Involutivity of sine-Gordon, pKdV and mKdV maps

Dinh T. Tran, P.H. van der Kamp, G.R.W. Quispel
Department of Mathematics, La Trobe University, Victoria, 3086, Australia
E-mail:td2tran@students.latrobe.edu.au

July 9, 2009

Abstract

We prove the involutivity of integrals of sine-Gordon, modified Korteweg-de Vries and potential Korteweg-de Vries maps which are obtained as $(1,p)$ traveling wave reductions of corresponding partial difference equations.

1 Introduction

Integrable systems boast a long and venerable history, and have famous members such as the Kepler system, the Korteweg-de Vries equation, and the sine-Gordon equation. More recently, interest in integrable systems has expanded to include systems with discrete time, e.g. integrable ordinary difference equations (or maps) and integrable partial difference equations.

For many of these systems, the way to prove complete integrability has been to show that there exist a sufficient number of functionally independent integrals that are in involution, that is, they Poisson commute.

In this paper we study the involutivity of integrals of a certain class of integrable maps related to the discrete sine-Gordon, modified Korteweg-de Vries(mKdV) and potential Korteweg-de Vries(pKdV) equations. Integrable maps typically come in an infinite family of increasing dimension, and for this reason it is not feasible to calculate Poisson brackets one by one and show that they all vanish. One way to circumvent this problem is to use the so-called Yang-Baxter structure, and that is the approach taken in [3, 6]. Here we take a different approach. Starting from recently found symplectic structures [4, 9], and recently obtained closed-form expressions for integrals of our family of sine-Gordon, mKdV and pKdV maps [5, 11], we proceed to prove involutivity of the integrals directly by induction on the dimension of the maps.

Recall that a $2n$-dimensional discrete map $L : x \mapsto x^{'}$ is said to be completely integrable if, cf. [4],

- there is a $2n \times 2n$ anti-symmetric non-degenerate matrix Ω such that the Jacobian identity holds:

$$\sum_{l} \left(\Omega_{li}\frac{\partial}{\partial x_l}\Omega_{jk} + \Omega_{lj}\frac{\partial}{\partial x_l}\Omega_{ki} + \Omega_{lk}\frac{\partial}{\partial x_l}\Omega_{ij}\right) = 0,$$

- the Jacobian dL of the map L satisfies $dL(x)\Omega(x)dL^T(x) = \Omega(x^{'})$,

- there exist n functionally independent integrals $I_1, I_2, \ldots, I_n$ satisfying $\triangledown I^T \Omega \triangledown I = 0_{n\times n}$, that is,

$$\{I_r, I_s\} = \sum_{i,j} \frac{\partial I_r}{\partial x_i}\Omega_{ij}\frac{\partial I_s}{\partial x_j} = 0, \ \forall\ 1 \leq r, s \leq n.$$

The families of discrete sine-Gordon, mKdV and pKdV equations obtained from the $(1,p)$-reduction [5, 8] are given as follows

$$\text{sG}: \alpha_1(v_n v_{n+p+1} - v_{n+1}v_{n+p}) + \alpha_2 v_n v_{n+1} v_{n+p} v_{n+p+1} - \alpha_3 = 0, \tag{1}$$

$$\text{mKdV}: \beta_1(v_n v_{n+p} - v_{n+1}v_{n+p+1}) + \beta_2 v_n v_{n+1} - \beta_3 v_{n+p} v_{n+p+1} = 0, \tag{2}$$

$$\text{pKdV}: (v_n - v_{n+p+1})(v_{n+1} - v_{n+p}) - \gamma = 0. \tag{3}$$

The corresponding $d = p+1$ dimensional maps are $L : \mathbb{R}^d \to \mathbb{R}^d$,

$$(v_1, v_1, \ldots, v_d) \mapsto (v_2, v_3, \ldots, v_{d+1}), \tag{4}$$

where

$$v_{d+1} = v_1^{-1}\frac{\alpha_1 v_2 v_d + \alpha_3}{\alpha_2 v_2 v_d + \alpha_1}, \quad v_{d+1} = v_1 \frac{\beta_1 v_d + \beta_2 v_2}{\beta_1 v_2 + \beta_3 v_d}, \quad v_{d+1} = v_1 - \frac{\gamma}{v_2 - v_d}$$

respectively. The integrals of sine-Gordon and mKdV maps are given by [5]

$$I_r = \alpha_1\big(\frac{v_d}{v_1}\Theta^{1,d-1}_{2r,1} + \frac{v_1}{v_d}\Theta^{1,d-1}_{2r,0}\big) + \alpha_2\Theta^{1,d-1}_{2r+1,1} + \alpha_3\Theta^{1,d-1}_{2r+1,0}, \tag{5}$$

$$\widetilde{I}_r = \beta_1\big(v_1 v_d \Theta^{1,d-1}_{2r-1,0} + \frac{1}{v_1 v_d}\Theta^{1,d-1}_{2r-1,1}\big) + \beta_2\Theta^{1,d-1}_{2r,1} + \beta_3\Theta^{1,d-1}_{2r,0}, \tag{6}$$

where $0 \le 2r \le d-1$ and $0 \le 2r \le d$ respectively, and

$$\Theta^{a,b}_{r,\epsilon}\{f_i\} := \sum_{a \le i_1 < i_2 < \cdots < i_r \le b} \prod_{j=1}^{r} (f_{i_j})^{(-1)^{j+\epsilon}}, \tag{7}$$

with $f_i = v_i v_{i+1}$. For the pKdV map integrals are given by

$$I_r = \Psi^{2,d-2}_{r-1} + (v_d - v_2)\Psi^{2,d-3}_{r-1} + (v_{d-1} - v_1)\Psi^{3,d-2}_{r-1} + \Psi^{3,d-3}_{r-2} + \big((v_{d-1} - v_1)(v_d - v_2) - \gamma\big)\Psi^{2,d-2}_{r}, \tag{8}$$

where $0 \le r \le \lfloor (d-3)/2 \rfloor$ and

$$\Psi^{a,b}_r = \Big(\sum_{a \le i_1, i_1+1 < i_2, i_2+1, \ldots, < i_r \le b} \prod_{j=1}^{r} f_{i_j}\Big) \prod_{i=a}^{b+1} c_i, \tag{9}$$

with $c_i = v_{i-1} - v_{i+1}$ and $f_i = 1/(c_i c_{i+1})$. We prove that the integrals (5),(6) and (8) are in involution with respect to accompanying symplectic structures.

This paper is organized as follows. In section 2, we prove the involutivity of integrals of the sine-Gordon map. Firstly, we consider the odd-dimensional maps. We introduce a transformation to reduce the dimension of the map by one and we present a symplectic structure of the reduced map. Then we give properties of Theta with respect to the Poisson bracket associated to this symplectic structure. To prove the involutivity of the integrals, we write the Poisson bracket $\{I_r, I_s\}$ as a polynomial in $\alpha_1, \alpha_2, \alpha_3$ and prove that all the coefficients of this polynomial vanish. Secondly, we consider the even-dimensional maps and we give a symplectic structure for them. Next, we show the relationship between the two symplectic structures. Therefore, many properties of Theta with respect to the new Poisson bracket can be obtained directly from the ones with respect to the old Poisson bracket. The proof of involutivity is similar to the first case.

In section 3, we present relationships between symplectic structures of the sine-Gordon and mKdV maps. We use these relationships to derive analogous properties of Theta with respect to the Poisson bracket of the mKdV maps. Involutivity of the integrals follows from these properties.

In section 4, we prove that the integrals of the pKdV map are in involution (with respect to symplectic structures). We also distinguish even and odd dimensional maps and present a relationship of symplectic structures between the two cases. For the even-dimensional map, the properties of multi-sums of products, Ψ, with respect to the symplectic structure are proved by induction. For the other case, the properties of Ψ with respect to its symplectic structure are derived from the previous case. The involutiviy of integrals (8) is proved by using these properties.

2 Involutivity of sine-Gordon

In this section, we distinguish two cases: the odd-dimensional and even-dimensional sine-Gordon maps. It is shown that for the even-dimensional map, we have enough integrals for integrability [5]. For the other case, we need a reduction to reduce the dimension of the map by one. The involutivity of integrals (5) is proved by writing a Poisson bracket of 2 integrals $\{I_r, I_s\}$ in terms of polynomial of parameters $\alpha_1, \alpha_2, \alpha_3$.

2.1 The case d=2n+1

Using a reduction $f_i = v_i v_{i+1}$, we obtain a $2n$-dimensional map sG:

$$(f_1, f_2, \dots, f_{2n}) \mapsto (f_2, f_3, \dots, f_{2n}, f_{2n+1}), \tag{10}$$

where

$$f_{2n+1} = \frac{f_2 f_4 \dots f_{2n}(\alpha_1 f_2 f_4 \dots f_{2n} + \alpha_3 f_3 f_5 \dots f_{2n-1})}{f_1 f_3 \dots f_{2n-1}(\alpha_2 f_2 f_4 \dots f_{2n} + \alpha_3 f_3 f_5 \dots f_{2n-1})} \tag{11}$$

This map has n integrals given by

$$I_r = \alpha_1\big(\frac{f_2 f_4 \dots f_{2n}}{f_1 f_3 \dots f_{2n-1}}\Theta^{1,2n}_{2r,1} + \frac{f_1 f_3 \dots f_{2n-1}}{f_2 f_4 \dots f_{2n}}\Theta^{1,2n}_{2r,0}\big) + \alpha_2 \Theta^{1,2n}_{2r+1,1} + \alpha_3 \Theta^{1,2n}_{2r+1,0}, \tag{12}$$

where the argument of Θ is f_i and $0 \le r \le n-1$.

A symplectic structure for the map (10) is given by, cf. [4, 9]

$$\Omega_{\mathrm{sG},2n} = \begin{pmatrix} 0 & f_1 f_2 & f_1 f_3 & f_1 f_4 & \cdots & f_1 f_{2n-1} & f_1 f_{2n} \\ -f_2 f_1 & 0 & f_2 f_3 & f_2 f_4 & \cdots & f_2 f_{2n-2} & f_2 f_{2n} \\ -f_3 f_1 & -f_3 f_2 & 0 & f_3 f_4 & \cdots & f_3 f_{2n-1} & f_3 f_{2n} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ -f_{2n} f_1 & -f_{2n} f_2 & -f_{2n} f_3 & -f_{2n} f_4 & \cdots & -f_{2n} f_{2n-1} & 0 \end{pmatrix}. \tag{13}$$

Let g and h be functions differentiable with respect to the f_i's. We define a Poisson bracket

$$\{g, h\}_f := \sum_{i<j} \big(\frac{\partial g}{\partial f_i}\frac{\partial h}{\partial f_j} f_i f_j - \frac{\partial g}{\partial f_j}\frac{\partial h}{\partial f_i} f_j f_i\big). \tag{14}$$

We will prove that integrals (12) are involution with respect to this symplectic structure, i.e $\{I_r, I_s\}_f = 0$, for all $0 \le r, s \le n-1$.

2.1.1 Properties of Theta

In this section, we give explicit expressions of the Poisson bracket between two Θs. These will be proven in Appendix A. First, we introduce a differential operator $E_f := \sum_{i\geq 1} f_i \frac{\partial}{\partial f_i}$. We can see that when E acts on each term in a multi-sum $\Theta^{1,p}_{r,\epsilon}$, we obtain a scale factor which equals the total degree of the term. It is easy to see that every term in the multi-sum has total degree 0 if r is even and degree $(-1)^{\epsilon+1}$ if r is odd. Therefore, we obtain the property of Θ as follows

$$E_f \Theta^{1,p}_{r,\epsilon} = \begin{cases} 0 & \text{if } r \text{ even} \\ (-1)^{\epsilon+1}\Theta^{1,p}_{r,\epsilon} & \text{if } r \text{ odd} \end{cases}. \tag{15}$$

Now we give properties of Theta with respect to the Poisson bracket (14).

Lemma 1. *Let* $0 \leq r, s \leq p$ *and* $\epsilon \in \{0, 1\}$. *We have*

$$\{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s,\epsilon}\}_f = \begin{cases} 0 & r, s \textit{ are both odd or even} \\ \sum_{i=0}^{s} (-1)^i \Theta^{1,p}_{r+i,\epsilon}\Theta^{1,p}_{s-i,\epsilon} & r \textit{ even, } s \textit{ odd and } r > s \\ \sum_{i=1}^{r} (-1)^{i-1} \Theta^{1,p}_{r-i,\epsilon}\Theta^{1,p}_{s+i,\epsilon} & r \textit{ even, } s \textit{ odd and } r < s \end{cases} \tag{16}$$

Lemma 2. *Let* $0 \leq r, s \leq p$.

1. *If* $r \equiv s \pmod 2$, *we have*

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_f = \begin{cases} \sum_{0\leq 2\lfloor i/2\rfloor \leq r-1} (-1)^i \Theta^{1,p}_{r-1-2\lfloor i/2\rfloor, i}\Theta^{1,p}_{s+1+2\lfloor i/2\rfloor, i+1}, & r \leq s \\ \sum_{0\leq 2\lfloor i/2\rfloor \leq s-1} (-1)^i \Theta^{1,p}_{s-1-2\lfloor i/2\rfloor, i}\Theta^{1,p}_{r+1+2\lfloor i/2\rfloor, i+1}, & r > s \end{cases}. \tag{17}$$

2. *If* $r \not\equiv s \pmod 2$, *we have*

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_f = \begin{cases} \sum_{i=0}^{r} (-1)^i \Theta^{1,p}_{s+i,i+1}\Theta^{0,p}_{r-i,i}, & r \equiv 1 \mod 2, s \equiv 0 \mod 2, \\ \sum_{i=0}^{s} (-1)^{i-1} \Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i,i}, & r \equiv 0 \mod 2, s \equiv 1 \mod 2 \end{cases}. \tag{18}$$

Using Lemma 1 and Lemma 2, we have the following corollary.

