1}^{1,2n-1} - C_2\Psi_{s-1}^{1,2n-2}\}_c,\\
A_2 &:= -\{\Psi_{r-1}^{1,2n-1}, C_1\Psi_{s-1}^{2,2n-1}\}_c - \{C_1\Psi_{r-1}^{2,2n-1}, \Psi_{s-1}^{1,2n-1}\}_c + \{C_1\Psi_{r-1}^{2,2n-1}, C_1\Psi_{s-1}^{2,2n-1}\}_c,\\
A_3 &:= \{\Psi_{r-1}^{1,2n-1}, \Psi_{s-2}^{2,2n-2}\}_c + \{\Psi_{r-2}^{2,2n-2}, \Psi_{s-1}^{1,2n-1}\}_c + \{\Psi_{r-2}^{2,2n-2}, \Psi_{s-2}^{2,2n-2}\}_c,\\
A_4 &:= \{C_2\Psi_{r-1}^{1,2n-2}, C_1\Psi_{s-1}^{2,2n-1}\}_c + \{C_1\Psi_{r-1}^{2,2n-1}, C_2\Psi_{s-1}^{1,2n-2}\}_c,\\
A_5 &:= -\{C_2\Psi_{r-1}^{1,2n-2}, \Psi_{s-2}^{2,2n-2}\}_c - \{\Psi_{r-2}^{2,2n-2}, C_2\Psi_{s-1}^{1,2n-2}\}_c,\\
A_6 &:= -\{C_1\Psi_{r-1}^{2,2n-1}, \Psi_{s-2}^{2,2n-2}\}_c - \{\Psi_{r-2}^{2,2n-2}, C_1\Psi_{s-1}^{2,2n-1}\}_c,\\
A_7 &:= -\{\Psi_{r-1}^{1,2n-1}, (C_1C_2-\gamma)\Psi_s^{1,2n-1}\}_c - \{(C_1C_2-\gamma)\Psi_r^{1,2n-1}, \Psi_{s-1}^{1,2n-1}\}_c,\\
A_8 &:= -\{C_2\Psi_{r-1}^{1,2n-2}, (C_1C_2-\gamma)\Psi_s^{1,2n-1}\}_c - \{(C_1C_2-\gamma)\Psi_r^{1,2n-1}, C_2\Psi_{s-1}^{1,2n-2}\}_c,\\
A_9 &:= -\{C_1\Psi_{r-1}^{2,2n-1}, (C_1C_2-\gamma)\Psi_s^{1,2n-1}\}_c - \{(C_1C_2-\gamma)\Psi_r^{1,2n-1}, C_1\Psi_{s-1}^{2,2n-1}\}_c,\\
A_{10} &:= \{\Psi_{r-2}^{2,2n-2}, (C_1C_2-\gamma)\Psi_s^{1,2n-1}\}_c + \{(C_1C_2-\gamma)\Psi_r^{1,2n-1}, \Psi_{s-2}^{2,2n-2}\}_c,\\
A_{11} &:= \{(C_1C_2-\gamma)\Psi_r^{1,2n-1}, (C_1C_2-\gamma)\Psi_s^{1,2n-1}\}_c.
\end{aligned}$$

Using Lemma 11, Corollary 17 and formulas (47), we have

$$\begin{aligned}
A_1 &= \Psi_{r-1}^{1,2n-2}\Psi_{s-1}^{2,2n-1} - \Psi_{s-1}^{1,2n-2}\Psi_{r-1}^{2,2n-1} + C_2\left(\Psi_{s-1}^{1,2n-2}\Psi_{r-1}^{2,2n-2} - \Psi_{r-1}^{1,2n-2}\Psi_{s-1}^{2,2n-2}\right),\\
A_2 &= \Psi_{s-1}^{2,2n-1}\Psi_{r-1}^{1,2n-2} - \Psi_{r-2}^{2,2n-1}\Psi_{s-1}^{1,2n-2} + C_1\left(\Psi_{r-1}^{2,2n-1}\Psi_{s-1}^{2,2n-2} - \Psi_{s-1}^{2,2n-1}\Psi_{r-1}^{2,2n-2}\right),\\
A_3 &= \Psi_{r-1}^{2,2n-1}\Psi_{s-1}^{1,2n-2} - \Psi_{s-1}^{2,2n-1}\Psi_{r-1}^{1,2n-2},\\
A_4 &= -\Psi_{r-1}^{1,2n-2}\Psi_{s-1}^{2,2n-1} + \Psi_{r-1}^{2,2n-1}\Psi_{s-1}^{1,2n-2},\\
A_5 &= -C_2\left(\Psi_{s-1}^{1,2n-2}\Psi_{r-1}^{2,2n-2} - \Psi_{r-1}^{1,2n-2}\Psi_{s-1}^{2,2n-2}\right),\\
A_6 &= -C_1\left(\Psi_{r-1}^{2,2n-1}\Psi_{s-1}^{2,2n-2} - \Psi_{s-1}^{2,2n-1}\Psi_{r-1}^{2,2n-2}\right).
\end{aligned}$$

It follows that $A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 0$. Now we show that $A_7 + A_8 + A_9 + A_{10} + A_{11} = 0$. We also have

$$\begin{aligned}
A_7 &= C_1\left(\Psi_s^{1,2n-1}\Psi_{r-1}^{2,2n-1} - \Psi_r^{1,2n-1}\Psi_{s-1}^{2,2n-1}\right) + C_2\left(\Psi_r^{1,2n-1}\Psi_{s-1}^{1,2n-2} - \Psi_s^{1,2n-1}\Psi_{r-1}^{1,2n-2}\right),\\
A_8 &= C_2(C_1C_2-\gamma)\left(\Psi_s^{1,2n-1}\Psi_r^{1,2n-2} - \Psi_r^{1,2n-1}\Psi_s^{1,2n-2}\right) + C_2\left(\Psi_{r-1}^{1,2n-2}\Psi_s^{1,2n-1} - \Psi_{s-1}^{1,2n-2}\Psi_r^{1,2n-1}\right)\\
&\quad + C_1C_2\left(\Psi_r^{1,2n-1}\Psi_{s-1}^{2,2n-2} - \Psi_s^{1,2n-1}\Psi_{r-1}^{2,2n-1}\right) + (C_1C_2-\gamma)\left(\Psi_{r-1}^{1,2n-2}\Psi_s^{2,2n-1} - \Psi_{s-1}^{1,2n-2}\Psi_r^{2,2n-1}\right),\\
A_9 &= C_1(C_1C_2-\gamma)\left(\Psi_r^{1,2n-1}\Psi_s^{2,2n-1} - \Psi_s^{1,2n-1}\Psi_r^{2,2n-1}\right) + C_1\left(\Psi_{s-1}^{2,2n-1}\Psi_r^{1,2n-1} - \Psi_{r-1}^{2,2n-1}\Psi_s^{1,2n-1}\right)\\
&\quad + C_1C_2\left(\Psi_s^{1,2n-1}\Psi_{r-1}^{2,2n-2} - \Psi_r^{1,2n-1}\Psi_{s-1}^{2,2n-2}\right) + (C_1C_2-\gamma)\left(\Psi_{s-1}^{2,2n-1}\Psi_r^{1,2n-2} - \Psi_{r-1}^{2,2n-1}\Psi_s^{1,2n-2}\right),\\
A_{10} &= (C_1C_2-\gamma)\left(\Psi_r^{2,2n-1}\Psi_{s-1}^{1,2n-2} - \Psi_s^{2,2n-1}\Psi_{r-1}^{1,2n-2} + \Psi_{r-1}^{2,2n-1}\Psi_s^{1,2n-2} - \Psi_{s-1}^{2,2n-1}\Psi_r^{1,2n-2}\right),\\
A_{11} &= (C_1C_2-\gamma)C_1\left(\Psi_s^{1,2n-1}\Psi_r^{2,2n-1} - \Psi_r^{1,2n-1}\Psi_s^{2,2n-1}\right) + \left(\Psi_s^{1,2n-2}\Psi_r^{1,2n-1} - \Psi_s^{1,2n-1}\Psi_r^{1,2n-2}\right)\\
&\quad (C_1c_2-\gamma)C_2.
\end{aligned}$$

This implies $A_7 + A_8 + A_9 + A_{10} + A_{11} = 0$. Therefore, we have $\{I_r, I_s\} = 0$. □

4.2 The case $d = 2n+1$

We introduce a reduction $u_i = v_i - v_{i+1}$. We obtain a $2n$-dimensional map

$$\widetilde{\mathrm{pKdV}}:\ (u_1,u_1,\ldots,u_{2n}) \mapsto (u_2,u_3,\ldots,u_{2n},\frac{\gamma}{u_2+u_3+\ldots+u_{2n}} - u_1 - u_2 - \ldots - u_{2n}) \tag{48}$$

with n integrals ($0 \le r \le n-1$)

$$\begin{aligned}
I_r^{\widetilde{\mathrm{pKdV}}} &= \Psi_{r-1}^{1,2n-2} - (u_2+u_3+\ldots+u_{2n})\Psi_{r-1}^{1,2n-3} - (u_1+u_2+\ldots+u_{2n-1})\Psi_{r-1}^{2,2n-2}\\
&\quad + \Psi_{r-2}^{2,2n-3} + \left((u_2+u_3+\ldots+u_{2n})(u_1+u_2+\ldots+u_{2n-1}) - \gamma\right)\Psi_r^{1,2n-2},
\end{aligned} \tag{49}$$

where the argument of Ψ is $f_i = 1/(c_ic_{i+1})$ with $c_i := u_i + u_{i+1}$. Based on the method given in [12], we obtain a symplectic structure $\Omega_{2n}^{\widetilde{\mathrm{pKdV}}}$ for the map (48), where

$$\Omega_p^{\widetilde{\mathrm{pKdV}}} = \begin{pmatrix} 0 & 1 & -1 & \cdots & (-1)^p\\ -1 & 0 & 1 & \cdots & (-1)^{p-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ (-1)^{p-1} & (-1)^p & \cdots & 0 & 1\\ (-1)^p & (-1)^{p-1} & \cdots & -1 & 0 \end{pmatrix}. \tag{50}$$

The Poisson bracket is denoted $\{g,h\}_u = \nabla_u(g)\widetilde{\Omega_{2n}^{\mathrm{pKdV}}}(\nabla_u(h))^T$. Next we present a relationship between the two symplectic structures (44) and (50) and the corresponding Poisson brackets. Consider the map

$$Q_p : (u_1, u_2, \ldots, u_p) \mapsto (u_1 + u_2, u_2 + u_3, \ldots, u_{p-1} + u_p).$$

Lemma 13. *The map Q_p is a Poisson map, i.e.*

$$\{f \circ Q_p, g \circ Q_p\}_u = \{f, g\}_{c=Q_p(u)}, \tag{51}$$

where $f(c)$ and $g(c)$ are differentiable functions.

Proof. By calculation we obtain

$$dQ_p \widetilde{\Omega_p^{\mathrm{pKdV}}} dQ_p^T = \Omega_{p-1}^{\mathrm{pKdV}}. \tag{52}$$

□

Theorem 14. *Let I_r, I_s be given by (49). Then, for all $0 \le r, s \le n-1$ we have*

$$\{I_r, I_s\}_u = 0.$$

Proof. As the following formulas hold,

$$\{g, u_2 + u_3 + \ldots + u_{2n}\}_u = \frac{\partial g}{\partial u_1}, \tag{53}$$

$$\{g, u_1 + u_2 + \ldots + u_{2n-1}\}_u = -\frac{\partial g}{\partial u_{2n}}, \tag{54}$$

and the properties of Ψ with respect to the bracket $\{,\}_u$ which are the same as those with respect to the bracket $\{,\}_c$, one can prove the involutivity of the integrals (49) similarly to what we did for the case $d = 2n+2$. □

5 Conclusion

In this paper, we have proved the involutivity of integrals of sine-Gordon, pKdV and mKdV maps directly by using induction and using recently found symplectic structures of these maps. In order to prove these maps are completely integrable in the sense of Louville-Arnold [2, 17], one also needs to prove the functional independence of their integrals. We briefly discuss some results that are based on different techniques which fall outside the scope of this paper and will be published elsewhere [14].

To prove functional independence, due to the analyticity of the integrals, it suffices to prove linear independence of the gradients of the integrals at a certain point. It turns out that we can evaluate the multi-sums of products at certain points in terms of binomial coefficients (counting its number of terms). Also, using the recursive formulas we can find the gradients at these points. The proof of functional independence thus boils down to LU-decomposition of a matrix whose entries are expressed in binomials coefficients. This has been performed in [14]: For the sine-Gordon the integrals are independent at $v_1 = v_2 = \cdots = v_d = c$ when $\alpha_2 c^4 \neq \alpha_3$, from which the result follows by varying c. For the mKdV map we proved functional independence except when $\beta_2 \neq \beta_3$. And for the pKdV map the functional independence has been established for the generic case where $\gamma \neq \frac{(d-2)^2}{4} + \frac{d-2}{2}$. In these case we conclude the integrability of the equations (2, 3, 4) for arbitrary order $d = p+1$.

We note that the integrals of maps obtained as $(p,-1)$-reductions of the equations in the ABS list [1], with the exception of Q_4, can be expressed in terms of multi-sums of products, Ψ [13]. It would be interesting to study their symplectic structures and furthermore their complete integrability.

Acknowledgment

This research has been funded by the Australian Research Council through the Centre of Excellence for Mathematics and Statistics of Complex Systems. DTT acknowledges the support of two scholarships, one from La Trobe University and the other from the Endeavour IPRS programme.

A Properties of Θ with respect to the Poisson brackets

In this Appendix, we prove Lemma 1 and Lemma 2. First of all, the following lemma follows from a property of the operator E_f (22).

Lemma 15.

$$\{\Theta_{r,\epsilon}^{1,p}, f_{p+1}^{(-1)^\delta}\}_f = \begin{cases} 0 & r \ \textit{even} \\ (-1)^{\delta+\epsilon+1} f_{p+1}^{(-1)^\delta} \Theta_{r,\epsilon}^{1,p} & r \ \textit{odd} \end{cases} \tag{55}$$

Proof. It is because

$$\{\Theta_{r,\epsilon}^{1,p}, f_{p+1}^{(-1)^\delta}\}_f = (-1)^\delta f_{p+1}^{(-1)^\delta} E_f \Theta_{r,\epsilon}^{1,p}.$$

□

Remark 16. *If we introduce $t_i = 1/f_i$, then we get*

$$\Theta_{r,\epsilon}^{1,p}[f_i] = \Theta_{r,\epsilon+1}^{1,p}[t_i].$$

We have

$$\begin{aligned} \{\Theta_{r,\epsilon}^{1,p}, \Theta_{s,\delta}^{1,p}\}_f &= \sum_{i<j} \left(\frac{\partial \Theta_{r,\epsilon}^{1,p}}{\partial f_i} \frac{\partial \Theta_{s,\delta}^{1,p}}{\partial f_j} - \frac{\partial \Theta_{r,\epsilon}^{1,p}}{\partial f_j} \frac{\partial \Theta_{s,\delta}^{1,p}}{\partial f_i} \right) f_i f_j \\ &= \sum_{i<j} \left(\frac{\partial \Theta_{r,\epsilon+1}^{1,p}}{\partial t_i} \frac{\partial \Theta_{s,\delta+1}^{1,p}}{\partial t_j} t_i^2 t_j^2 - \frac{\partial \Theta_{r,\epsilon+1}^{0,p}}{\partial t_j} \frac{\partial \Theta_{s,\delta+1}^{1,p}}{\partial t_i} t_i^2 t_j^2 \right) \frac{1}{t_i t_j} |_{t=T(f)} \\ &= \{\Theta_{r,\epsilon+1}^{1,p}, \Theta_{s,\delta+1}^{1,p}\}_{f=T(f)}, \end{aligned}$$

where T is defined as follows

$$T : (f_1, f_2, \dots, f_p) \mapsto \left(\frac{1}{f_1}, \frac{1}{f_2}, \dots, \frac{1}{f_p} \right).$$