Corollary 3. *Let* r *and* s *be both even or both odd and let* $\epsilon \in \{0, 1\}$. *We have*

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_f + \{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,0}\}_f = 0, \tag{19}$$

$$\{\Theta^{1,p}_{r-1,\epsilon}, \Theta^{1,p}_{s,\epsilon}\}_f + \{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon}\}_f = \begin{cases} 0, & r, s \textit{ even,} \\ \Theta^{1,p}_{r-1,\epsilon}\Theta^{1,p}_{s,\epsilon} - \Theta^{1,p}_{s-1,\epsilon}\Theta^{1,p}_{r,\epsilon}, & r, s \textit{ odd} \end{cases}, \tag{20}$$

$$\{\Theta^{1,p}_{r-1,\epsilon\pm 1}, \Theta^{1,p}_{s,\epsilon}\}_f + \{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon\pm 1}\}_f = \begin{cases} 0, & r, s \textit{ even} \\ \Theta^{1,p}_{s-1,\epsilon\pm 1}\Theta^{1,p}_{r,\epsilon} - \Theta^{1,p}_{s,\epsilon}\Theta^{1,p}_{r-1,\epsilon\pm 1}, & r, s \textit{ odd} \end{cases}. \tag{21}$$

2.1.2 Involutivity of integrals

This section will prove the involutivity of integrals (12) with respect to the symplectic structure (13).

Theorem 4. *Let $0 \le r, s \le n-1$. Let I_r, I_s be given by formula (12). Then*

$$\{I_r, I_s\}_f = 0.$$

Proof. First of all, we denote

$$W = \frac{f_1 f_3 \dots f_{2n-1}}{f_2 f_4 \dots f_{2n}}.$$

It is easy to see that, for any $g(f_1, f_2, \dots, f_{2n})$

$$\{W^{\pm 1}, g\}_f = \pm W^{\pm 1} E g. \tag{22}$$

Now we expand $\{I_r, I_s\}_f$ in terms of polynomials in $\alpha_1, \alpha_2, \alpha_3$ as follows

$$\{I_r, I_s\}_f = \alpha_1^2 A_1 + \alpha_2^2 A_2 + \alpha_3^2 A_3 + \alpha_1\alpha_2 A_{12} + \alpha_1\alpha_3 A_{13} + \alpha_2\alpha_3 A_{23},$$

where

$$\begin{aligned}
A_1 &= \{W^{-1}\Theta_{2r,1}^{1,2n} + W\Theta_{2r,0}^{1,2n}, W^{-1}\Theta_{2s,1}^{1,2n} + W\Theta_{2s,0}^{1,2n}\}_f \\
A_2 &= \{\Theta_{2r+1,1}^{1,2n}, \Theta_{2s+1,1}^{1,2n}\}_f \\
A_3 &= \{\Theta_{2r+1,0}^{1,2n}, \Theta_{2s+1,0}^{1,2n}\}_f \\
A_{12} &= \{W^{-1}\Theta_{2r,1}^{1,2n} + W\Theta_{2r,0}^{1,2n}, \Theta_{2s+1,1}^{1,2n}\}_f + \{\Theta_{2r+1,1}^{1,2n}, W^{-1}\Theta_{2s,1}^{1,2n} + W\Theta_{2s,0}^{1,2n}\}_f \\
A_{13} &= \{W^{-1}\Theta_{2r,1}^{1,2n} + W\Theta_{2r,0}^{1,2n}, \Theta_{2s+1,0}^{1,2n}\}_f + \{\Theta_{2r+1,0}^{1,2n}, W^{-1}\Theta_{2s,1}^{1,2n} + W\Theta_{2s,0}^{1,2n}\}_f \\
A_{23} &= \{\Theta_{2r+1,1}^{1,2n}, \Theta_{2s+1,0}^{1,2n}\}_f + \{\Theta_{2r+1,0}^{1,2n}, \Theta_{2s+1,1}^{1,2n}\}_f.
\end{aligned}$$

We prove that all these coefficients equal 0. Using Lemma 1 and Corollary 3, we have $A_2 = A_3 = A_{23} = 0$. We now expand A_1, A_{12} and B_{13}, We have

$$\begin{aligned}
A_1 &= W^{-1}\Big(\Theta_{2s,1}^{1,2n}\{\Theta_{2r,1}^{1,2n}, W^{-1}\}_f + \Theta_{2r,1}^{1,2n}\{W^{-1}, \Theta_{2s,1}^{1,2n}\}_f + \Theta_{2s,0}^{1,2n}\{\Theta_{2r,1}^{1,2n}, W\}_f + \Theta_{2r,0}^{1,2n}\{W, \Theta_{2s,1}^{1,2n}\}_f\Big) \\
&\quad + W\Big(\Theta_{2r,1}^{1,2n}\{W^{-1}, \Theta_{2s,0}^{1,2n}\}_f + \Theta_{2s,1}^{1,2n}\{\Theta_{2r,0}^{1,2n}, W^{-1}\}_f + \Theta_{2r,0}^{1,2n}\{W, \Theta_{2s,0}^{1,2n}\}_f + \Theta_{2s,0}^{1,2n}\{\Theta_{2r,0}^{1,2n}, W\}_f\Big) \\
&= W^{-2}\big(\Theta_{2s,1}^{1,2n} E_f \Theta_{2r,1}^{1,2n} - \Theta_{2r,1}^{1,2n} E_f \Theta_{2s,1}^{1,2n}\big) - \Theta_{2s,0}^{1,2n} E_f \Theta_{2r,1}^{0,2k-1} + \Theta_{2r,0}^{1,2n} E_f \Theta_{2s,1}^{1,2n} - \Theta_{2r,1}^{1,2n} E \Theta_{2s,0}^{1,2n} + \Theta_{2r,1}^{1,2n} E \Theta_{2s,0}^{1,2n} \\
&\quad + W^2 \Theta_{2r,0}^{1,2n}\big(E_f \Theta_{2s,0}^{1,2n} - \Theta_{2s,0}^{1,2n} E_f \Theta_{2r,0}^{1,2n}\big) \\
&= 0,
\end{aligned}$$

where we have used (15). We also have

$$\begin{aligned}
A_{12} = {}& W^{-1}\big(\{\Theta_{2r,1}^{1,2n}, \Theta_{2s+1,1}^{1,2n}\}_f + \{\Theta_{2r+1,1}^{1,2n}, \Theta_{2s,1}^{1,2n}\}\big)_f + W\big(\{\Theta_{2r,0}^{1,2n}, \Theta_{2s+1,1}^{1,2n}\}_f + \{\Theta_{2r+1,1}^{1,2n}, \Theta_{2s,0}^{1,2n}\}_f\big) \\
& + \Theta_{2r,1}^{1,2n}\{W^{-1}, \Theta_{2s+1,1}^{1,2n}\}_f + \Theta_{2r,0}^{1,2n}\{W, \Theta_{2s+1,1}^{1,2n}\}_f + \Theta_{2s,1}^{1,2n}\{\Theta_{2r+1,1}^{1,2n}, W^{-1}\}_f + \Theta_{2s,0}^{1,2n}\{\Theta_{2r+1,1}^{1,2n}, W\}_f \\
= {}& W^{-1}\big(\Theta_{2r,1}^{1,2n}\Theta_{2s+1,1}^{1,2n} - \Theta_{2r+1,1}^{1,2n}\Theta_{2s,1}^{1,2n}\big) + W\big(\Theta_{2s,0}^{1,2n}\Theta_{2r+1,1}^{1,2n} - \Theta_{2s+1,1}^{1,2n}\Theta_{2r,0}^{1,2n}\big) \\
& - W^{-1}\Theta_{2r,1}^{1,2n} E_f \Theta_{2s+1,1}^{1,2n} + W\Theta_{2r,0}^{1,2n} E_f \Theta_{2s+1,1}^{1,2n} + W^{-1}\Theta_{2s,1}^{1,2n}\Theta_{2r+1,1}^{1,2n} - W\Theta_{2s,0}^{1,2n}\Theta_{2r+1,1}^{1,2n} \\
= {}& 0, .
\end{aligned}$$

here we have used (15) again. Similarly we have $A_{13} = 0$. Therefore, we have $\{I_r, I_s\} = 0$. □

2.2 The case where $d = 2n$

In this case, we consider a $2n$-dimensional map

$$\mathrm{sG}^{'} : (v_1, v_2, \cdots, v_{2n}) \mapsto (v_2, v_3, \ldots, v_{2n+1}), \tag{23}$$

where

$$v_{2n+1} = G(v_1, v_2, \ldots, v_{2n}) = v_1^{-1} \frac{\alpha_1 v_2 v_{2n} + \alpha_3}{\alpha_2 v_2 v_{2n} + \alpha_1}. \tag{24}$$

This map has n integrals given by

$$I_r = \alpha_1 \Big(\frac{v_{2n}}{v_0} \Theta^{1,2n-1}_{2r,1} + \frac{v_1}{v_{2n}} \Theta^{1,2n-1}_{2r,0} \Big) + \alpha_2 \Theta^{1,2n-1}_{2r+1,1} + \alpha_3 \Theta^{1,2n-1}_{2r+1,0} \tag{25}$$

where $0 \leq r \leq n-1$ and the argument of Theta is $f_i = v_i v_{i+1}$.

The sine-Gordon map (23) has a symplectic structure, cf [4, 9],

$$\Omega^{'}_{sG,2k} = \begin{pmatrix} 0 & v_1 v_2 & 0 & v_1 v_4 & \ldots & 0 & v_1 v_{2n} \\ -v_2 v_1 & 0 & v_2 v_3 & 0 & \ldots & v_2 v_{2n-1} & 0 \\ 0 & -v_3 v_3 & 0 & v_3 v_4 & \ldots & 0 & v_3 v_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -v_{2n} v_1 & 0 & -v_{2n} v_3 & 0 & \ldots & -v_{2n} v_{2n-1} & 0 \end{pmatrix}. \tag{26}$$

Therefore, we define a bracket as below.

Definition 5. *Let g and h be functions differentiable with respect to v_i's . We define*

$$\{g, h\}_v = \sum_{i<j,\ i \not\equiv j \pmod 2} \Big(\frac{\partial g}{\partial v_i} \frac{\partial h}{\partial v_j} - \frac{\partial g}{\partial v_j} \frac{\partial h}{\partial v_i} \Big) v_i v_j. \tag{27}$$

We will prove that the integrals (25) are involution with respect to this bracket, i.e we prove $\{I_r, I_s\}_v = 0, \forall\ 0 \leq r, s \leq n-1$. We give properties of Theta with respect to this symplectic structure and this bracket in the next section.

2.2.1 Properties of Theta

We present explicit expressions of the Poisson bracket (27) between two Θs. Similar to (15), we have the following property. Let $\epsilon, r, p \in \mathbb{N}$. We have

$$E_v \Theta^{1,p}_{r,\epsilon} = \begin{cases} 0, & r \text{ even}, \\ 2(-1)^{\epsilon+1} \Theta^{1,p}_{r,\epsilon} & r \text{ odd} \end{cases}. \tag{28}$$

We consider the map G_p, where

$$(v_1, v_2, \ldots, v_p) \mapsto (f_1, f_2, \ldots, f_{p-1})$$

with $f_i = v_i v_{i+1}$ and the Jacobian of this map is denoted by $dG_p = \dfrac{\partial(f_1, f_2, \ldots, f_{p-1})}{\partial(v_1, v_2, \ldots, v_p)}$. By direct calculation, we have

$$2\Omega_{\mathrm{sG},p-1}|_{f_i = v_i v_{i+1}} = dG_p \Omega^{'}_{\mathrm{sG},p} (dG_p)^T. \tag{29}$$

Now we give the relationship between the two brackets. Let g, h be differential functions on $\mathbb{R}^{p-1}$. We consider the functions $g \circ G_p$ and $h \circ G_p$. We have $g \circ G_p(v_1, v_2, \ldots, v_p) = g(f_1, f_2, \ldots, f_{p-1})|_{f_i = v_i v_{i+1}}$. Thus, we get

$$(\frac{\partial(g \circ G_p)}{\partial v_1}, \frac{\partial(g \circ G_p)}{\partial v_2}, \ldots, \frac{\partial(g \circ G_p)}{\partial v_p}) = (\frac{\partial g}{\partial f_1}, \frac{\partial g}{\partial f_2}, \ldots, \frac{\partial g}{\partial f_{p-1}}) dG_p|_{f_i = v_i v_{i+1}}.$$

Similarly, we have

$$(\frac{\partial(h \circ G_p)}{\partial v_1}, \frac{\partial(h \circ G_p)}{\partial v_2}, \ldots, \frac{\partial(h \circ G_p)}{\partial v_p}) = (\frac{\partial h}{\partial f_1}, \frac{\partial h}{\partial f_2}, \ldots, \frac{\partial h}{\partial f_{p-1}}) dG_p|_{f_i = v_i v_{i+1}}.$$

Thus, we have

$$\begin{aligned}
\{g \circ G_p, h \circ G_p\}_v &= (\frac{\partial(g \circ G_p)}{\partial v_1}, \frac{\partial(g \circ G_P)}{\partial v_2}, \ldots, \frac{\partial(g \circ G_p)}{\partial v_p}) \Omega'_{\mathrm{sG},p} (\frac{\partial(h \circ G_p)}{\partial v_1}, \frac{\partial(h \circ G_p)}{\partial v_2}, \ldots, \frac{\partial(h \circ G_p)}{\partial v_n})^T \\
&= (\frac{\partial g}{\partial f_1}, \frac{\partial g}{\partial f_2}, \ldots, \frac{\partial g}{\partial f_{p-1}}) dG_p \Omega'_{\mathrm{sG},p} (dG_p)^T (\frac{\partial h}{\partial f_1}, \frac{\partial h}{\partial f_2}, \ldots, \frac{\partial h}{\partial f_{p-1}})^T |_{f_i = v_i v_{i+1}} \\
&= 2(\frac{\partial g}{\partial f_1}, \frac{\partial g}{\partial f_2}, \ldots, \frac{\partial g}{\partial f_{p-1}}) \Omega_{\mathrm{sG},p-1} (\frac{\partial h}{\partial f_1}, \frac{\partial h}{\partial f_1}, \ldots, \frac{\partial h}{\partial f_{p-1}})^T |_{f_i = v_i v_{i+1}} \\
&= 2\{g, h\}_f |_{f_i = v_i v_{i+1}}.
\end{aligned}$$

We have obtained the following lemmas.