A.1 Proof of Lemma 1

Proof. We will prove this lemma by induction. The following properties, given in [16, 13] will be used in our proof:

$$\Theta_{r,\epsilon}^{a,b} = \Theta_{r,\epsilon}^{a,b-1} + f_b^{(-1)^{\epsilon+r}} \Theta_{r-1,\epsilon}^{a,b-1}, \tag{56}$$

$$\Theta_{r,\epsilon}^{a,b} = \Theta_{r,\epsilon}^{a+1,b} + f_a^{(-1)^{\epsilon\pm1}} \Theta_{r-1,\epsilon\pm1}^{a+1,b}. \tag{57}$$

Using Remark 16, it is sufficient to prove for the case $\epsilon = 1$. One verifies that (16) holds for $p = 1, 2$ and for all $1 \le r, s \le p$. Suppose that (16) holds for $p-1$ and p ($p \ge 2$). We will prove that (16) holds for $p+1$.

Using identity (56), we expand the left hand side of (16) and we obtain

$$
\begin{aligned}
&\{\Theta_{r,1}^{1,p+1}, \Theta_{s,1}^{1,p+1}\}_f \\
=&\{\Theta_{r,1}^{1,p}, \Theta_{s,1}^{1,p}\}_f + f_{p+1}^{(-1)^{r+1}}\{\Theta_{r-1,1}^{1,p}, \Theta_{s,1}^{1,p}\}_f + \Theta_{r-1,1}^{1,p}\{f_{p+1}^{(-1)^{p+1}}, \Theta_{s,1}^{1,p}\}_f + f_{p+1}^{(-1)^{s+1}}\{\Theta_{r,1}^{1,p}, \Theta_{s-1,1}^{1,p}\}_f \\
&+ \Theta_{s-1,1}^{1,p}\{\Theta_{r,1}^{1,p}, f_{p+1}^{(-1)^{s+1}}\}_f + f_{p+1}^{(-1)^{r+1}+(-1)^{s+1}}\{\Theta_{r-1,1}^{1,p}, \Theta_{s-1,1}^{1,p}\}_f + \Theta_{r-1,1}^{1,p}\Theta_{s-1,1}^{1,p}\{f_{p+1}^{(-1)^{r+1}}, f_{p+1}^{(-1)^{s+1}}\}_f \\
&+ f_{p+1}^{(-1)^{s+1}}\Theta_{r-1,1}^{1,p}\{f_{p+1}^{(-1)^{r+1}}, \Theta_{s-1,1}^{1,p}\}_f + f_{p+1}^{(-1)^{r+1}}\Theta_{s-1,1}^{1,p}\{\Theta_{r-1,1}^{1,p}, f_{p+1}^{(-1)^{s+1}}\}_f. \qquad (58)
\end{aligned}
$$

The case $r = s$ is trivial. Now we distinguish 3 cases.

1. r and s are both even or both odd. Since $\{\Theta_{r,1}^{1,p+1}, \Theta_{s,1}^{1,p+1}\}_f = -\{\Theta_{s,1}^{1,p+1}, \Theta_{r,1}^{1,p+1}\}_f$, without loss of generality we assume that $r > s$.
   If both r and s are even, on the right hand side of (58) the first, third, fifth, sixth, seventh terms vanish. Thus, we have

$$
\begin{aligned}
\{\Theta_{r,1}^{1,p+1}, \Theta_{s,1}^{1,p+1}\}_f =& f_{p+1}^{-1}\left(\{\Theta_{r-1,1}^{1,p}, \Theta_{s,1}^{1,p}\}_f + \{\Theta_{r,1}^{1,p}, \Theta_{s-1,1}^{1,p}\}_f\right) + f_{p+1}^{-1}\Theta_{r-1,1}^{1,p}\{f_{p+1}^{-1}, \Theta_{s-1,1}^{1,p}\}_f \\
&+ f_{p+1}^{-1}\Theta_{s-1,1}^{1,p}\{\Theta_{r-1,1}^{1,p}, f_{p+1}^{-1}\}_f \\
=& f_{p+1}^{-1}\left(\sum_{i\geq 1}(-1)^i\Theta_{s-i,1}^{1,p}\Theta_{r+i-1,1}^{1,p} + \sum_{i\geq 0}(-1)^i\Theta_{r+i,1}^{1,p}\Theta_{s-1-i,1}^{1,p}\right) \\
&+ f_{p+1}^{-2}\Theta_{r-1,1}^{1,p}\Theta_{s-1,1}^{1,p} - f_{p+1}^{-2}\Theta_{s-1,1}^{1,p}\Theta_{r-1,1}^{1,p} \\
=& f_{p+1}^{-1}\left(\sum_{j\geq 0}(-1)^{j+1}\Theta_{s-j-1,1}^{1,p}\Theta_{r+j,1}^{1,p} + \sum_{i\geq 0}(-1)^i\Theta_{r+i,1}^{1,p}\Theta_{s-1-i,1}^{1,p}\right) \\
=& 0.
\end{aligned}
$$

   If both r and s are odd and assuming $r > s$, on the right hand side of (58) the first, sixth, seventh, eighth, and ninth terms vanish. Therefore, we have

$$
\begin{aligned}
&\{\Theta_{r,1}^{1,p+1}, \Theta_{s,1}^{1,p+1}\}_f \\
=& f_{p+1}\left(\{\Theta_{r-1,1}^{1,p}, \Theta_{s,1}^{1,p}\}_f + \{\Theta_{r,1}^{1,p}, \Theta_{s-1,1}^{1,p}\}_f\right) + \Theta_{r-1,1}^{1,p}\{f_{p+1}, \Theta_{s,1}^{1,p}\}_f + \Theta_{s-1,1}^{1,p}\{\Theta_{r,1}^{1,p}, f_{p+1}\}_f \\
=& f_{p+1}\left(\sum_{i\geq 0}(-1)^i\Theta_{r-1+i,1}^{1,p}\Theta_{s-i,1}^{1,p} - \sum_{i\geq 1}(-1)^{i-1}\Theta_{s-1-i,1}^{1,p}\Theta_{r+i,1}^{1,p}\right) - f_{p+1}\Theta_{r-1,1}^{1,p}\Theta_{s,1}^{1,p} + f_{p+1}\Theta_{s-1,1}^{1,p}\Theta_{r,1}^{1,p} \\
=& f_{p+1}\left(\sum_{i\geq 0}(-1)^i\Theta_{r-1+i,1}^{1,p}\Theta_{s-i,1}^{1,p} - \sum_{j\geq 2}(-1)^j\Theta_{s-j,1}^{1,p}\Theta_{r+j-1,1}^{1,p} - \Theta_{r-1,1}^{1,p}\Theta_{s,1}^{1,p} + \Theta_{s-1,1}^{1,p}\Theta_{r,1}^{1,p}\right) \\
=& f_{p+1}\left(\Theta_{r-1,1}^{1,p}\Theta_{s,1}^{1,p} - \Theta_{r,1}^{1,p}\Theta_{s-1,1}^{1,p} - \Theta_{r-1,1}^{1,p}\Theta_{s,1}^{1,p} + \Theta_{s-1,1}^{1,p}f_{p+1}\Theta_{r,1}^{1,p}\right) \\
=& 0.
\end{aligned}
$$

2. r is even, s is odd and $r > s$. We have

$$
\begin{aligned}
&\{\Theta^{1,p+1}_{r,1}, \Theta^{1,p+1}_{s,1}\}_f \\
=&\{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,1}\}_f + \{\Theta^{1,p}_{r-1,1}, \Theta^{1,p}_{s-1,1}\}_f + f^{-1}_{p+1}\Theta^{1,p}_{s-1,1}\{\Theta^{1,p}_{r-1,1}, f_{p+1}\}_f + \Theta^{1,p}_{r-1,1}\{f^{-1}_{p+1}, \Theta^{1,p}_{s,1}\}_f \\
=&\sum_{i\geq 0}(-1)^i\Theta^{1,p}_{r+i,1}\Theta^{1,p}_{s-i,1} - \left(\sum_{i\geq 1}(-1)^{i-1}\Theta^{1,p}_{s-i-1,1}\Theta^{1,p}_{r-1+i,1}\right) + \Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r-1,1} + f^{-1}_{p+1}\Theta^{1,p}_{r-1,1}\Theta^{1,p}_{s,1} \\
=&\sum_{i\geq 0}(-1)^i\Theta^{1,p}_{r+i,1}\Theta^{1,p}_{s-i,1} + \sum_{i\geq 0}(-1)^i\Theta^{1,p}_{s-i-1,1}\Theta^{1,p}_{r-1+i,1} + f^{-1}_{p+1}\Theta^{1,p}_{r-1,1}\Theta^{1,p}_{s,1} \\
=&\sum_{i\geq 0}(-1)^i\Theta^{1,p+1}_{r+i,1}\Theta^{1,p+1}_{s-i,1},
\end{aligned}
$$

where in the last step we used (56).

3. r is even, s is odd and $r < s$. We do similarly as in the previous case. Therefore, with $\epsilon = 1$ identity (16) holds for $p+1$. Then, it holds for all $p \geq 0$.

□

A.2 Proof of Lemma 2

Proof. The proof proceeds by induction again. It is easy to see that (17) can be rewritten as follows

$$
\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_f = \begin{cases} \sum_{i\geq 0}\left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1} - \Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0}\right) & r \leq s, \\ \sum_{i\geq 0}\left(\Theta^{1,p}_{s-2i-1,0}\Theta^{1,p}_{r+2i+1,1} - \Theta^{1,p}_{s-2i-1,1}\Theta^{1,p}_{r+2i+1,0}\right) & r > s. \end{cases} \tag{59}
$$

Identities (17) (or (59)) and (18) hold for $p = 1, 2$. Suppose that they hold for $p-1$ and p ($p \geq 2$). We will prove that they hold for $p+1$.

Using identity (56), expanding the left hand sides of (17) (or (59)) and (18), we have

$$
\begin{aligned}
&\{\Theta^{1,p+1}_{r,0}, \Theta^{1,p+1}_{s,1}\}_f \\
=&\{\Theta^{1,p}_{r,0} + f^{(-1)^r}_{p+1}\Theta^{1,p}_{r-1,0}, \Theta^{1,p}_{s,1} + f^{(-1)^{s+1}}_{p+1}\Theta^{1,p}_{s-1,1}\}_f \\
=&\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_f + f^{(-1)^r+(-1)^{s+1}}_{p+1}\{\Theta^{1,p}_{r-1,0}, \Theta^{1,p}_{s-1,1}\}_f + \Theta^{1,p}_{r,0}\Theta^{1,p}_{s,1}\{f^{(-1)^r}_{p+1}, f^{(-1)^{s+1}}_{p+1}\}_f \\
&+ f^{(-1)^{s+1}}_{p+1}\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s-1,1}\}_f + f^{(-1)^r}_{p+1}\{\Theta^{1,p}_{r-1,0}, \Theta^{1,p}_{s,1}\}_f + \Theta^{1,p}_{s-1,1}\{\Theta^{1,p}_{r,0}, f^{(-1)^{s+1}}_{p+1}\}_f \\
&+ \Theta^{1,p}_{r-1,0}\{f^{(-1)^r}_{p+1}, \Theta^{1,p}_{s,1}\}_f + f^{(-1)^r}_{p+1}\Theta^{1,p}_{s-1,1}\{\Theta^{1,p}_{r-1,0}, f^{(-1)^{s+1}}_{p+1}\}_f + f^{(-1)^{s+1}}_{p+1}\Theta^{1,p}_{r-1,0}\{(f^{(-1)^r}_{p+1}, \Theta^{1,p}_{s-1,1}\}_f.
\end{aligned} \tag{60}
$$

1. $r \equiv s \pmod 2$. We distinguish 2 cases.
   Case 1: $s - r = k \geq 0$, we first prove the following

$$
\sum_{i\geq 0}(-1)^{i+\epsilon}\Theta^{1,p}_{s-1-i,i+\epsilon}\Theta^{1,p}_{r+i,i+\epsilon+1} = \sum_{i\geq 0}(-1)^{i+\epsilon}\Theta^{1,p}_{r-i-1,i+\epsilon}\Theta^{1,p}_{s+i,i+\epsilon+1}. \tag{61}
$$

The left hand side of this identity equals

$$\begin{aligned}
&\sum_{i=k}^{s-1}(-1)^{i+\epsilon}\Theta^{1,p}_{s-1-i,i+\epsilon}\Theta^{1,p}_{r+i,i+\epsilon+1}+\sum_{i=0}^{k-1}(-1)^{i+\epsilon}\Theta^{1,p}_{s-1-i,i+\epsilon}\Theta^{1,p}_{r+i,i+\epsilon+1}\\
=&\sum_{j=0}^{r-1}(-1)^{k+j+\epsilon}\Theta^{1,p}_{r-j-1,k+j+\epsilon}\Theta^{1,p}_{s+j,k+j+\epsilon+1}\\
&+\sum_{i=0}^{\frac{k}{2}-1}\Big((-1)^{i+\epsilon}\Theta^{1,p}_{s-1-i,i+\epsilon}\Theta^{1,p}_{r+i,i+\epsilon+1}+(-1)^{k-1-i+\epsilon}\Theta^{1,p}_{s-1-(k-i-1),k-i-1+\epsilon}\Theta^{1,p}_{r+k-i-1,k-i+\epsilon}\Big)\\
=&\sum_{i=0}^{r-1}(-1)^{j+\epsilon}\Theta^{1,p}_{r-j-1,j+\epsilon}\Theta^{1,p}_{s+j,j+\epsilon+1}+\sum_{i=0}^{\frac{k}{2}-1}\Big((-1)^{i+\epsilon}\Theta^{1,p}_{s-1-i,i+\epsilon}\Theta^{1,p}_{r+i,i+\epsilon+1}+(-1)^{i+\epsilon+1}\Theta^{1,p}_{r+i,i+\epsilon+1}\Theta^{1,p}_{s-i-1,i+\epsilon}\Big)\\
=&\sum_{i=0}^{r-1}(-1)^{i+\epsilon}\Theta^{1,p}_{r-i-1,i+\epsilon}\Theta^{1,p}_{s+i,i+\epsilon+1},
\end{aligned}$$

which is the right hand side of (61).