Lemma 6. *Let $0 \le r, s \le p$. We have*

$$\{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,1}\}_v = \begin{cases} 0, & r, s \text{ both odd or both even}, \\ 2\sum_{i=0}^{s} (-1)^i \Theta^{1,p}_{r+i,1} \Theta^{1,p}_{s-i,1}, & r \text{ even}, \ s \text{ odd and } r > s, \\ 2\sum_{i=1}^{r} (-1)^{i-1} \Theta^{1,p}_{r-i,1} \Theta^{1,p}_{s+i,1}, & r \text{ even}, \ s \text{ odd and } r < s \end{cases}. \tag{30}$$

Lemma 7. *Let $0 \le r, s \le p$.*

- *If $r \equiv s \pmod 2$, we have*

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_v = \begin{cases} 2 \sum_{0 \le 2\lfloor i/2 \rfloor \le r-1} (-1)^i \Theta^{1,p}_{r-1-2\lfloor i/2 \rfloor, i} \Theta^{1,p}_{s+1+2\lfloor i/2 \rfloor, i+1}, & r \le s \\ 2 \sum_{0 \le 2\lfloor i/2 \rfloor \le s-1} (-1)^i \Theta^{1,p}_{s-1-2\lfloor i/2 \rfloor, i} \Theta^{1,p}_{r+1+2\lfloor i/2 \rfloor, i+1}, & r > s \end{cases}. \tag{31}$$

- *If $r \not\equiv s \pmod 2$, we have*

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_1 = \begin{cases} 2\sum_{i=0}^{r} (-1)^i \Theta^{1,p}_{s+i,i+1} \Theta^{1,p}_{r-i,i}, & r \equiv 1 \mod 2, s \equiv 0 \mod 2, \\ 2\sum_{i=0}^{s} (-1)^{i-1} \Theta^{1,p}_{s-i,i+1} \Theta^{1,p}_{r+i,i}, & r \equiv 0 \mod 2, s \equiv 1 \mod 2 \end{cases}. \tag{32}$$

Lemma 8. *Let r and s both be even or odd and let $\epsilon \in \{0,1\}$. We have*

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_1 + \{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,0}\}_1 = 0, \tag{33}$$

$$\{\Theta^{1,p}_{r-1,\epsilon}, \Theta^{1,p}_{s,\epsilon}\}_1 + \{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon}\}_1 = \begin{cases} 0, & r,s \ \textit{even}, \\ 2\Theta^{1,p}_{r-1,\epsilon}\Theta^{1,p}_{s,\epsilon} - 2\Theta^{1,p}_{s-1,\epsilon}\Theta^{1,p}_{r,\epsilon}, & r,s \ \textit{odd} \end{cases}, \tag{34}$$

$$\{\Theta^{1,p}_{r-1,\epsilon\pm1}, \Theta^{1,p}_{s,\epsilon}\}_1 + \{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon\pm1}\}_1 = \begin{cases} 0, & r,s \ \textit{even} \\ 2\Theta^{1,p}_{s-1,\epsilon\pm1}\Theta^{1,p}_{r,\epsilon} - 2\Theta^{1,p}_{s,\epsilon}\Theta^{1,p}_{r-1,\epsilon\pm1}, & r,s \ \textit{odd} \end{cases}. \tag{35}$$

2.2.2 Involutivity of integrals

Now we will prove the involutivity of integrals of the sine-Gordon map (23).

Theorem 9. *Let I_r and I_s, with $0 \leq r,s \leq n-1$, be given by the formula (5). Then we have*

$$\{I_r, I_s\}_v = 0.$$

Proof. With $V = \frac{v_1}{v_{2n}}$, for any $g(v_1, v_2, \ldots, v_{2n})$, we have

$$\{V^{\pm1}, g\}_v = \pm V^{\pm1} E_v g. \tag{36}$$

We write the Poisson bracket between 2 integrals as follows

$$\{I_r, I_s\}_v = \alpha_1^2 B_1 + \alpha_2^2 B_2 + \alpha_3^2 B_3 + \alpha_1\alpha_2 B_{12} + \alpha_1\alpha_3 B_{13} + \alpha_2\alpha_3 B_{23},$$

where

$$\begin{aligned}
B_1 &= \{\frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2r,1} + \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2r,0}, \frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2s,1} + \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2s,0}\}_v, \\
B_2 &= \{\Theta^{1,2n-1}_{2r+1,1}, \Theta^{1,2n-1}_{2s+1,1}\}_v, \\
B_3 &= \{\Theta^{1,2n-1}_{2r+1,0}, \Theta^{1,2n-1}_{2s+1,0}\}_v, \\
B_{12} &= \{\frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2r,1} + \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2r,0}, \Theta^{1,2n-1}_{2s+1,1}\}_v + \{\Theta^{1,2n-1}_{2r+1,1}, \frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2s,1} + \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2s,0}\}_v, \\
B_{13} &= \{\frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2r,1} + \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2r,0}, \Theta^{1,2n-1}_{2s+1,0}\}_v + \{\Theta^{1,2n-1}_{2r+1,0}, \frac{v_{2n}}{v_1}\Theta^{1,2n}_{2s,1} + \frac{v_1}{v_n}\Theta^{1,2n-1}_{2s,0}\}_v, \\
B_{23} &= \{\Theta^{1,2n-1}_{2r+1,1}, \Theta^{1,2n-1}_{2s+1,0}\}_v + \{\Theta^{1,2n-1}_{2r+1,0}, \Theta^{1,2n-1}_{2s+1,1}\}_v.
\end{aligned}$$

By Lemmas 6 and 8 , we have $B_2 = 0, B_3 = 0$ and $B_{23} = 0$. Now we simplify B_1 by using (28) and (36) and Lemma 8. We have

$$\begin{aligned}
B_1 = {} & \frac{v_{2n}}{v_1}\Big(\Theta^{1,2n-1}_{2r,1}\{\frac{v_{2n}}{v_1}, \Theta^{1,2n-1}_{2s,1}\}_v + \Theta^{1,2n-1}_{2s,1}\{\Theta^{1,2n-1}_{2r,1}, \frac{v_{2n}}{v_1}\}_v + \Theta^{1,2n-1}_{2r,0}\{\frac{v_1}{v_{2n}}, \Theta^{1,2n-1}_{2s,1}\}_v \\
& + \Theta^{1,2n-1}_{2s,0}\{\Theta^{1,2n-1}_{2r,1}, \frac{v_1}{v_{2n}}\}_v\Big) + \frac{v_1}{v_{2n}}\Big(\Theta^{1,2n-1}_{2s,1}\{\Theta^{1,2n-1}_{2r,0}, \frac{v_{2n}}{v_1}\}_v + \Theta^{1,2n-1}_{2r,1}\{\frac{v_{2n}}{v_1}, \Theta^{1,2n-1}_{2s,0}\}_v \\
& + \Theta^{1,2n-1}_{2r,0}\{\frac{v_1}{v_{2n}}, \Theta^{1,2n-1}_{2s,0}\}_v + \Theta^{1,2n-1}_{2s,0}\{\Theta^{1,2n-1}_{2r,0}, \frac{v_1}{v_{2n}}\}_v\Big) \\
= {} & \frac{v_{2n}}{v_1}\Big(- \frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2r,1}E_v\Theta^{1,2n-1}_{2s,1} + \frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2s,1}E_v\Theta^{1,2n-1}_{2r,1} + \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2r,0}\Theta^{1,2n-1}_{2s,1} - \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2s,0}E_v\Theta^{1,2n-1}_{2r,1}\Big) \\
& + \frac{v_1}{v_{2n}}\Big(\frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2s,1}E_v\Theta^{1,2n-1}_{2r,0} - \frac{v_{2n}}{v_1}\Theta^{1,2n-1}_{2r,1}E_v\Theta^{1,2n-1}_{2s,0} \\
& + \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2r,0}E_v\Theta^{1,2n-1}_{2s,0} - \frac{v_1}{v_{2n}}\Theta^{1,2n-1}_{2s,0}E_v\Theta^{1,2n-1}_{2r,0}\Big) \\
= {} & 0.
\end{aligned}$$

We also have

$$\begin{aligned}
B_{12} &= \frac{v_{2n}}{v_1}\Big(\{\Theta^{1,2n-1}_{2r,1},\Theta^{1,2n-1}_{2s+1,1}\}_v + \{\Theta^{1,2n-1}_{2r+1},\Theta^{1,2n-2}_{2s,1}\}_v\Big) + \frac{v_1}{v_{2n}}\Big(\{\Theta^{1,2n-1}_{2r,0},\Theta^{1,2n-1}_{2s+1,1}\}_v + \{\Theta^{1,2n-1}_{2r+1,1},\Theta^{1,2n-1}_{2s,0}\}_v\Big)\\
&\quad + \Theta^{1,2n-1}_{2r,1}\{\frac{v_{2n}}{v_1},\Theta^{1,2n-1}_{2s+1,1}\}_v + \Theta^{1,2n-1}_{2r,0}\{\frac{v_1}{v_{2n}},\Theta^{1,2n-1}_{2s+1,1}\}_v + \Theta^{1,2n-1}_{2s,1}\{\Theta^{1,2n-1}_{2r+1,1},\frac{v_{2n}}{v_1}\}_v\\
&\quad + \Theta^{1,2n-1}_{2s,0}\{\Theta^{1,2n-1}_{2r+1,1},\frac{v_1}{v_{2n}}\}_v\\
&= \frac{2v_{2n}}{v_1}\big(\Theta^{1,2n-1}_{2r,1}\Theta^{1,2n-1}_{2s+1,1} - \Theta^{1,2n-1}_{2s,1}\Theta^{1,2n-1}_{2r+1,1}\big) + \frac{2v_1}{v_{2n}}\big(\Theta^{1,2n-1}_{2s,0}\Theta^{1,2n-1}_{2r+1,1} - \Theta^{1,2n-1}_{2s+1,1}\Theta^{1,2n-1}_{2r,0}\big)\\
&\quad - \frac{2v_{2n}}{v_1}\Theta^{1,2n-1}_{2r,1}\Theta^{1,2n-1}_{2s,1} + \frac{2v_1}{v_{2n}}\Theta^{1,2n-1}_{2r,0}\Theta^{1,2n-1}_{2s+1,1} + \frac{2v_{2n}}{v_1}\Theta^{1,2n-1}_{2s,1}\Theta^{1,2n-1}_{2r+1,1} - \frac{2v_1}{v_{2n}}\Theta^{1,2n-1}_{2s,0}\Theta^{1,2n-1}_{2r+1,1}\\
&= 0.
\end{aligned}$$

Similarly, we obtain $B_{13} = 0$. Therefore, $\{I_r, I_s\}_1 = 0$. $\square$

3 Involutivity of the mKdV map

We consider the mKdV map (4), where

$$v_{d+1} = v_1\frac{\beta_1 v_d + \beta_2 v_2}{\beta_1 v_2 + \beta_3 v_d}. \tag{37}$$

As shown in [5], this map has $\lfloor (d-1)/2 \rfloor$ integrals given by the formula (6) with $0 < 2r < d$. If $d = 2n+1$, we have a $2n+1$ dimensional map (37). This map reduces to a $2n$-dimensional map with exactly n integrals via reduction $z_i = v_{i+1}/v_i$. For the other case, where $d = 2n+2$, the $2n+2$ map (37) has only n integrals which are not enough for integrability. However, this map can reduce to a $2n$-dimensional map with exactly n integrals by using a reduction $z_i = v_{i+2}/v_i$. We show that the integrals of these reduced maps. In each case, we present a relationship between a corresponding symplectic structures and the symplectic structure of the sine-Gordon map in the even case (13). This relation can be used to derive properties of Theta with new symplectic structures. are involution with respect to symplectic structures.

3.1 $d = 2n+1$

Using the reduction $z_i = v_{i+1}/v_i$, we obtain the map

$$(z_1, z_2, \ldots, z_{2n}) \mapsto (z_2, z_3, \ldots, z_{2n+1}), \tag{38}$$

where

$$z_{2n+1} = \frac{1}{z_1 z_2 \ldots z_{2n}}\cdot\frac{\alpha_1 z_2 z_3 \ldots z_{2n} + \alpha_2}{\alpha_1 + \alpha_3 z_2 z_3 \ldots z_{2n}}.$$

The integrals of this map are given by

$$I_r = \alpha_1\big(z_1 z_2 \ldots z_{2n}\Theta^{1,2n}_{2r-1,0} + \frac{1}{z_1 z_2 \ldots z_{2n}}\Theta^{1,2n}_{2r-1,1}\big) + \alpha_2\Theta^{1,2n}_{2r,1} + \alpha_3\Theta^{1,2n}_{2r,0}, \tag{39}$$

where arguments for Theta are

$$f_1 = z_1,\ f_i = z_1^2 z_2^2 \ldots z_{i-1}^2 z_i \text{ for } i > 1. \tag{40}$$

We obtain a symplectic structure $\Omega_{\mathrm{mKdV},p}$ for the map (38) with $p = 2n$, cf[4, 9], where

$$\Omega_{\mathrm{mKdV},p} := \begin{pmatrix} 0 & z_1 z_2 & -z_1 z_3 & z_0 z_3 & \dots & (-1)^p z_1 z_p \\ -z_2 z_1 & 0 & z_2 z_3 & z_2 z_4 & \dots & (-1)^{p-1} z_2 z_p \\ z_3 z_1 & -z_3 z_2 & 0 & z_3 z_4 & \dots & (-1)^{p-2} z_3 z_p \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ (-1)^{p-1} z_p z_0 & (-1)^{p-2} z_{p-1} z_1 & (-1)^{p-3} z_{p-1} z_2 & (-1)^{p-4} z_{p-1} z_3 & \dots & 0 \end{pmatrix} \tag{41}$$

Therefore, a Poisson bracket in this case is defined as

$$\{g, h\}_z = \sum_{i<j} (-1)^{i+j} \big(\frac{\partial g}{\partial z_j} \frac{\partial h}{\partial z_i} z_i z_j - \frac{\partial g}{\partial z_i} \frac{\partial h}{\partial z_j} z_i z_j \big). \tag{42}$$

Now we consider the map M_p: $z \to f$ given by (40)

$$(z_1, z_2, \dots, z_p) \mapsto (f_1, f_2, \dots, f_p).$$

Let dM_p be a Jacobian matrix of this map. We have the following relation

$$dM_p.\Omega_{\mathrm{mKdV},p}.dM_p^T = \Omega_{\mathrm{sG},p} \mid_{f_1 = z_1, f_i = z_1^2 z_2^2 \dots z_{i-1}^2 z_i} . \tag{43}$$

This implies

$$\{g, h\}_z = \{g, h\}_f \mid_{f_1 = z_1, f_i = z_1^2 z_2^2 \dots z_{i-1}^2 z_i} .$$

Therefore, properties of Theta with respect to the Poisson bracket (42) are derived from those in Lemma 1 and Corollary 3 with respect to Poisson bracket (14). Hence, we have the following corollary.