Now using (56), we expand the right hand side of the first identity of (59). We have

$$\begin{aligned}
&\sum_{i\geq 0}\Big(\Theta^{1,p+1}_{r-2i-1,0}\Theta^{1,p+1}_{s+2i+1,1}-\Theta^{1,p+1}_{r-2i-1,1}\Theta^{1,p+1}_{s+2i+1,0}\Big)\\
=&\sum_{i\geq 0}\Big(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1}-\Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0}+\Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i,1}-\Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i,0}\Big)\\
&+f_{p+1}^{(-1)^{r-1}}\sum_{i\geq 0}\Big(\Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i+1,1}-\Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i,0}\Big)\\
&+f_{p+1}^{(-1)^{s}}\sum_{i\geq 0}\Big(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i,1}-\Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i+1,0}\Big)\\
=&\sum_{i\geq 0}\Big(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1}-\Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0}+\Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i,1}-\Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i,0}\Big)\\
&+f_{p+1}^{(-1)^{s+1}}\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{r-1-i,i+1}\Theta^{1,p}_{s+i,i}+f_{p+1}^{(-1)^{s}}\sum_{i\geq 0}(-1)^{i}\Theta^{1,p}_{r-1-i,i}\Theta^{1,p}_{s+i,i+1}. \qquad (62)
\end{aligned}$$

If r and s are both even. Using (60) and the induction assumption, we have

$$\begin{aligned}
&\{\Theta^{1,p+1}_{r,0},\Theta^{1,p+1}_{s,1}\}_f\\
=&\sum_{i\geq 0}\Big(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1}-\Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0}\Big)+\sum_{i\geq 0}\Big(\Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i,1}-\Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i,0}\Big)\\
&+f_{p+1}^{(-1)^{r}}\sum_{i\geq 0}(-1)^{i}\Theta^{1,p}_{s+i,i+1}\Theta^{1,p}_{r-1-i,i}+\Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r-1,0}-\Theta^{1,p}_{r-1,0}\Theta^{1,p}_{s-1,1}\\
&+f_{p+1}^{(-1)^{s+1}}\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-1-i,i+1}\Theta^{1,p}_{r+i,i},
\end{aligned}$$

which equals (62) by using (61) with $\epsilon=1$.

If r and s are both odd, we have

$$
\begin{aligned}
&\{\Theta^{1,p+1}_{r,0},\Theta^{1,p+1}_{s,1}\}_f\\
=&\sum_{i\geq 0}\left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1}-\Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0}\right)+\sum_{i\geq 0}\left(\Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i,1}-\Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i,0}\right)\\
&+f^{(-1)^{s+1}}_{p+1}\sum_{i\geq 0}(-1)^i\Theta^{1,p}_{s-1+i,i+1}\Theta^{1,p}_{r-i,i}+f^{(-1)^r}_{p+1}\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r-1+i,i}-f^{(-1)^{s+1}}_{p+1}\Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r,0}\\
&+f^{(-1)^r}_{p+1}\Theta^{1,p}_{r-1,0}\Theta^{1,p}_{s,1}\\
=&\sum_{i\geq 0}\left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1}-\Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0}\right)+\sum_{i\geq 0}\left(\Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i,1}-\Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i,0}\right)\\
&+f^{(-1)^{s+1}}_{p+1}\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s+i,i}\Theta^{1,p}_{r-1-i,i+1}+f^{(-1)^r}_{p+1}\sum_{i\geq 0}(-1)^i\Theta^{1,p}_{s-1-i,i}\Theta^{1,p}_{r+i,i+1}.
\end{aligned}
$$

which equals (62) by using (61) with $\epsilon=0$. Thus, the first identity of (17) (or (59)) holds for $p+1$.

Case 2: $s-r=-k<0$, identity (17) (or (59)) also holds by using Remark 16.

2. $r\not\equiv s \pmod 2$. Case 1: $s-r=k>0$ With r even, s odd and $r<s$, we now expand the right hand side of the first identity of (18) with $p+1$. Similar as (61), we have the following identities

$$\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i,i}=\sum_{i\geq 1}(-1)^i\Theta^{1,p}_{s-1-i,i+1}\Theta^{1,p}_{r+i-1,i} \tag{63}$$

$$
\begin{aligned}
\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i-1,i+1}\Theta^{1,p}_{r+i,i}+\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i-1,i}=&\sum_{i\geq 1}(-1)^i\Theta^{1,p}_{r-1-i,i}\Theta^{1,p}_{s+i,i+1}\\
&+\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{r-1-i,i+1}\Theta^{1,p}_{s+i,i}.
\end{aligned}
\tag{64}
$$

We now expand the right hand side of (18) by using (56, we obtain

$$
\begin{aligned}
\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p+1}_{s-i,i+1}\Theta^{1,p+1}_{r+i,i}=&\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i,i}+f^2_{p+1}\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i-1,i+1}\Theta^{1,p}_{r+i-1,i}\\
&+f_{p+1}\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i-1,i+1}\Theta^{1,p}_{r+i,i}+f_{p+1}\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i-1,i}.
\end{aligned}
\tag{65}
$$

For the left hand side of (18), using (60) and the induction assumption, we have

$$
\begin{aligned}
&\{\Theta^{1,p+1}_{r,0}, \Theta^{1,p+1}_{s,1}\}_f \\
=&\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i,i} + f^2_{p+1}\sum_{i\geq 0}(-1)^i\Theta^{1,p}_{s-1+i,i+1}\Theta^{1,p}_{r-1-i,i} \\
&+ f_{p+1}\sum_{i\geq 0}\left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s-1+2i+1,1} - \Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s-1+2i+1,0}\right) \\
&+ f_{p+1}\sum_{i\geq 0}\left(\Theta^{1,p}_{r-1-2i-1,0}\Theta^{1,p}_{s+2i+1,1} - \Theta^{1,p}_{r-1-2i-1,1}\Theta^{1,p}_{s+2i+1,0}\right) - f_{p+1}\Theta^{1,p}_{r-1,0}\Theta^{1,p}_{s,1} - f^2_{p+1}\Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r-1,0} \\
=&\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i,i} + f^2_{p+1}\sum_{i\geq 1}(-1)^i\Theta^{1,p}_{s-1+i,i+1}\Theta^{1,p}_{r-1-i,i} + f_{p+1}\sum_{i\geq 1}(-1)^i\Theta^{1,p}_{r-1-i,i}\Theta^{1,p}_{s+i,i+1} \\
&+ f_{p+1}\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p}_{r-1-i,i+1}\Theta^{1,p}_{s+i,i} \\
=&\sum_{i\geq 0}(-1)^{i-1}\Theta^{1,p+1}_{s-i,i+1}\Theta^{1,p+1}_{r+i,i}
\end{aligned}
$$

where in the final equality we used (63) and (64). That implies that the second identity of (18) holds for $p+1$.

With r odd, s even and $r<s$, we do similarly to what we did in the previous case.

Case 2: $s=r=-k<0$, identity (18) still holds by using Remark 16.

□

B Properties of Ψ with respect to the Poisson brackets

B.1 Proof of Lemma 11

Proof. We prove (45) and (46) simultaneously by induction. We use the following property

$$\Psi^{a,b+1}_r = c_{b+2}\Psi^{a,b}_r + \Psi^{a,b-1}_{r-1} \tag{66}$$

and therefore we get

$$\frac{\partial \Psi^{a,b+1}_r}{\partial c_{b+2}} = \Psi^{a,b}_r. \tag{67}$$

We see that (1) and (2) hold for $p=1,2,3$. Suppose (1) and (2) hold for $p-2, p-1$ and p. We need to prove that (1) and (2) hold for $p+1$. Expanding the right hand side of the first identity, we have

$$
\begin{aligned}
\{\Psi^{1,p+1}_r, \Psi^{1,p+1}_s\}_c &= \{c_{p+2}\Psi^{1,p}_r + \Psi^{1,p-1}_{r-1}, c_{p+2}\Psi^{1,p}_s + \Psi^{1,p-1}_{s-1}\}_c \\
&= \{c_{p+2}\Psi^{1,p}_r, c_{p+2}\Psi^{1,p}_s\}_c + \{c_{p+2}\Psi^{1,p}_r, \Psi^{1,p-1}_{s-1}\}_c + \{\Psi^{1,p-1}_{r-1}, c_{p+2}\Psi^{1,p}_s\}_c + \{\Psi^{1,p-1}_{r-1}, \Psi^{1,p-1}_{s-1}\}_c \\
&= c_{p+2}\Psi^{1,p}_s\{\Psi^{1,p}_r, c_{p+2}\}_c + c_{p+2}\Psi^{1,p}_r\{c_{p+2}, \Psi^{1,p}_s\}_c + \Psi^{1,p}_r\{c_{p+2}, \Psi^{1,p-1}_s\}_c \\
&\quad + \Psi^{1,p}_s\{\Psi^{1,p-1}_r, c_{p+2}\}_c + c_{p+2}\left(\{\Psi^{1,p}_r, \Psi^{1,p-1}_{s-1}\}_c + \{\Psi^{1,p-1}_{r-1}, \Psi^{1,p}_s\}_c\right) \\
&= c_{p+2}\left(\Psi^{1,p}_s\Psi^{1,p-1}_r - \Psi^{1,p}_r\Psi^{1,p-1}_s + \{\Psi^{1,p}_r, \Psi^{1,p-1}_{s-1}\}_c + \{\Psi^{1,p-1}_{r-1}, \Psi^{1,p}_s\}_c\right).
\end{aligned}
$$