Corollary 10.

$$\{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s,\epsilon}\}_z = 0 \tag{44}$$

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_z + \{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,0}\}_z = 0 \tag{45}$$

$$\{\Theta^{1,p}_{r-1,\epsilon}, \Theta^{1,p}_{s,\epsilon}\}_z + \{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon}\}_z = \begin{cases} 0, & r, s \textit{ even} \\ \Theta^{1,p}_{r-1,\epsilon} \Theta^{1,p}_{s,\epsilon} - \Theta^{1,p}_{s-1,\epsilon} \Theta^{1,p}_{r,\epsilon}, & r, s \textit{ odd} \end{cases} \tag{46}$$

$$\{\Theta^{1,p}_{r-1,\epsilon\pm1}, \Theta^{1,p}_{s,\epsilon}\}_z + \{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon\pm1}\}_z = \begin{cases} 0, & r, s \textit{ even} \\ \Theta^{1,p}_{s-1,\epsilon\pm1} \Theta^{1,p}_{r,\epsilon} - \Theta^{1,p}_{s,\epsilon} \Theta^{1,p}_{r-1,\epsilon\pm1}, & r, s \textit{ odd} \end{cases} . \tag{47}$$

We also have

$$\frac{\partial \Theta^{1,p}_{r,\epsilon}}{\partial z_i} z_i = \frac{\partial \Theta^{1,p}_{r,\epsilon}}{\partial f_i} f_i + 2 \sum_{j>i} \frac{\partial \Theta^{1,p}_{r,\epsilon}}{\partial f_j} f_j .$$

This implies that

$$\sum_{i \text{ even}} \frac{\partial \Theta^{1,p}_{r,\epsilon}}{\partial z_i} z_i - \sum_{i \text{ odd}} \frac{\partial \Theta^{1,p}_{r,\epsilon}}{\partial z_i} z_i = \sum_{i=1}^{p} \frac{\partial \Theta^{1,p}_{r,\epsilon}}{\partial f_i} f_i \mid_{f_1 = z_1, f_i = z_1^2 z_1^2 \dots z_{i-1}^2 z_i} = E_f \Theta^{1,p}_{r,s} \mid_{f_1 = z_1, f_i = z_1^2 z_2^2 \dots z_{i-1}^2 z_i} .$$

Thus, by (15), we obtain the following properties.

Proposition 11.

$$\sum_i (-1)^i \frac{\partial \Theta^{1,p}_{r,\epsilon}}{\partial z_i} z_i = \begin{cases} 0 & r \textit{ even} \\ (-1)^{\epsilon+1} \Theta^{1,p}_{r,\epsilon}, & r \textit{ odd} \end{cases} . \tag{48}$$

The involutivity of the integrals (39) are obtained by using Corollary 10 and Proposition 11.

3.2 $d = 2n + 2$

Now using a reduction $w_i = v_{i+2}/v_i$, we obtain the map

$$(w_1, w_2, \dots, w_{2n}) \mapsto (w_2, w_3, \dots, w_{2n+1}), \tag{49}$$

where

$$w_{2n+1} = \frac{1}{w_1 w_3 \dots w_{2n-1}} \cdot \frac{\beta_1 w_2 w_4 \dots w_{2n} + \beta_2}{\beta_1 + \beta_3 w_2 w_4 4 \dots w_{2n}}.$$

Integrals of this map are given by

$$I_r = \alpha_1 \big(w_2 w_4 \dots w_{2n} \Theta^{1,2n+1}_{2r-1,0} + \frac{1}{w_2 w_4 \dots w_{2n}} \Theta^{1,2n+1}_{2r-1,1} \big) + \alpha_2 \Theta^{1,2n+1}_{2r,1} + \alpha_3 \Theta^{1,2n+1}_{2r,0}, \tag{50}$$

where $1 \leq r \leq n$ arguments of Θ are

$$f_1 = 1, \;\; f_{i+1} = w_1 w_2 \dots w_i.$$

The map (49) has a symplectic structure, with $p = 2n$,

$$\Omega^{'}_{\mathrm{mKdV},p} = \begin{pmatrix} 0 & w_1 w_2 & 0 & 0 & \dots & 0 \\ -w_2 w_1 & 0 & w_1 w_2 & 0 & \dots & 0 \\ 0 & -w_3 w_2 & 0 & w_2 w_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -w_p w_{p-1} & 0 \end{pmatrix}. \tag{51}$$

Therefore, we define a Poisson bracket

$$\{g, h\}_w = (\triangledown g)^T \Omega^{'}_p \triangledown h = \sum_i \big(\frac{\partial g}{\partial w_i} \frac{\partial h}{\partial w_{i+1}} w_i w_{i+1} - \frac{\partial g}{\partial w_{i+1}} \frac{\partial h}{\partial w_i} w_i w_{i+1} \big), \tag{52}$$

where g and h are differentiable functions.

Now we give a relationship between the symplectic structure (41) and the symplectic structure (13). We consider the map K_p, given as follows

$$(z_1, z_2, \dots, z_p) \mapsto (f_2, \dots, f_{p+1}),$$

where $f_i = z_1 z_2 \dots z_{i-1}$ and the Jacobian matrix of this map is denoted by dK_p. By direct calculation we have

$$dK_p \Omega^{'}_p dK_p^T = S(\Omega_{\mathrm{sG},p}) \mid_{f_i = z_1 z_2 \dots z_{i-1}}, \tag{53}$$

where S is a shift operator. Therefore, we have

$$\{g, h\}_w (w_1, w_2, \dots, w_p) = \{g, h\}_f \mid_{f_i = w_1 w_2 \dots w_{i-1},\; i \geq 2}, \tag{54}$$

where the left hand side is defined in 52, and the right hand side is defined in (14). Using this identity, Lemma 1 and Corollary 3, we obtain the following corollary for r, s are both even or both odd.

Corollary 12.

$$\{\Theta^{2,p}_{r,\epsilon}, \Theta^{2,p}_{s,\epsilon}\}_w = 0, \tag{55}$$

$$\{\Theta^{2,p}_{r,0}, \Theta^{2,p}_{s,1}\}_w + \{\Theta^{2,p}_{r,1}, \Theta^{2,p}_{s,0}\}_w = 0, \tag{56}$$

$$\{\Theta^{2,p}_{r-1,\epsilon}, \Theta^{2,p}_{s,\epsilon}\}_w + \{\Theta^{2,p}_{r,\epsilon}, \Theta^{2,p}_{s-1,\epsilon}\}_w = \begin{cases} 0, & r, s \;\; \mathit{even} \\ \Theta^{2,p}_{r-1,\epsilon} \Theta^{2,p}_{s,\epsilon} - \Theta^{2,p}_{s-1,\epsilon} \Theta^{2,p}_{r,\epsilon}, & r, s \;\; \mathit{odd} \end{cases}, \tag{57}$$

$$\{\Theta^{2,p}_{r-1,\epsilon\pm1}, \Theta^{2,p}_{s,\epsilon}\}_w + \{\Theta^{2,p}_{r,\epsilon}, \Theta^{2,p}_{s-1,\epsilon\pm1}\}_w = \begin{cases} 0, & r, s \;\; \mathit{even} \\ \Theta^{2,p}_{s-1,\epsilon\pm1} \Theta^{2,p}_{r,\epsilon} - \Theta^{2,p}_{s,\epsilon} \Theta^{2,p}_{r-1,\epsilon\pm1}, & r, s \;\; \mathit{odd} \end{cases}. \tag{58}$$

Now since $f_1 = 1$, we have $\Theta^{1,p}_{r,\epsilon} = \Theta^{2,p}_{r,\epsilon} + \Theta^{2,p}_{r-1,\epsilon\pm1}$. Using this identity and Corollary 12, we obtain the next corollary.

Corollary 13.

$$\{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s,\epsilon}\}_w = \begin{cases} 0, & r, s \ \textit{even} \\ \Theta^{2,p}_{s-1,\epsilon\pm1}\Theta^{2,p}_{r,\epsilon} - \Theta^{2,p}_{r-1,\epsilon\pm1}\Theta^{2,p}_{s,\epsilon}, & r, s \ \textit{odd} \end{cases}, \tag{59}$$

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_w + \{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,0}\}_w = \begin{cases} 0, & r, s \ \textit{even} \\ \Theta^{2,p}_{r-1,0}\Theta^{2,p}_{s,0} - \Theta^{2,p}_{s-1,0}\Theta^{2,p}_{r,0} + \Theta^{2,p}_{r-1,1}\Theta^{2,p}_{s,1} - \Theta^{2,p}_{s-1,1}\Theta^{2,p}_{r,1}, & r, s \ \textit{odd} \end{cases}, \tag{60}$$

$$\{\Theta^{1,p}_{r,\epsilon}, \Theta^{1,p}_{s-1,\epsilon\pm1}\}_w + \{\Theta^{1,p}_{r-1,\epsilon\pm1}, \Theta^{1,p}_{s,\epsilon}\}_w = \begin{cases} \Theta^{2,p}_{s-2,\epsilon}\Theta^{2,p}_{r-1,\epsilon\pm1} - \Theta^{2,p}_{s-1,\epsilon\pm1}\Theta^{2,p}_{r-2,\epsilon}, & r, s \ \textit{even} \\ \Theta^{2,p}_{s-1,\epsilon\pm1}\Theta^{2,p}_{r,\epsilon} - \Theta^{2,p}_{r-1,\epsilon\pm1}\Theta^{2,p}_{s,\epsilon}, & r, s \ \textit{odd} \end{cases}, \tag{61}$$

$$\{\Theta^{1,p}_{r,}, \Theta^{1,p}_{s-1,0}\}_w + \{\Theta^{1,p}_{r-1,0}, \Theta^{1,p}_{s,0}\}_w = \begin{cases} \Theta^{2,p}_{r-2,1}\Theta^{2,p}_{s-1,1} - \Theta^{2,p}_{s-2,1}\Theta^{2,p}_{r-1,1} + \Theta^{2,p}_{s-1,1}\Theta^{2,p}_{r-1,0} - \Theta^{2,p}_{r-1,1}\Theta^{2,p}_{s-1,0} \\ (\ r, s \ \textit{even}) \\ \Theta^{2,p}_{r-1,0}\Theta^{2,p}_{s,0} - \Theta^{2,p}_{s,0}\Theta^{2,p}_{r-1,0} + \Theta^{2,p}_{r-1,0}\Theta^{2,p}_{s-1,1} - \Theta^{2,p}_{r-1,1}\Theta^{2,p}_{s-1,0} \\ (\ r, s \ \textit{odd}) \end{cases}, \tag{62}$$

$$\{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s-1,1}\}_w + \{\Theta^{1,p}_{r-1,1}, \Theta^{1,p}_{s,1}\}_w = \begin{cases} \Theta^{2,p}_{r-2,0}\Theta^{2,p}_{s-1,0} - \Theta^{2,p}_{s-2,0}\Theta^{2,p}_{r-1,0} + \Theta^{2,p}_{s-1,0}\Theta^{2,p}_{r-1,1} - \Theta^{2,p}_{r-1,0}\Theta^{2,p}_{s-1,1}, \\ \ r, s \ \textit{even}) \\ \Theta^{2,p}_{r-1,1}\Theta^{2,p}_{s,1} - \Theta^{2,p}_{s,1}\Theta^{2,p}_{r-1,1} + \Theta^{2,p}_{r-1,1}\Theta^{2,p}_{s-1,0} - \Theta^{2,p}_{r-1,0}\Theta^{2,p}_{s-1,1} \\ (\ r, s \ \textit{odd}) \end{cases}. \tag{63}$$

Using this Corollary, one can prove that the integrals given by (50) are in involution with respect to the symplectic structure (51).

4 Involutivity of the pKdV map

In this section, we give a proof of the involutivity of integrals of the pKdV map (3). Similar to the sine-Gordon map, we consider two cases where the dimension d of the map is even or odd. For both, there are not enough integrals for integrability, therefore we need to introduce reductions. We present a symplectic structure in each case for the corresponding reduced map and give a relationship between these symplectic structures. For the case where d is even, properties of multi- sums of products, Ψ, with respect to its symplectic structure are proved by induction. For the case where d is odd, properties of Ψ are derived from those in the even case and the relationship between the two symplectic structures.

4.1 $d = 2n + 2$

We have a $2n+2$-dimensional map (4). The integrals I_r of this map are given by (8) with $0 \leq r \leq n-1$ which are not enough integrals for integrability in the sense of Liouville-Arnold. Therefore, we use a reduction $c_i = v_{i-1} - v_{i+1}$ to reduce the dimension of the map by 2. From the equation (3) and rewrite the indices, we obtain the following map L:

$$(c_1, c_2, \ldots, c_{2n}) \mapsto (c_2, c_3, \ldots, c_{2n}, c_{2n+1}), \tag{64}$$

where

$$c_{2n+1} = \frac{\gamma}{c_2 + c_4 + \ldots + c_{2n}} - c_1 - c_3 - \ldots - c_{2n-1}. \tag{65}$$

This map has exactly n integrals given by

$$\begin{aligned} I_r = {} & \Psi_{r-1}^{1,2n-1} - (c_2 + c_4 + \ldots + c_{2n})\Psi_{r-1}^{1,2n-2} - (c_1 + c_3 + \ldots + c_{2n-1})\Psi_{r-1}^{2,2n-1} \\ & + \Psi_{r-2}^{2,2n-2} + \big((c_1 + c_3 + \ldots + c_{2n-1})(c_2 + c_4 + \ldots + c_{2n}) - \gamma\big)\Psi_r^{1,2n-1}. \end{aligned} \tag{66}$$

The following symplectic structure Ω_{pKdV} of the map (64) is given in [4, 9]

$$\Omega_{\mathrm{pKdV},2n} = \begin{pmatrix} 0 & 1 & 0 & 0\ldots & 0 \\ -1 & 0 & 1 & 0\ldots & 0 \\ 0 & -1 & 0 & 1\ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & 0 \end{pmatrix}. \tag{67}$$

Hence, we define a Poisson bracket as follows.

Definition 14. *With differentiable functions g and h, we define*

$$\{g, h\}_c := \sum_i \Big(\frac{\partial g}{\partial c_i}\frac{\partial h}{\partial c_{i+1}} - \frac{\partial g}{\partial c_{i+1}}\frac{\partial h}{\partial c_i}\Big). \tag{68}$$

We prove that the integrals of the map L are in involution with respect to this Poisson bracket.

4.1.1 Properties of Psi

Similar to the sine-Gordon map, we present properties of Psi with respect to the Poisson bracket (68). We use induction to prove these properties.

Lemma 15. *Let $p \geq 1$ and $0 \leq r, s \leq \lfloor (p+1)/2 \rfloor$. Then we have the following identities*

$$\{\Psi_r^{1,p}, \Psi_s^{1,p}\}_c = 0, \tag{69}$$

$$\{\Psi_r^{1,p}, \Psi_s^{1,p-1}\}_c + \{\Psi_r^{1,p-1}, \Psi_s^{1,p}\}_c = 0. \tag{70}$$

From this lemma we have following corollaries.

Corollary 16. *1.* $\{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{1,p}\Psi_s^{1,p-1} - \Psi_s^{1,p}\Psi_r^{1,p-1}$

2. $\{\Psi_r^{a,b}, \Psi_s^{a,b}\}_c = 0$ *with* $0 \leq r, s \leq \lfloor (b-a)/2 \rfloor + 1$

3. $\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}_c = \Psi_s^{1,n}\Psi_r^{2,p} - \Psi_r^{1,p}\Psi_s^{2,p}$

4. $\{\Psi_r^{1,p}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p}\}_c = 0$

5. $\{\Psi_r^{1,p}, \Psi_s^{2,p+1}\}_c + \{\Psi_r^{2,p+1}, \Psi_s^{1,p}\}_c = 0$

6. $\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{2,p}\Psi_s^{1,p-1} - \Psi_s^{2,p}\Psi_r^{1,p-1}.$

7. $\{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{1,p}\Psi_s^{1,p-1} - \Psi_s^{1,p}\Psi_r^{1,p-1}$

8. $\{\Psi_r^{1,p+1}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p+1}\}_c = 0$

9. $\{\Psi_r^{1,p}, \Psi_{s-2}^{2,p-1}\}_c + \{\Psi_{r-2}^{2,p-1}, \Psi_s^{1,p}\}_c = \Psi_r^{2,p}\Psi_{s-1}^{1,p-1} - \Psi_s^{2,p}\Psi_{r-1}^{1,p-1} + \Psi_{r-1}^{2,p}\Psi_s^{1,p-1} - \Psi_{s-1}^{2,p}\Psi_r^{1,p-1}$

4.1.2 Involutivity of the integrals

We prove that the integrals of the pKdV maps are in involution with respect its Poisson bracket (14) by using the above mentioned properties of Ψ.

Theorem 17. *For all $0 \leq r, s \leq n-1$, we have $\{I_r, I_s\}_c = 0$, where I_r, I_s are given by (66).*

Proof. To prove this theorem we need the following formulas. Let $g(c_1, c_2, \ldots, c_{2n})$ be a differentiable function on $\mathbb{R}^{2n}$. Denote

$$C_1 = c_1 + c_3 + \cdots + c_{2n-1},\ C_2 = c_2 + c_4 + \cdots + c_{2n},$$

we have

$$\{g, C_1\}_c = -\frac{\partial g}{\partial c_{2n}},\ \{g, C_2\}_c = \frac{\partial g}{\partial c_1}. \tag{71}$$

We write $\{I_r, I_s\}_c = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11}$ where

$$\begin{aligned}
A_1 &:= \{\Psi_{r-1}^{1,2n-1} - C_2\Psi_{r-1}^{1,2n-2}, \Psi_{s-1}^{1,2n-1} - C_2\Psi_{s-1}^{1,2n-2}\}_c \\
A_2 &:= -\{\Psi_{r-1}^{1,2n-1}, C_1\Psi_{s-1}^{2,2n-1}\}_c - \{C_1\Psi_{r-1}^{2,2n-1}, \Psi_{s-1}^{1,2n-1}\}_c + \{C_1\Psi_{r-1}^{2,2n-1}, C_1\Psi_{s-1}^{2,2n-1}\}_c \\
A_3 &:= \{\Psi_{r-1}^{1,2n-1}, \Psi_{s-2}^{2,2n-2}\}_c + \{\Psi_{r-2}^{2,2n-2}, \Psi_{s-1}^{1,2n-1}\}_c + \{\Psi_{r-2}^{2,2n-2}, \Psi_{s-2}^{2,2n-2}\}_c \\
A_4 &:= \{C_2\Psi_{r-1}^{1,2n-2}, C_1\Psi_{s-1}^{2,2n-1}\}_c + \{C_1\Psi_{r-1}^{2,2n-1}, C_2\Psi_{s-1}^{1,2n-2}\}_c \\
A_5 &:= -\{C_2\Psi_{r-1}^{1,2n-2}, \Psi_{s-2}^{2,2n-2}\}_c - \{\Psi_{r-2}^{2,2n-2}, C_2\Psi_{s-1}^{1,2n-2}\}_c \\
A_6 &:= -\{C_1\Psi_{r-1}^{2,2n-1}, \Psi_{s-2}^{2,2n-2}\}_c - \{\Psi_{r-2}^{2,2n-2}, C_1\Psi_{s-1}^{2,2n-1}\}_c \\
A_7 &:= -\{\Psi_{r-1}^{1,2n-1}, (C_1C_2 - \gamma)\Psi_s^{1,2n-1}\}_c - \{\big(C_1C_2 - \gamma\big)\Psi_r^{1,2n-1}, \Psi_{s-1}^{1,2n-1}\}_c \\
A_8 &:= -\{C_2\Psi_{r-1}^{1,2n-2}, \big(C_1C_2 - \gamma\big)\Psi_s^{1,2n-1}\}_c - \{\big(C_1C_2 - \gamma\big)\Psi_r^{1,2n-1}, C_2\Psi_{s-1}^{1,2n-2}\}_c \\
A_9 &:= -\{C_1\Psi_{r-1}^{2,2n-1}, (C_1C_2 - \gamma)\Psi_s^{1,2n-1}\}_c - \{\big(C_1C_2 - \gamma\big)\Psi_r^{1,2n-1}, C_1\Psi_{s-1}^{2,2n-1}\}_c \\
A_{10} &:= \{\Psi_{r-2}^{2,2n-2}, (C_1C_2 - \gamma)\Psi_s^{1,2n-1}\}_c + \{\big(C_1C_2 - \gamma\big)\Psi_r^{1,2n-1}, \Psi_{s-2}^{2,2n-2}\}_c \\
A_{11} &:= \{(C_1C_2 - \gamma)\Psi_r^{1,2n-1}, (C_1C_2 - \gamma)\Psi_s^{1,2n-1}\}_c
\end{aligned}$$

Using Lemma 15, Corollary 16 and formulas (71), we have

$$\begin{aligned}
A_1 &= \Psi_{r-1}^{1,2n-2}\Psi_{s-1}^{2,2n-1} - \Psi_{s-1}^{1,2n-2}\Psi_{r-1}^{2,2n-1} + C_2\big(\Psi_{s-1}^{1,2n-2}\Psi_{r-1}^{2,2n-2} - \Psi_{r-1}^{1,2n-2}\Psi_{s-1}^{2,2n-2}\big), \\
A_2 &= \Psi_{s-1}^{2,2n-1}\Psi_{r-1}^{1,2n-2} - \Psi_{r-2}^{2,2n-1}\Psi_{s-1}^{1,2n-1} + C_1\big(\Psi_{r-1}^{2,2n-1}\Psi_{s-1}^{2,2n-2} - \Psi_{s-1}^{2,2n-1}\Psi_{r-1}^{2,2n-2}\big), \\
A_3 &= \Psi_{r-1}^{2,2n-1}\Psi_{s-1}^{1,2n-2} - \Psi_{s-1}^{2,2n-1}\Psi_{r-1}^{1,2n-2}, \\
A_4 &= -\Psi_{r-1}^{1,2n-2}\Psi_{s-1}^{2,2n-1} + \Psi_{r-1}^{2,2n-1}\Psi_{s-1}^{1,2n-2}, \\
A_5 &= -C_2\big(\Psi_{s-1}^{1,2n-2}\Psi_{r-1}^{2,2n-2} - \Psi_{r-1}^{1,2n-2}\Psi_{s-1}^{2,2n-2}\big), \\
A_6 &= -C_1\big(\Psi_{r-1}^{2,2n-1}\Psi_{s-1}^{2,2n-2} - \Psi_{s-1}^{2,2n-1}\Psi_{r-1}^{2,2n-2}\big).
\end{aligned}$$

It follows that $A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 0$. Now we show that $A_7 + A_8 + A_9 + A_{10} + A_{11} = 0$.

We also have

$$
\begin{aligned}
A_7 &= C_1\big(\Psi_s^{1,2n-1}\Psi_{r-1}^{2,2n-1} - \Psi_r^{1,2n-1}\Psi_{s-1}^{2,2n-1}\big) + C_2\big(\Psi_r^{1,2n-1}\Psi_{s-1}^{1,2n-2} - \Psi_s^{1,2n-1}\Psi_{r-1}^{1,2n-2}\big),\\
A_8 &= C_2(C_1C_2-\gamma)\big(\Psi_s^{1,2n-1}\Psi_r^{1,2n-2} - \Psi_r^{1,2n-1}\Psi_s^{1,2n-2}\big) + C_2\big(\Psi_{r-1}^{1,2n-2}\Psi_s^{1,2n-1} - \Psi_{s-1}^{1,2n-2}\Psi_r^{1,2n-1}\big)\\
&\quad + C_1C_2\big(\Psi_r^{1,2n-1}\Psi_{s-1}^{2,2n-2} - \Psi_s^{1,2n-1}\Psi_{r-1}^{2,2n-1}\big) + (C_1C_2-\gamma)\big(\Psi_{r-1}^{1,2n-2}\Psi_s^{2,2n-1} - \Psi_{s-1}^{1,2n-2}\Psi_r^{2,2n-1}\big),\\
A_9 &= C_1(C_1C_2-\gamma)\big(\Psi_r^{1,2n-1}\Psi_s^{2,2n-1} - \Psi_s^{1,2n-1}\Psi_r^{2,2n-1}\big) + C_1\big(\Psi_{s-1}^{2,2n-1}\Psi_r^{1,2n-1} - \Psi_{r-1}^{2,2n-1}\Psi_s^{1,2n-1}\big)\\
&\quad + C_1C_2\big(\Psi_s^{1,2n-1}\Psi_{r-1}^{2,2n-2} - \Psi_r^{1,2n-1}\Psi_{s-1}^{2,2n-2}\big) + (C_1C_2-\gamma)\big(\Psi_{s-1}^{2,2n-1}\Psi_r^{1,2n-2} - \Psi_{r-1}^{2,2n-1}\Psi_s^{1,2n-2}\big),\\
A_{10} &= (C_1C_2-\gamma)\big(\Psi_r^{2,2n-1}\Psi_{s-1}^{1,2n-2} - \Psi_s^{2,2n-1}\Psi_{r-1}^{1,2n-2} + \Psi_{r-1}^{2,2n-1}\Psi_s^{1,2n-2} - \Psi_{s-1}^{2,2n-1}\Psi_r^{1,2n-2}\big),\\
A_{11} &= (C_1C_2-\gamma)C_1\big(\Psi_s^{1,2n-1}\Psi_r^{2,2n-1} - \Psi_r^{1,2n-1}\Psi_s^{2,2n-1}\big) + \big(\Psi_s^{1,2n-2}\Psi_r^{1,2n-1} - \Psi_s^{1,2n-1}\Psi_r^{1,2n-2}\big)\\
&\qquad (C_1c_2-\gamma)C_2.
\end{aligned}
$$

This implies $A_7 + A_8 + A_9 + A_{10} + A_{11} = 0$. Therefore, we have $\{I_r, I_s\} = 0$. □

4.2 $d = 2n + 1$

We introduce a reduction $u_i = v_i - v_{i+1}$. We obtain a $2n$-dimensional map

$$(u_1, u_1, \ldots, u_{2n}) \mapsto (u_2, u_3, \ldots, u_{2n}, \frac{\gamma}{u_2 + u_3 + \ldots + u_{2n}} - u_1 - u_2 - \ldots - u_{2n}) \tag{72}$$

with n integrals $(0 \le r \le n-1)$

$$
\begin{aligned}
I_r = \Psi_{r-1}^{2,2n-1} &- (u_2 + u_3 + \ldots + u_{2n})\Psi_{r-1}^{2,2n-2} - (u_1 + u_2 + \ldots + u_{2n-1})\Psi_{r-1}^{3,2n-1}\\
&+ \Psi_{r-2}^{3,2n-2} + \big((u_2 + u_3 + \ldots + u_{2n})(u_1 + u_2 + \ldots + u_{2n-1}) - \gamma\big)\Psi_r^{2,2n-1},
\end{aligned} \tag{73}
$$

where the argument of Ψ is $f_i = 1/(c_ic_{i+1})$ with $c_i = u_{i-1} + u_i$. Based on the method given in [9], we obtain a symplectic structure $\Omega'_{\mathrm{pKdV},2n}$ of the map (72) with $p = 2n$

$$
\Omega'_{\mathrm{pKdV},p} = \begin{pmatrix}
0 & 1 & -1 & 1\ldots & (-1)^p\\
-1 & 0 & 1 & -1\ldots & -1^{p-1}\\
1 & -1 & 0 & 1\ldots & 1\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
(-1)^p & (-1)^{p-1} & \ldots & -1 & 0
\end{pmatrix}, \tag{74}
$$

and this defines a bracket

$$
\begin{aligned}
\{g, h\}_u &= (\frac{\partial g}{\partial u_1}, \ldots, \frac{\partial g}{\partial u_p})\Omega'_{\mathrm{pKdV},p}(\frac{\partial h}{\partial u_1}, \ldots, \frac{\partial h}{\partial u_n})^T\\
&= \sum_{i<j}(-1)^{i+j}(\frac{\partial g}{\partial u_i}\frac{\partial h}{\partial u_j} - \frac{\partial g}{\partial u_j}\frac{\partial h}{\partial u_i}).
\end{aligned} \tag{75}
$$

Now we present a relationship between the two symplectic structures (67) and (**??**). We consider a map P as below

$$(u_1, u_2, \ldots, u_n) \mapsto (c_2, c_3, \ldots, c_n),$$

where $c_i = u_{i-1} + u_i$. We denote a Jacobian matrix

$$dP = \frac{\partial(c_2, c_3, \ldots, c_n)}{\partial(u_1, u_2, \ldots, u_n})).$$

By calculation we obtain

$$dP\Omega'_{\mathrm{pKdV},n}dP^T = \Omega_{\mathrm{pKdV},n-1}. \tag{76}$$

Similar to the sine-Gordon map, we have a relation

$$\{f,g\}_u = \{f,g\}_c \,|_{c_i=u_{i-1}+u_i}, \tag{77}$$

where $f(c_2,c_3,\ldots,c_n)$ and $g(c_2,c_3,\ldots,c_n)$ are differentiable functions. It follows that all the properties of Psi with respect to the first bracket (68) are preserved with respect to the second bracket (75). These properties are used to prove the following theorem.