For the second identity, we also have

$$
\begin{aligned}
\{\Psi^{1,p+1}_r, \Psi^{1,p}_s\}_c + \{\Psi^{1,p}_r, \Psi^{1,p+1}_s\}_c &= \{c_{p+2}\Psi^{1,p}_r + \Psi^{1,p-1}_{r-1}, \Psi^{1,p}_s\}_c + \{\Psi^{1,p}_r, c_{p+2}\Psi^{1,p}_s + \Psi^{1,p-1}_{s-1}\}_c \\
&= \{\Psi^{1,p-1}_{r-1}, \Psi^{1,p}_s\}_c + \{\Psi^{1,p}_r, \Psi^{1,p-1}_{s-1}\}_c - \Psi^{1,p}_r\Psi^{1,p-1}_{s-1} + \Psi^{1,p}_s\Psi^{1,p-1}_{r-1}
\end{aligned}
$$

Now to prove (1) and (2) hold for $p+1$, we only need to prove that

$$T := \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c + \{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c - \Psi_r^{1,p}\Psi_{s-1}^{1,p-1} + \Psi_s^{1,p}\Psi_{r-1}^{1,p-1} = 0.$$

Using (66) and the induction assumption to expand T, we obtain

$$\begin{aligned}
T &= (c_{p+1}\Psi_s^{1,p-1} + \Psi_{s-1}^{1,p-2})\Psi_r^{1,p-1} - c_{p+2}(c_{p+1}\Psi_r^{1,p-1} + \Psi_{r-1}^{1,p-2})\Psi_s^{1,p-1} \\
&\quad + \{c_{p+1}\Psi_r^{1,p-1} + \Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, c_{p+1}\Psi_s^{1,p-1} + \Psi_{s-1}^{1,p-2}\}_c \\
&= \Psi_{s-1}^{1,p-2}\Psi_r^{1,p-1} - \Psi_{r-1}^{1,p-2}\Psi_s^{1,p-1} + \Big(\{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_{s-1}^{1,p-2}\}_c\Big) \\
&\quad + \Psi_r^{1,p-1}\{c_{p+1}, \Psi_{s-1}^{1,p-1}\} + \Psi_s^{1,p-1}\{\Psi_{r-1}^{1,p-1}, c_{p+1}\}_c \\
&= \Psi_{s-1}^{1,p-2}\Psi_r^{1,p-1} - c_{p+2}\Psi_{r-1}^{1,p-2}\Psi_s^{1,p-1} + \Big(\{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_{s-1}^{1,p-2}\}_c\Big) \\
&\quad - \Psi_r^{1,p-1}\Psi_{s-1}^{1,p-2} + \Psi_s^{1,p-1}\Psi_{r-1}^{1,p-2} \\
&= \Big(\{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{1,p-1}\} + \{\Psi_{r-1}^{1,p-1}, \Psi_{s-1}^{1,p-2}\}\Big) \\
&= 0.
\end{aligned}$$

That means (1) and (2) hold for $p+1$. □

B.2 Corollary of lemma 11

We obtain the following formulas for (sums of) Poisson brackets of Ψs.

Corollary 17. *Let $p \geq 1$ and let $r, s \in \mathbb{Z}$. Then,*

1. $\{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{1,p}\Psi_s^{1,p-1} - \Psi_s^{1,p}\Psi_r^{1,p-1}$,
2. $\{\Psi_r^{a,b}, \Psi_s^{a,b}\}_c = 0$ *with* $0 \leq r, s \leq \lfloor (b-a)/2 \rfloor + 1$,
3. $\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}_c = \Psi_s^{1,n}\Psi_r^{2,p} - \Psi_r^{1,p}\Psi_s^{2,p}$,
4. $\{\Psi_r^{1,p}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p}\}_c = 0$,
5. $\{\Psi_r^{1,p}, \Psi_s^{2,p+1}\}_c + \{\Psi_r^{2,p+1}, \Psi_s^{1,p}\}_c = 0$,
6. $\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{2,p}\Psi_s^{1,p-1} - \Psi_s^{2,p}\Psi_r^{1,p-1}$,
7. $\{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{1,p}\Psi_s^{1,p-1} - \Psi_s^{1,p}\Psi_r^{1,p-1}$,
8. $\{\Psi_r^{1,p+1}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p+1}\}_c = 0$,
9. $\{\Psi_r^{1,p}, \Psi_{s-2}^{2,p-1}\}_c + \{\Psi_{r-2}^{2,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{2,p}\Psi_{s-1}^{1,p-1} - \Psi_s^{2,p}\Psi_{r-1}^{1,p-1} + \Psi_{r-1}^{2,p}\Psi_s^{1,p-1} - \Psi_{s-1}^{2,p}\Psi_r^{1,p-1}$.

Proof. 1. It follows from the proof of Lemma 11.

2. It follows from Lemma 11.

3. Using (2) we have $\{\Psi_r^{0,p}, \Psi_s^{0,p}\}_c = 0$. On the other hand, we have

$$\begin{aligned}
\{\Psi_r^{0,p}, \Psi_s^{0,p}\}_c &= \{c_0\Psi_r^{1,p} + \Psi_{r-1}^{2,p}, c_0\Psi_s^{1,p} + \Psi_{s-1}^{2,p}\}_c \\
&= \{c_0\Psi_r^{1,p}, c_0\Psi_s^{1,p}\}_c + \{c_0\Psi_s^{1,p}, \Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p}, c_0\Psi_s^{1,p}\}_c + \{\Psi_{r-1}^{2,p}, \Psi_{s-1}^{2,p}\}_c \\
&= c_0\Psi_s^{1,p}\{\Psi_r^{1,p}, c_0\} + c_0\Psi_r^{1,p}\{c_0, \Psi_s^{1,p}\}_c + c_0\left(\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}_c\right) \\
&\quad + \Psi_s^{1,p}\{\Psi_{r-1}^{2,p}, c_0\}_c + \Psi_r^{1,p}\{c_0, \Psi_{s-2}^{2,p}\}_c \\
&= -c_0\Psi_s^{1,p}\Psi_r^{2,p} + c_0\Psi_r^{1,p}\Psi_s^{2,p} + c_0\left(\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p}\} + \{\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}\right)
\end{aligned}$$

Therefore, we get $\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}_c = \Psi_s^{1,p}\Psi_r^{2,p} - \Psi_r^{1,p}\Psi_s^{2,p}$.