Theorem 18. *Let I_r, I_s be given by (73). Then for all $0 \leq r,s \leq n-1$, we have*

$$\{I_r, I_s\}_u = 0.$$

Proof. Firstly, we need the following formulas

$$\{g, u_2+u_3+\ldots+u_{2n}\}_u = \frac{\partial g}{\partial u_1}, \tag{78}$$

$$\{g, u_1+u_2+\ldots+u_{2n-1}\}_u = -\frac{\partial g}{\partial u_{2n}},. \tag{79}$$

Using these formulas, and properties of Psi with respect to the second bracket which are the same as those with respect to the first bracket one can proved the involutivity of the integrals (73) similarly as we did for the case $d = 2n+2$. □

5 Discussion

This paper has proved the involutivity of the sine-Gordon, pKdV and mKdV maps directly by using induction and recently found symplectic structures of these maps. In order to prove these maps are completely integrable in the sense of Louville-Arnold [2], we also need to prove functionally independence of their integrals which we hope to be publish elsewhere [12].

It should be noted that the integrals of all equations in the ABS list[1], [11], with the exception of Q_4, are expressed in terms of multi-sums of products, Ψ. Therefore, it would be interesting to study symplectic structures and integrability of the equations in the ABS list.

Acknowledgment

This research has been funded by the Australian Research Council through the Centre of Excellence for Mathematics and Statistics of Complex Systems. Dinh T. Tran acknowledges the support of two scholarships, one from La Trobe University and the other from the Endeavour IPRS programme.

A Proving properties of Θ

First of all, the following lemma follows directly from (15).

Lemma 19.

$$\{\Theta^{1,p}_{r,\epsilon}, f^{(-1)^s}_{p+1}\}_f = \begin{cases} 0 & r \text{ even} \\ (-1)^{s+\epsilon+1} f^{(-1)^s}_{p+1}\Theta^{1,p}_{r,\epsilon} & r \text{ odd} \end{cases} \tag{80}$$

A.1 Proof of Lemma 1

Proof. We will prove this lemma by induction. The following properties, given in [5, 11], will be used in our proof:

$$\Theta^{a,b}_{r,\epsilon} = \Theta^{a,b-1}_{r,\epsilon} + f_b^{(-1)^{\epsilon+r}} \Theta^{a,b-1}_{r-1,\epsilon} \tag{81}$$

$$\Theta^{a,b}_{r,\epsilon} = \Theta^{a+1,b}_{r,\epsilon} + f_a^{(-1)^{\epsilon\pm1}} \Theta^{a+1,b}_{r-1,\epsilon\pm1}. \tag{82}$$

We first assume $\epsilon = 1$. One verifies that (16) holds for $p = 1, 2$ and for all $0 \leq r, s \leq p$. Suppose that (16) hold for $p-1$ and p ($p \geq 2$). We will prove that (16) holds for $p+1$.

Using identity (81), we expand the left hand side of (16) and we obtain

$$\begin{aligned}
&\{\Theta^{1,p+1}_{r,1}, \Theta^{1,p+1}_{s,1}\}_f \\
=&\{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,1}\}_f + f_{p+1}^{(-1)^{r+1}} \{\Theta^{1,p}_{r-1,1}, \Theta^{1,p}_{s,1}\}_f + \Theta^{1,p}_{r-1,1}\{f_{p+1}^{(-1)^{p+1}}, \Theta^{1,p}_{s,1}\}_f + f_{p+1}^{(-1)^{s+1}} \{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s-1,1}\}_f \\
&+ \Theta^{1,p}_{s-1,1}\{\Theta^{1,p}_{r,1}, f_{p+1}^{(-1)^{s+1}}\}_f + f_{p+1}^{(-1)^{r+1}+(-1)^{s+1}} \{\Theta^{1,p}_{r-1,1}, \Theta^{1,p}_{s-1,1}\}_f + \Theta^{1,p}_{r-1,1}\Theta^{1,p}_{s-1,1}\{f_{p+1}^{(-1)^{r+1}}, f_{p+1}^{(-1)^{s+1}}\}_f \\
&+ f_{p+1}^{(-1)^{s+1}} \Theta^{1,p}_{r-1,1}\{f_{p+1}^{(-1)^{r+1}}, \Theta^{1,p}_{s-1,1}\}_f + f_{p+1}^{(-1)^{r+1}} \Theta^{1,p}_{s-1,1}\{\Theta^{1,p}_{r-1,1}, f_{p+1}^{(-1)^{s+1}}\}_f.
\end{aligned} \tag{83}$$

The case $r = s$ is trivial. Now we distinguish 3 cases.

1. r and s are both even or both odd. Since $\{\Theta^{1,p+1}_{r,1}, \Theta^{1,p+1}_{s,1}\}_f = -\{\Theta^{1,p+1}_{s,1}, \Theta^{1,p+1}_{r,1}\}_f$, without loss of generality we assume that $r > s$. If both r and s are even, on the right hand side of (83) the first, third, fifth sixth, seventh terms vanish. Thus, we have

$$\begin{aligned}
\{\Theta^{1,p+1}_{r,1}, \Theta^{1,p+1}_{s,1}\}_f =& f^{-1}_{p+1}\big(\{\Theta^{1,p}_{r-1,1}, \Theta^{1,p}_{s,1}\}_f + \{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s-1,1}\}_f\big) + f^{-1}_{p+1}\Theta^{1,p}_{r-1,1}\{f^{-1}_{p+1}, \Theta^{1,p}_{s-1,1}\}_f \\
&+ f^{-1}_{p+1}\Theta^{1,p}_{s-1,1}\{\Theta^{1,p}_{r-1,1}, f^{-1}_{p+1}\}_f \\
=& f^{-1}_{p+1}\big(\sum_{i=1}^{s}(-1)^i\Theta^{1,p}_{s-i,1}\Theta^{1,p}_{r-i+1} + \sum_{i=0}^{s-1}(-1)^i\Theta^{1,p}_{r+i,1}\Theta^{1,p}_{s-1-i,1}\big) + f^{-2}_{p+1}\Theta^{1,p}_{r-1,1}\Theta^{1,p}_{s_1,1} \\
&- f^{-2}_{p+1}\Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r-1,1} \\
=& f^{-1}_{p+1}\Big(\sum_{j=0}^{s-1}(-1)^{j+1}\Theta^{1,p}_{s-j-1,1}\Theta^{1,p}_{r-j}\big) + \sum_{i=0}^{s-1}(-1)^i\Theta^{1,p}_{r+i,1}\Theta^{1,p}_{s-1-i,1}\Big) \\
=&0.
\end{aligned}$$

With $r > s$ and r, s are both odd, on the right hand side of (83) the first, sixth, seventh, eighth, and ninth terms vanish. Therefore, we have

$$\begin{aligned}
&\{\Theta^{1,p+1}_{r,1}, \Theta^{1,p+1}_{s,1}\}_f \\
=&f_{p+1}\big(\{\Theta^{1,p}_{r-1,1}, \Theta^{1,p}_{s,1}\}_f + \{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s-1,1}\}_f\big) + \Theta^{1,p}_{r-1,1}\{f_{p+1}, \Theta^{1,p}_{s,1}\}_f + \Theta^{1,p}_{s-1,1}\{\Theta^{1,p}_{r,1}, f_{p+1}\}_f \\
=&f_{p+1}\Big(\sum_{i=0}^{s}(-1)^i\Theta^{1,p}_{r-1+i,1}\Theta^{1,p}_{s-i,1} - \big(\sum_{i=1}^{s-1}(-1)^{i-1}\Theta^{1,p}_{s-1-i,1}\Theta^{1,p}_{r+i,1}\big)\Big) - f_{p+1}\Theta^{1,p}_{r-1,1}\Theta^{1,p}_{s,1} + f_{p+1}\Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r,1} \\
=&f_{p+1}\Big(\sum_{i=0}^{s}(-1)^i\Theta^{1,p}_{r-1+i,1}\Theta^{1,p}_{s-i,1} - \big(\sum_{j=2}^{s}(-1)^j\Theta^{1,p}_{s-j,1}\Theta^{1,p}_{r+j-1,1}\big)\Big) - f_{p+1}\Theta^{1,p}_{r-1,1}\Theta^{1,p}_{s,1} + f_{p+1}\Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r,1} \\
=&f_{p+1}\big(\Theta^{1,p}_{r-1,1}\Theta^{1,p}_{s,1} - \Theta^{1,p}_{r,1}\Theta^{1,p}_{s-1,1}\big) - f_{p+1}\Theta^{1,p}_{r-1,1}\Theta^{1,p}_{s,1} + \Theta^{1,p}_{s-1,1}f_{p+1}\Theta^{1,p}_{r,1} \\
=&0.
\end{aligned}$$

2. r is even, s is odd and $r > s$. We have

$$\begin{aligned}
&\{\Theta^{1,p+1}_{r,1}, \Theta^{1,p+1}_{s,1}\}_f \\
=&\{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,1}\}_f + \{\Theta^{1,p}_{r-1,1}, \Theta^{1,p}_{s-1,1}\}_f + f^{-1}_{p+1}\Theta^{1,p}_{s-1,1}\{\Theta^{1,p}_{r-1,1}, f_{p+1}\}_f + \Theta^{1,p}_{r-1,1}\{f^{-1}_{p+1}, \Theta^{1,p}_{s,1}\}_f \\
=&\sum_{i=0}^{s}(-1)^i\Theta^{1,p}_{r+i,1}\Theta^{1,p}_{s-i,1} - \big(\sum_{i=1}^{s-1}(-1)^{i-1}\Theta^{1,p}_{s-i-1,1}\Theta^{1,p}_{r-1+i,1}\big) + \Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r-1,1} + f^{-1}_{p+1}\Theta^{1,p}_{r-1,1}\Theta^{1,p}_{s,1} \\
=&\sum_{i=0}^{s}(-1)^i\Theta^{1,p}_{r+i,1}\Theta^{1,p}_{s-i,1} + \sum_{i=0}^{s-1}(-1)^i\Theta^{1,p}_{s-i-1,1}\Theta^{1,p}_{r-1+i,1} + f^{-1}_{p+1}\Theta^{1,p}_{r-1,1}\Theta^{1,p}_{s,1} \\
=&\sum_{i=0}^{s}(-1)^i\Theta^{1,p+1}_{r+i,1}\Theta^{1,p+1}_{s-i,1},
\end{aligned}$$

   where in the final equality we used (81).

3. r is even, s is odd and $r < s$. We do similarly as in the previous case.. Therefore, with $\epsilon = 1$ identity (16) holds for $p+1$. Then, it holds for all $p \geq 0$.

Now with the case $\epsilon = 0$, we introduce $t_i = 1/f_i$, then we get

$$\Theta^{1,p}_{r,0}(\{f_i\}) = \Theta^{1,p}_{r,1}(\{t_i\}).$$

We have

$$\begin{aligned}
\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,0}\}_f(\{f_i\}) &= \sum_{i=1}^{p}\big(\frac{\partial\Theta^{1,p}_{r,0}}{\partial f_i}\frac{\partial\Theta^{1,p}_{s,0}}{\partial f_j} - \frac{\partial\Theta^{1,p}_{r,0}}{\partial f_j}\frac{\partial\Theta^{1,p}_{s,0}}{\partial f_i}\big)f_if_j \\
&= \big(\frac{\partial\Theta^{1,p}_{r,1}}{\partial t_i}\frac{\partial\Theta^{1,p}_{s,1}}{\partial t_j}t_i^2t_j^2 - \frac{\partial\Theta^{0,p}_{r,1}}{\partial t_j}\frac{\partial\Theta^{1,p}_{s,1}}{\partial t_i}t_i^2t_j^2\big)\frac{1}{t_it_j} \\
&= \{\Theta^{1,p}_{r,1}, \Theta^{1,p}_{s,1}\}_t(\{t_i = 1/f_i\}).
\end{aligned}$$

This proves our statement with $\epsilon = 0$. ☐

A.2 Proof of Lemma 2

Proof. We prove by induction again. It is easy to see that (17) can be rewritten as the following (respectively)

$$\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_f = \begin{cases} \sum_{i=0}^{\lfloor (r-1)/2\rfloor}\big(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1} - \Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0}\big), & r < s \\ \sum_{i=0}^{\lfloor (s-1)/2\rfloor}\big(\Theta^{1,p}_{s-2i-1,0}\Theta^{1,p}_{r+2i+1,1} - \Theta^{1,p}_{s-2i-1,1}\Theta^{1,p}_{r+2i+1,0}\big), & r > s \end{cases}. \tag{84}$$

Identities (17) (or (84)) and (18) hold for $p = 1, 2$. Suppose that they hold for $p-1$ and p ($p \geq 2$). We will prove that they hold for $p+1$.

Using identity (81), expanding the left hand sides of (17) (or (84)) and (18) we have

$$\begin{aligned}
&\{\Theta^{1,p+1}_{r,0}, \Theta^{1,p+1}_{s,1}\}_f \\
=&\{\Theta^{1,p}_{r,0} + f^{(-1)^r}_{p+1}\Theta^{1,p}_{r-1,0}, \Theta^{1,p}_{s,1} + f^{(-1)^{s+1}}_{p+1}\Theta^{1,p}_{s-1,1}\}_f \\
=&\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s,1}\}_f + f^{(-1)^r+(-1)^{s+1}}_{p+1}\{\Theta^{1,p}_{r-1,0}, \Theta^{1,p}_{s-1,1}\}_f + \Theta^{1,p}_{r,0}\Theta^{1,p}_{s,1}\{f^{(-1)^r}_{p+1}, f^{(-1)^{s+1}}_{p+1}\}_f \\
&+ f^{(-1)^{s+1}}_{p+1}\{\Theta^{1,p}_{r,0}, \Theta^{1,p}_{s-1,1}\}_f + f^{(-1)^r}_{p+1}\{\Theta^{1,p}_{r-1,0}, \Theta^{1,p}_{s,1}\}_f + \Theta^{1,p}_{s-1,1}\{\Theta^{1,p}_{r,0}, f^{(-1)^{s+1}}_{p+1}\}_f \\
&+ \Theta^{1,p}_{r-1,0}\{f^{(-1)^r}_{p+1}, \Theta^{1,p}_{s,1}\}_f + f^{(-1)^r}_{p+1}\Theta^{1,p}_{s-1,1}\{\Theta^{1,p}_{r-1,0}, f^{(-1)^{s+1}}_{p+1}\}_f + f^{(-1)^{s+1}}_{p+1}\Theta^{1,p}_{r-1,0}\{(f^{(-1)^r}_{p+1}, \Theta^{1,p}_{s-1,1}\}_f.
\end{aligned} \tag{85}$$

1. $r \equiv s \pmod 2$. We distinguish 2 cases.
   Case 1: $s - r = k \geq 0$, we first prove the following

$$\sum_{i=0}^{s-1}(-1)^{i+\epsilon}\Theta^{1,p}_{s-1-i,i+\epsilon}\Theta^{1,p}_{r+i,i+\epsilon+1} = \sum_{i=0}^{r-1}(-1)^{i+\epsilon}\Theta^{1,p}_{r-i-1,i+\epsilon}\Theta^{1,p}_{s+i,i+\epsilon+1}. \tag{86}$$

The left hand side of this identity equals

$$\begin{aligned}
&\sum_{i=k}^{s-1}(-1)^{i+\epsilon}\Theta^{1,p}_{s-1-i,i+\epsilon}\Theta^{1,p}_{r+i,i+\epsilon+1} + \sum_{i=0}^{k-1}(-1)^{i+\epsilon}\Theta^{1,p}_{s-1-i,i+\epsilon}\Theta^{1,p}_{r+i,i+\epsilon+1}\\
=&\sum_{j=0}^{r-1}(-1)^{k+j+\epsilon}\Theta^{1,p}_{r-j-1,k+j+\epsilon}\Theta^{1,p}_{s+j,k+j+\epsilon+1}\\
&+\sum_{i=0}^{\frac{k}{2}-1}\left((-1)^{i+\epsilon}\Theta^{1,p}_{s-1-i,i+\epsilon}\Theta^{1,p}_{r+i,i+\epsilon+1} + (-1)^{k-1-i+\epsilon}\Theta^{1,p}_{s-1-(k-i-1),k-i-1+\epsilon}\Theta^{1,p}_{r+k-i-1,k-i+\epsilon}\right)\\
=&\sum_{i=0}^{r-1}(-1)^{j+\epsilon}\Theta^{1,p}_{r-j-1,j+\epsilon}\Theta^{1,p}_{s+j,j+\epsilon+1} + \sum_{i=0}^{\frac{k}{2}-1}\left((-1)^{i+\epsilon}\Theta^{1,p}_{s-1-i,i+\epsilon}\Theta^{1,p}_{r+i,i+\epsilon+1} + (-1)^{i+\epsilon+1}\Theta^{1,p}_{r+i,i+\epsilon+1}\Theta^{1,p}_{s-i-1,i+\epsilon}\right)\\
=&\sum_{i=0}^{r-1}(-1)^{i+\epsilon}\Theta^{1,p}_{r-i-1,i+\epsilon}\Theta^{1,p}_{s+i,i+\epsilon+1},
\end{aligned}$$

which is the right hand side of (86).

Now using (81), we expand the right hand side of the first identity of (84). We have

$$\begin{aligned}
&\sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\left(\Theta^{1,p+1}_{r-2i-1,0}\Theta^{1,p+1}_{s+2i+1,1} - \Theta^{1,p+1}_{r-2i-1,1}\Theta^{1,p+1}_{s+2i+1,0}\right)\\
=&\sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1} - \Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0} + \Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i,1} - \Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i,0}\right)\\
&+ f^{(-1)^{r-1}}_{p+1}\sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\left(\Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i+1,1} - \Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i,0}\right)\\
&+ f^{(-1)^{s}}_{p+1}\sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i,1} - \Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i+1,0}\right)\\
=&\sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1} - \Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0} + \Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i,1} - \Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i,0}\right)\\
&+ f^{(-1)^{s+1}}_{p+1}\sum_{i=0}^{r-1}(-1)^{i-1}\Theta^{1,p}_{r-1-i,i+1}\Theta^{1,p}_{s+i,i} + f^{(-1)^{s}}_{p+1}\sum_{i=0}^{r-1}(-1)^{i}\Theta^{1,p}_{r-1-i,i}\Theta^{1,p}_{s+i,i+1}.
\end{aligned} \tag{87}$$

If r and s are both even. Using (85) and the induction assumption, we have

$$
\begin{aligned}
&\{\Theta^{1,p+1}_{r,0}, \Theta^{1,p+1}_{s,1}\}_f \\
&= \sum_{i=0}^{r/2-1} \left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1} - \Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0}\right) + \sum_{i=0}^{r/2-1} \left(\Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i,1} - \Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i,0}\right) \\
&\quad + f_{p+1}^{(-1)^r} \sum_{i=0}^{r-1} (-1)^i \Theta^{1,p}_{s+i,i+1}\Theta^{1,p}_{r-1-i,i} + \Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r-1,0} - \Theta^{1,p}_{r-1,0}\Theta^{1,p}_{s-1,1} \\
&\quad + f_{p+1}^{(-1)^{s+1}} \sum_{i=0}^{s-1} (-1)^{i-1} \Theta^{1,p}_{s-1-i,i+1}\Theta^{1,p}_{r+i,i},
\end{aligned}
$$

which equals (87) by using (86) with $\epsilon = 1$.

If r and s are both odd, we have

$$
\begin{aligned}
&\{\Theta^{1,p+1}_{r,0}, \Theta^{1,p+1}_{s,1}\}_f \\
&= \sum_{i=0}^{\lfloor \frac{r-1}{2} \rfloor} \left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1} - \Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0}\right) + \sum_{i=0}^{\lfloor \frac{r-2}{2} \rfloor} \left(\Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i,1} - \Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i,0}\right) \\
&\quad + f_{p+1}^{(-1)^{s+1}} \sum_{i=0}^{r} (-1)^i \Theta^{1,p}_{s-1+i,i+1}\Theta^{1,p}_{r-i,i} + f_{p+1}^{(-1)^r} \sum_{i=0}^{s} (-1)^{i-1} \Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r-1+i,i} - f_{p+1}^{(-1)^{s+1}} \Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r,0} \\
&\quad + f_{p+1}^{(-1)^r} \Theta^{1,p}_{r-1,0}\Theta^{1,p}_{s,1} \\
&= \sum_{i=0}^{\lfloor \frac{r-1}{2} \rfloor} \left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s+2i+1,1} - \Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s+2i+1,0}\right) + \sum_{i=0}^{\lfloor \frac{r-2}{2} \rfloor} \left(\Theta^{1,p}_{r-2-2i,0}\Theta^{1,p}_{s+2i,1} - \Theta^{1,p}_{r-2-2i,1}\Theta^{1,p}_{s+2i,0}\right) \\
&\quad + f_{p+1}^{(-1)^{s+1}} \sum_{i=0}^{r-1} (-1)^{i-1} \Theta^{1,p}_{s+i,i}\Theta^{1,p}_{r-1-i,i+1} + f_{p+1}^{(-1)^r} \sum_{i=0}^{s-1} (-1)^i \Theta^{1,p}_{s-1-i,i}\Theta^{1,p}_{r+i,i+1}.
\end{aligned}
$$

which equals (87) by using (86) with $\epsilon = 0$. Thus, the first identity of (17) (or (84)) holds for $p+1$.

Case 2: $s - r = -k < 0$, we have

$$
\{\Theta^{1,p+1}_{r,0}, \Theta^{1,p+1}_{s,1}\}_f\{f_i\} = \{\Theta^{1,p+1}_{r,1}, \Theta^{1,p+1}_{s,0}\}_t\{t_i = 1/f_i\} = -\{\Theta^{1,p+1}_{s,0}, \Theta^{1,p+1}_{r,1}\}_t\{t_i = 1/f_i\}.
$$

This implies that identity (17) (or (84)) also holds for $p+1$.

2. $r \not\equiv s \pmod 2$. Case 1: $s - r = k > 0$ With r even, s odd and $r < s$, we now expand the right hand side of the first identity of (18) with $p+1$. Similar as (86), we have the following identities

$$
\sum_{i=0}^{s} (-1)^{i-1} \Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i,i} = \sum_{i=1}^{r-1} (-1)^i \Theta^{1,p}_{s-1-i,i+1}\Theta^{1,p}_{r+i-1,i} \tag{88}
$$

$$
\begin{aligned}
\sum_{i=0}^{s-1} (-1)^{i-1} \Theta^{1,p}_{s-i-1,i+1}\Theta^{1,p}_{r+i,i} + \sum_{i=0}^{s} (-1)^{i-1} \Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i-1,i} &= \sum_{i=1}^{r-1} (-1)^i \Theta^{1,p}_{r-1-i,i}\Theta^{1,p}_{s+i,i+1} \\
&\quad + \sum_{i=0}^{r-1} (-1)^{i-1} \Theta^{1,p}_{r-1-i,i+1}\Theta^{1,p}_{s+i,i}.
\end{aligned} \tag{89}
$$

We now expand the right hand side of (18) by using (81, we obtain

$$\begin{aligned}\sum_{i=0}^{s}(-1)^{i-1}\Theta^{1,p+1}_{s-i,i+1}\Theta^{1,p+1}_{r+i,i} =&\sum_{i=0}^{s}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i,i} + f_{p+1}^{2}\sum_{i=0}^{s-1}(-1)^{i-1}\Theta^{1,p}_{s-i-1,i+1}\Theta^{1,p}_{r+i-1,i}\\ &+ f_{p+1}\sum_{i=0}^{s-1}(-1)^{i-1}\Theta^{1,p}_{s-i-1,i+1}\Theta^{1,p}_{r+i,i} + f_{p+1}\sum_{i=0}^{s}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i-1,i}.\end{aligned} \tag{90}$$

For the left hand side of (18), using (85) and the induction assumption, we have

$$\begin{aligned}&\{\Theta^{1,p+1}_{r,0},\Theta^{1,p+1}_{s,1}\}_f\\ =&\sum_{i=0}^{s}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i,i} + f_{p+1}^{2}\sum_{i=0}^{r-1}(-1)^{i}\Theta^{1,p}_{s-1+i,i+1}\Theta^{1,p}_{r-1-i,i}\\ &+ f_{p+1}\sum_{i=0}^{\lfloor\frac{r-1}{2}\rfloor}\left(\Theta^{1,p}_{r-2i-1,0}\Theta^{1,p}_{s-1+2i+1,1}-\Theta^{1,p}_{r-2i-1,1}\Theta^{1,p}_{s-1+2i+1,0}\right)\\ &+ f_{p+1}\sum_{i=0}^{\lfloor\frac{r-2}{2}\rfloor}\left(\Theta^{1,p}_{r-1-2i-1,0}\Theta^{1,p}_{s+2i+1,1}-\Theta^{1,p}_{r-1-2i-1,1}\Theta^{1,p}_{s+2i+1,0}\right) - f_{p+1}\Theta^{1,p}_{r-1,0}\Theta^{1,p}_{s,1} - f_{p+1}^{2}\Theta^{1,p}_{s-1,1}\Theta^{1,p}_{r-1,0}\\ =&\sum_{i=0}^{s}(-1)^{i-1}\Theta^{1,p}_{s-i,i+1}\Theta^{1,p}_{r+i,i} + f_{p+1}^{2}\sum_{i=1}^{r-1}(-1)^{i}\Theta^{1,p}_{s-1+i,i+1}\Theta^{1,p}_{r-1-i,i} + f_{p+1}\sum_{i=1}^{r-1}(-1)^{i}\Theta^{1,p}_{r-1-i,i}\Theta^{1,p}_{s+i,i+1}\\ &+ f_{p+1}\sum_{i=0}^{r-1}(-1)^{i-1}\Theta^{1,p}_{r-1-i,i+1}\Theta^{1,p}_{s+i,i}\\ =&\sum_{i=0}^{s}(-1)^{i-1}\Theta^{1,p+1}_{s-i,i+1}\Theta^{1,p+1}_{r+i,i}\end{aligned}$$

where in the final equality we used (88) and (89). That implies that the second identity of (18) holds for $p+1$.

With r odd, s even and $r<s$. We do similarly as the previous case.