4. For this identity, we expand the left hand side. We have

$$\begin{aligned}
\text{LHS} &= \{\Psi_r^{1,p}, \Psi_{s+1}^{0,p} - c_0\Psi_{s+1}^{1,p}\}_c + \{\Psi_{r+1}^{0,p} - c_0\Psi_{r+1}^{1,p}, \Psi_s^{1,p}\}_c \\
&= \{\Psi_r^{1,p}, \Psi_{s+1}^{0,p}\}_c + \{\Psi_{r+1}^{0,p}, \Psi_s^{1,p}\}_c - \Psi_{s+1}^{1,p}\{\Psi_r^{1,p}, c_0\}_c - \Psi_{r+1}^{1,p}\{c_0, \Psi_s^{1,p}\}_c \\
&= \{\Psi_r^{1,p}, \Psi_{s+1}^{0,p}\}_c + \{\Psi_{r+1}^{0,p}, \Psi_s^{1,p}\}_c + \Psi_{s+1}^{1,p}\Psi_r^{2,p} - \Psi_{r+1}^{1,p}\Psi_s^{2,p}.
\end{aligned}$$

By property (3) we have

$$\begin{aligned}
\{\Psi_r^{1,p}, \Psi_{s+1}^{0,p}\}_c + \{\Psi_{r+1}^{0,p}, \Psi_s^{1,p}\}_c &= \Psi_{s+1}^{0,p}\Psi_{r+1}^{1,p} - \Psi_{r+1}^{0,p}\Psi_{s+1}^{1,p} \\
&= (c_0\Psi_{r+1}^{1,p} + \Psi_r^{2,p})\Psi_{r+1}^{1,p} - (c_0\Psi_{r+1}^{1,p} + \Psi_r^{2,n})\Psi_{s+1}^{1,p} \\
&= \Psi_s^{2,p}\Psi_{r+1}^{1,p} - \Psi_r^{2,p}\Psi_{s+1}^{1,p}.
\end{aligned}$$

It means that $\{\Psi_r^{1,p}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p}\}_c = 0$.

5. One can verify that this identity holds for $p = 1, 2, 3$. Expanding the left hand side using (66), we have

$$\begin{aligned}
LHS &= \{\Psi_r^{1,p}, c_{p+2}\Psi_s^{2,p} + \Psi_{s-1}^{2,p-1}\}_c + \{c_{p+2}\Psi_r^{2,p} + \Psi_{r-1}^{2,p-1}, \Psi_s^{1,p}\}_c \\
&= c_{p+2}(\{\Psi_r^{1,p}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p}\}_c) + \Psi_s^{2,p}\{\Psi_r^{1,p}, c_{p+2}\}_c + \Psi_r^{2,p}\{c_{p+2}, \Psi_s^{1,p}\}_c \\
&\quad + \{\Psi_r^{1,p}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_s^{1,p}\}_c \\
&= \Psi_s^{2,p}\Psi_r^{1,p-1} - \Psi_r^{2,p}\Psi_s^{1,p-1} + \{c_{p+1}\Psi_r^{1,p-1} + \Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{2,p-1}\}_c \\
&\quad + \{\Psi_{r-1}^{2,p-1}, c_{p+1}\Psi_s^{1,p-1} + \Psi_{s-1}^{1,p-2}\}_c \\
&= \Psi_s^{2,p}\Psi_r^{1,p-1} - \Psi_r^{2,p}\Psi_s^{1,p-1} + c_{p+1}(\{\Psi_r^{1,p-1}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_s^{1,p-1}\}_c) \\
&\quad + \Psi_r^{1,p-1}\{c_{p+1}, \Psi_{s-1}^{2,p-1}\}_c + \Psi_s^{1,p-1}\{\Psi_{r-1}^{2,p-1}, c_{p+1}\}_c + \{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_{s-1}^{1,p-2}\}_c \\
&= \Psi_s^{2,p}\Psi_r^{1,p-1} - \Psi_r^{2,p}\Psi_s^{1,p-1} + c_{p+1}(\Psi_s^{1,p-1}\Psi_r^{2,p-1} - \Psi_r^{1,p-1}\Psi_s^{2,p-1}) \\
&\quad - \Psi_r^{1,p-1}\Psi_{s-1}^{2,p-2} + \Psi_s^{1,p-1}\Psi_{r-1}^{2,p-2} + \{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_{s-1}^{1,p-2}\}_c \\
&= \{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_{s-1}^{1,p-2}\}_c.
\end{aligned}$$

Therefore, using induction we prove our statement.

6. This identity follows from the proof of the identity in (5).

7. We have

$$\begin{aligned}\{\Psi_r^{1,p},\Psi_{s-1}^{1,p-1}\}_c+\{\Psi_{r-1}^{1,p-1},\Psi_s^{1,p}\}_c &= \{\Psi_r^{1,p},\Psi_s^{1,p+1}-c_{p+2}\Psi_s^{1,p}\}_c\\ &\quad+\{\Psi_r^{1,p+1}-c_{p+2}\Psi_r^{1,p},\Psi_s^{1,p}\}_c\\ &= -\Psi_s^{1,p}\{\Psi_r^{1,p},c_{p+2}\}_c-\Psi_r^{1,p}\{c_{p+2},\Psi_s^{1,p}\}_c\\ &= -\Psi_s^{1,p}\Psi_r^{1,p-1}+\Psi_r^{1,p}\Psi_s^{1,p-1}\end{aligned}$$

8. We prove this by induction. This identity holds for $p=1,2$. Suppose that this identity holds for $p-2$, and $p-1$ we need to prove that it holds for p. We have

$$\begin{aligned}&\{\Psi_r^{1,p+1},\Psi_s^{2,p}\}_c+\{\Psi_r^{2,p},\Psi_s^{1,p+1}\}_c\\ &=\{c_{p+2}\Psi_r^{1,p}+\Psi_{r-1}^{1,p-1},\Psi_s^{2,p}\}_c+\{\Psi_r^{2,p},c_{p+2}\Psi_s^{1,p}+\Psi_{s-1}^{1,p-1}\}_c\\ &=\{\Psi_{r-1}^{1,p-1},\Psi_s^{2,p}\}_c+\{\Psi_r^{2,p},\Psi_{s-1}^{1,p-1}\}_c-\Psi_r^{1,p}\Psi_s^{2,p-1}+\Psi_s^{1,p}\Psi_r^{2,p-1}\\ &=\{\Psi_{r-1}^{1,p-1},c_{p+1}\Psi_s^{2,p-1}+\Psi_{s-1}^{2,p-2}\}_c+\{c_{p+1}\Psi_r^{2,p-1}+\Psi_{r-1}^{2,p-2},\Psi_{s-1}^{1,p-1}\}_c\\ &\quad-\Psi_r^{1,p}\Psi_s^{2,p-1}+\Psi_s^{1,p}\Psi_r^{2,p-1}\\ &=\{\Psi_{r-1}^{1,p-1},\Psi_{s-2}^{2,p-2}\}_c+\{\Psi_{r-2}^{2,p-2},\Psi_{s-1}^{1,p-1}\}_c+c_{p+1}(\{\Psi_{r-1}^{1,p-1},\Psi_s^{2,p-1}\}_c+\{\Psi_r^{2,p-1},\Psi_{s-1}^{1,p-1}\}_c)\\ &\quad-\Psi_s^{2,p-1}\Psi_{r-1}^{1,p-2}+\Psi_r^{2,p-1}\Psi_{s-1}^{1,p-2}-\Psi_r^{1,p}\Psi_s^{2,p-1}+\Psi_s^{1,p}\Psi_r^{2,p-1}\\ &=c_{p+1}(\{\Psi_{r-1}^{1,p-1},\Psi_s^{2,p-1}\}_c+\{\Psi_r^{2,p-1},\Psi_{s-1}^{1,p-1}\}_c)+\Psi_s^{2,p-1}\Psi_{r-1}^{1,p-2}-\Psi_r^{2,p-1}\Psi_{s-1}^{1,p-2}\\ &\quad-\Psi_r^{1,p}\Psi_s^{2,p-1}+\Psi_s^{1,p}\Psi_r^{2,p-1}.\end{aligned}$$

Since $\{\Psi_{r-1}^{1,p-1},\Psi_s^{2,p-1}\}_c+\{\Psi_{r-1}^{2,p-1},\Psi_s^{1,p-1}\}_c=0$ (by (4)), we get $\{\Psi_{r-1}^{1,p-1},\Psi_s^{2,p-1}\}_c=\{\Psi_s^{1,p-1},\Psi_{r-1}^{2,p-1}\}_c$. Similarly, we obtain $\{\Psi_r^{2,p-1},\Psi_{s-1}^{1,p-1}\}_c=\{\Psi_{s-1}^{2,p-1},\Psi_r^{1,p-1}\}_c$. Therefore, we have

$$\begin{aligned}\{\Psi_{r-1}^{1,p-1},\Psi_s^{2,p-1}\}_c+\{\Psi_r^{2,p-1},\Psi_{s-1}^{1,p-1}\}_c &= \{\Psi_s^{1,p-1},\Psi_{r-1}^{2,p-1}\}_c+\{\Psi_{s-1}^{2,p-1},\Psi_r^{1,p-1}\}_c\\ &= \Psi_r^{1,p-1}\Psi_s^{2,p-1}-\Psi_s^{1,p-1}\Psi_r^{2,p-1}.\end{aligned}$$

Thus, we get $\{\Psi_r^{1,p+1},\Psi_s^{2,p}\}_c+\{\Psi_r^{2,p},\Psi_s^{1,p+1}\}_c=0$.

9. We have

$$\begin{aligned}LHS &= \{\Psi_r^{1,p},\Psi_{s-1}^{2,p+1}-c_{p+2}\Psi_{s-1}^{2,p}\}_c+\{\Psi_{r-1}^{2,p+1}-c_{p+2}\Psi_{r-1}^{2,p},\Psi_s^{1,p}\}_c\\ &= \{\Psi_r^{1,p},\Psi_{s-1}^{2,p+1}\}_c+\{\Psi_{r-1}^{2,p+1},\Psi_s^{1,p}\}_c-\Psi_{s-1}^{2,p}\Psi_r^{1,p-1}+\Psi_{r-1}^{2,p}\Psi_{s-1}^{1,p-1}-c_{p+2}(\{\Psi_r^{1,p},\Psi_{s-1}^{2,p}\}_c+\{\Psi_{r-1}^{2,p},\Psi_s^{1,p}\}_c)\\ &= \{\Psi_r^{1,p},\Psi_{s-1}^{2,p+1}\}_c+\{\Psi_{r-1}^{2,p+1},\Psi_s^{1,p}\}_c-\Psi_{s-1}^{2,p}\Psi_r^{1,p-1}+\Psi_{r-1}^{2,p}\Psi_{s-1}^{1,p-1}-c_{p+2}(\Psi_s^{1,p}\Psi_r^{2,p}-\Psi_r^{1,p}\Psi_s^{2,p}).\end{aligned}$$

Now we have

$$\begin{aligned}&\{\Psi_r^{1,p},\Psi_{s-1}^{2,p+1}\}_c+\{\Psi_{r-1}^{2,p+1},\Psi_s^{1,p}\}_c\\ &=\{\Psi_r^{1,p},\Psi_s^{0,p+1}-c_0\Psi_s^{1,p+1}\}_c+\{\Psi_r^{0,p+1}-c_0\Psi_r^{1,p+1},\Psi_s^{1,p}\}_c\\ &=\{\Psi_r^{1,p},\Psi_s^{0,p+1}\}_c+\{\Psi_r^{0,p+1},\Psi_s^{1,p}\}_c-c_0(\{\Psi_r^{1,p},\Psi_s^{1,p+1}\}_c+\Psi_r^{1,p+1},\Psi_s^{1,p})-\Psi_s^{1,p+1}\{\Psi_r^{1,p},c_0\}_c\\ &\quad-\Psi_r^{1,p+1}\{c_0,\Psi_s^{1,p}\}_c\\ &=\Psi_s^{1,p+1}\Psi_r^{2,p}-\Psi_r^{1,p+1}\Psi_s^{2,p}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}LHS &= \Psi_s^{1,p+1}\Psi_r^{2,p}-\Psi_r^{1,p+1}\Psi_s^{2,p}-c_{p+2}(\Psi_s^{1,p}\Psi_r^{2,p}-\Psi_r^{1,p}\Psi_s^{2,p})-\Psi_{s-1}^{2,p}\Psi_r^{1,p-1}+\Psi_{r-1}^{2,p}\Psi_{s-1}^{1,p-1}\\ &= \Psi_r^{2,p}\Psi_{s-1}^{1,p-1}-\Psi_s^{2,p}\Psi_{r-1}^{1,p-1}+\Psi_{r-1}^{2,p}\Psi_s^{1,p-1}-\Psi_{s-1}^{2,p}\Psi_r^{1,p-1}=RHS.\end{aligned}$$

□

References

[1] V Adler, A Bobenko, and Y Suris, *Classification of integrable equations on quad-graphs. The consistency approach*, Commun. Math. Phys. **3** (2003), no. 233, 513–543.

[2] M Bruschi, O Ragnisco, P M Santini, and T G Zhang, *Integrable symplectic maps*, Phys. D **49** (1991), no. 3, 273–294.

[3] H W Capel, F W Nijhoff, and V G Papageorgiou, *Complete integrability of Lagrangian mappings and lattices of KdV type*, Phys. Let. A **155** (1991), no.6-7, 377-387.

[4] B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, *Integrable systems. I*, in Dynamical systems, IV, Encyclopaedia Math. Sci., vol. 4, Springer, Berlin, (2001), 177–332.

[5] B Grammaticos and A Ramani, *Integrability – and how to detect it*, in “Integrability of Nonlinear Systems” (Y Kosmann-Schwarzbach, B Grammaticos, and K M Tamizhmani, eds.), Lect. Notes Phys., vol. 638, Springer, 2004, 31–94.

[6] R Hirota, *Non-linear partial difference equations I-IV*, J. Phys. Soc. Jpn. **43** (1977), 1423–1433, 2074–2078, 2079–2086, J. Phys. Soc. Jpn. **45** (1978), 321–332.

[7] A Iatrou *Higher dimensional integrable mappings* , Physica D **179** (2003), 229–253

[8] S Maeda, *Completely integrable symplectic mapping*, Proc. Japan Acad., Ser. A **6** (1983), 198–200

[9] F W Nijhoff, H W Capel, and V G Papageorgiou, *Integrable quantum mappings*, Phys. Rev. (3) **46** (1992), no. 4, 2155–2158 .

[10] G R W Quispel, H W Capel, V G Capel, and F W Nijhoff, *Integrable mappings derived from soliton equations*, Phys. A **173** (1991), 243–266.

[11] G R W Quispel, F W Nijhoff, , H W Capel, and J van der Linden, *Linear integral equations and non-linear difference-difference equations*, Physica A **125** (1984), no. 2-3, 344–380.

[12] J A G Roberts and G R W Quispel, *Constant Poisson structures for difference equations, including integrable maps*, (in preparation).

[13] D T Tran, P H van der Kamp, and G R W Quispel, *Closed-form expressions for integrals of traveling wave reductions of integrable lattice equations*, J.Phys A.: Math. Theor. **42** (2009), no. 22, 225201 (20pp).

[14] D T Tran, P H van der Kamp, and G R W Quispel, *Functional independence of integrals of sine-Gordon, mKdV and pKdV maps*, (in preparation).

[15] P H van der Kamp, *Initial value problems for lattice equations*, J. Phys. A: Math. Theor. **42** (2009), 404019–34

[16] P H van der Kamp, O Rojas, and G R W Quispel, *Closed-form expressions for integrals of mKdV and sine-Gordon maps*, J. Phys. A: Math. Theor. **39** (2007), 12789–12798.

[17] A P Veselov, *Integrable maps*, Russ. Math. Surveys **46** (1991), no. 5, 1–51.