Case 2: $s=r=-k<0$, identity (18) still holds since

$$\{\Theta^{1,p}_{r,0},\Theta^{1,p}_{s,1}\}_f\{f_i\} = \{\Theta^{1,p}_{r,1},\Theta^{1,p}_{s,0}\}_t\{t_i=1/f_i\} = -\{\Theta^{1,p}_{s,0},\Theta^{1,p}_{r,1}\}_t\{t_i=1/f_i\}.$$

□

B Proving properties of Psi

B.1 Proof of Lemma 15

Proof. We prove (69) and (70) simultaneously by induction. We use the following property

$$\Psi^{a,b+1}_{r} = c_{b+2}\Psi^{a,b}_{r} + \Psi^{a,b-1}_{r-1} \tag{91}$$

and therefore we get

$$\frac{\partial\Psi^{a,b+1}_{r}}{\partial c_{b+2}} = \Psi^{a,b}_{r}. \tag{92}$$

We see that (1) and (2) hold for $p = 1$ and $p = 2$. Suppose (1) and (2) hold for $p-2, p-1$ and p. We need to prove that (1) and (2) hold for $p+1$. Expanding the right hand side of the first identity, we have

$$\begin{aligned}
\{\Psi_r^{1,p+1}, \Psi_s^{1,p+1}\}_c &= \{c_{p+2}\Psi_r^{1,p} + \Psi_{r-1}^{1,p-1}, c_{p+2}\Psi_s^{1,p} + \Psi_{s-1}^{1,p-1}\}_c \\
&= \{c_{p+2}\Psi_r^{1,p}, c_{p+2}\Psi_s^{1,p}\}_c + \{c_{p+2}\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, c_{p+2}\Psi_s^{1,p}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_{s-1}^{1,p-1}\}_c \\
&= c_{p+2}\Psi_s^{1,p}\{\Psi_r^{1,p}, c_{p+2}\}_c + c_{p+2}\Psi_r^{1,p}\{c_{p+2}, \Psi_s^{1,p}\}_c + \Psi_r^{1,p}\{c_{p+2}, \Psi_s^{1,p-1}\}_c \\
&\quad + \Psi_s^{1,p}\{\Psi_r^{1,p-1}, c_{p+2}\}_c + c_{p+2}\big(\{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c\big) \\
&= c_{p+2}\Big(\Psi_s^{1,p}\Psi_r^{1,p-1} - \Psi_r^{1,p}\Psi_s^{1,p-1} + \{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c\Big).
\end{aligned}$$

For the second identity, we also have

$$\begin{aligned}
\{\Psi_r^{1,p+1}, \Psi_s^{1,p}\}_c + \{\Psi_r^{1,p}, \Psi_s^{1,p+1}\}_c &= \{c_{p+2}\Psi_r^{1,p} + \Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c + \{\Psi_r^{1,p}, c_{p+2}\Psi_s^{1,p} + \Psi_{s-1}^{1,p-1}\}_c \\
&= \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c + \{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c - \Psi_r^{1,p}\Psi_{s-1}^{1,p-1} + \Psi_s^{1,p}\Psi_{r-1}^{1,p-1}
\end{aligned}$$

Now to prove (1) and (2) hold for $p+1$, we only need to prove that

$$T := \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c + \{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c - \Psi_r^{1,p}\Psi_{s-1}^{1,p-1} + \Psi_s^{1,p}\Psi_{r-1}^{1,p-1} = 0.$$

Using (91) and the induction assumption to expand T, we obtain

$$\begin{aligned}
T &= (c_{p+1}\Psi_s^{1,p-1} + \Psi_{s-1}^{1,p-2})\Psi_r^{1,p-1} - c_{p+2}(c_{p+1}\Psi_r^{1,p-1} + \Psi_{r-1}^{1,p-2})\Psi_s^{1,p-1} \\
&\quad + \{c_{p+1}\Psi_r^{1,p-1} + \Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, c_{p+1}\Psi_s^{1,p-1} + \Psi_{s-1}^{1,p-2}\}_c \\
&= \Psi_{s-1}^{1,p-2}\Psi_r^{1,p-1} - \Psi_{r-1}^{1,p-2}\Psi_s^{1,p-1} + \big(\{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_{s-1}^{1,p-2}\}_c\big) \\
&\quad + \Psi_r^{1,p-1}\{c_{p+1}, \Psi_{s-1}^{1,p-1}\} + \Psi_s^{1,p-1}\{\Psi_{r-1}^{1,p-1}, c_{p+1}\}_c \\
&= \Psi_{s-1}^{1,p-2}\Psi_r^{1,p-1} - c_{p+2}\Psi_{r-1}^{1,p-2}\Psi_s^{1,p-1} + \big(\{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_{s-1}^{1,p-2}\}_c\big) \\
&\quad - \Psi_r^{1,p-1}\Psi_{s-1}^{1,p-2} + \Psi_s^{1,p-1}\Psi_{r-1}^{1,p-2} \\
&= \big(\{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{1,p-1}\} + \{\Psi_{r-1}^{1,p-1}, \Psi_{s-1}^{1,p-2}\}\big) \\
&= 0.
\end{aligned}$$

That means (1) and (2) hold for $p+1$. ☐

B.2 Proof of Corollary 16

Proof.

1. It follows from the proof of Lemma 15.

2. It follows from Lemma 15.

3. Using (2) we have $\{\Psi_r^{0,p}, \Psi_s^{0,p}\}_c = 0$. On the other hand, we have

$$\begin{aligned}
\{\Psi_r^{0,p}, \Psi_s^{0,p}\}_c &= \{c_0\Psi_r^{1,p} + \Psi_{r-1}^{2,p}, c_0\Psi_s^{1,p} + \Psi_{s-1}^{2,p}\}_c \\
&= \{c_0\Psi_r^{1,p}, c_0\Psi_s^{1,p}\}_c + \{c_0\Psi_s^{1,p}, \Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p}, c_0\Psi_s^{1,p}\}_c + \{\Psi_{r-1}^{2,p}, \Psi_{s-1}^{2,p}\}_c \\
&= c_0\Psi_s^{1,p}\{\Psi_r^{1,p}, c_0\} + c_0\Psi_r^{1,p}\{c_0, \Psi_s^{1,p}\}_c + c_0\big(\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}_c\big) \\
&\quad + \Psi_s^{1,p}\{\Psi_{r-1}^{2,p}, c_0\}_c + \Psi_r^{1,p}\{c_0, \Psi_{s-2}^{2,p}\}_c \\
&= -c_0\Psi_s^{1,p}\Psi_r^{2,p} + c_0\Psi_r^{1,p}\Psi_s^{2,p} + c_0\big(\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p}\} + \{\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}\big)
\end{aligned}$$

   Therefore, we get $\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}_c = \Psi_s^{1,p}\Psi_r^{2,p} - \Psi_r^{1,p}\Psi_s^{2,p}$.

4. For this identity, we expand the left hand side. We have

$$\begin{aligned}\text{LHS} &= \{\Psi_r^{1,p}, \Psi_{s+1}^{0,p} - c_0\Psi_{s+1}^{1,p}\}_c + \{\Psi_{r+1}^{0,p} - c_0\Psi_{r+1}^{1,p}, \Psi_s^{1,p}\}_c \\ &= \{\Psi_r^{1,p}, \Psi_{s+1}^{0,p}\}_c + \{\Psi_{r+1}^{0,p}, \Psi_s^{1,p}\}_c - \Psi_{s+1}^{1,p}\{\Psi_r^{1,p}, c_0\}_c - \Psi_{r+1}^{1,p}\{c_0, \Psi_s^{1,p}\}_c \\ &= \{\Psi_r^{1,p}, \Psi_{s+1}^{0,p}\}_c + \{\Psi_{r+1}^{0,p}, \Psi_s^{1,p}\}_c + \Psi_{s+1}^{1,p}\Psi_r^{2,p} - \Psi_{r+1}^{1,p}\Psi_s^{2,p}.\end{aligned}$$

By property (3) we have

$$\begin{aligned}\{\Psi_r^{1,p}, \Psi_{s+1}^{0,p}\}_c + \{\Psi_{r+1}^{0,p}, \Psi_s^{1,p}\}_c &= \Psi_{s+1}^{0,p}\Psi_{r+1}^{1,p} - \Psi_{r+1}^{0,p}\Psi_{s+1}^{1,p} \\ &= (c_0\Psi_{r+1}^{1,p} + \Psi_r^{2,p})\Psi_{r+1}^{1,p} - (c_0\Psi_{r+1}^{1,p} + \Psi_r^{2,n})\Psi_{s+1}^{1,p} \\ &= \Psi_s^{2,p}\Psi_{r+1}^{1,p} - \Psi_r^{2,p}\Psi_{s+1}^{1,p}.\end{aligned}$$

It means that $\{\Psi_r^{1,p}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p}\}_c = 0$.

5. One can verify that this identity hold for $p = 1, 2, 3$. Now we expand the left hand side but replacing p with $p + 1$. We have

$$\begin{aligned}LHS &= \{\Psi_r^{1,p}, c_{p+2}\Psi_s^{2,p} + \Psi_{s-1}^{2,p-1}\}_c + \{c_{p+2}\Psi_r^{2,p} + \Psi_{r-1}^{2,p-1}, \Psi_s^{1,p}\}_c \\ &= c_{p+2}(\{\Psi_r^{1,p}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p}\}_c) + \Psi_s^{2,p}\{\Psi_r^{1,p}, c_{p+2}\}_c + \Psi_r^{2,p}\{c_{p+2}, \Psi_s^{1,p}\}_c \\ &\quad + \{\Psi_r^{1,p}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_s^{1,p}\}_c \\ &= \Psi_s^{2,p}\Psi_r^{1,p-1} - \Psi_r^{2,p}\Psi_s^{1,p-1} + \{c_{p+1}\Psi_r^{1,p-1} + \Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{2,p-1}\}_c \\ &\quad + \{\Psi_{r-1}^{2,p-1}, c_{p+1}\Psi_s^{1,p-1} + \Psi_{s-1}^{1,p-2}\}_c \\ &= \Psi_s^{2,p}\Psi_r^{1,p-1} - \Psi_r^{2,p}\Psi_s^{1,p-1} + c_{p+1}(\{\Psi_r^{1,p-1}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_s^{1,p-1}\}_c) \\ &\quad + \Psi_r^{1,p-1}\{c_{p+1}, \Psi_{s-1}^{2,p-1}\}_c + \Psi_s^{1,p-1}\{\Psi_{r-1}^{2,p-1}, c_{p+1}\}_c + \{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_{s-1}^{1,p-2}\}_c \\ &= \Psi_s^{2,p}\Psi_r^{1,p-1} - \Psi_r^{2,p}\Psi_s^{1,p-1} + c_{p+1}(\Psi_s^{1,p-1}\psi_r^{2,p-1} - \Psi_r^{1,p-1}\Psi_s^{2,p-1}) \\ &\quad - \Psi_r^{1,p-1}\Psi_{s-1}^{2,p-2} + \Psi_s^{1,p-1}\Psi_{r-1}^{2,p-2} + \{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_{s-1}^{1,p-2}\}_c \\ &= \{\Psi_{r-1}^{1,p-2}, \Psi_{s-1}^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_{s-1}^{1,p-2}\}_c.\end{aligned}$$

Therefore, using induction we prove our statement.

6. This identity follows from the proof of the identity in (5).

7. We have

$$\begin{aligned}\{\Psi_r^{1,p}, \Psi_{s-1}^{1,p-1}\}_c + \{\Psi_{r-1}^{1,p-1}, \Psi_s^{1,p}\}_c &= \{\Psi_r^{1,p}, \Psi_s^{1,p+1} - c_{p+2}\Psi_s^{1,p}\}_c \\ &\quad + \{\Psi_r^{1,p+1} - c_{p+2}\Psi_r^{1,p}, \Psi_s^{1,p}\}_c \\ &= -\Psi_s^{1,p}\{\Psi_r^{1,p}, c_{p+2}\}_c - \Psi_r^{1,p}\{c_{p+2}, \Psi_s^{1,p}\}_c \\ &= -\Psi_s^{1,p}\Psi_r^{1,p-1} + \Psi_r^{1,p}\Psi_s^{1,p-1}\end{aligned}$$

8. We prove by induction. This identity holds for $p = 1, 2$. Suppose that this identity holds for $p-2$,

and $p-1$ we need to prove that it holds for p. We have

$$\begin{aligned}
&\{\Psi_r^{1,p+1}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p+1}\}_c \\
&= \{c_{p+2}\Psi_r^{1,p} + \Psi_{r-1}^{1,p-1}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, c_{p+2}\Psi_s^{1,p} + \Psi_{s-1}^{1,p-1}\}_c \\
&= \{\Psi_{r-1}^{1,p-1}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_{s-1}^{1,p-1}\}_c - \Psi_r^{1,p}\Psi_s^{2,p-1} + \Psi_s^{1,p}\Psi_r^{2,p-1} \\
&= \{\Psi_{r-1}^{1,p-1}, c_{p+1}\Psi_s^{2,p-1} + \Psi_{s-1}^{2,p-2}\}_c + \{c_{p+1}\Psi_r^{2,p-1} + \Psi_{r-1}^{2,p-2}, \Psi_{s-1}^{1,p-1}\}_c \\
&\quad - \Psi_r^{1,p}\Psi_s^{2,p-1} + \Psi_s^{1,p}\Psi_r^{2,p-1} \\
&= \{\Psi_{r-1}^{1,p-1}, \Psi_{s-2}^{2,p-2}\}_c + \{\Psi_{r-2}^{2,p-2}, \Psi_{s-1}^{1,p-1}\}_c + c_{p+1}(\{\Psi_{r-1}^{1,p-1}, \Psi_s^{2,p-1}\}_c + \{\Psi_r^{2,p-1}, \Psi_{s-1}^{1,p-1}\}_c) \\
&\quad - \Psi_s^{2,p-1}\Psi_{r-1}^{1,p-2} + \Psi_r^{2,p-1}\Psi_{s-1}^{1,p-2} - \Psi_r^{1,p}\Psi_s^{2,p-1} + \Psi_s^{1,p}\Psi_r^{2,p-1} \\
&= c_{p+1}(\{\Psi_{r-1}^{1,p-1}, \Psi_s^{2,p-1}\}_c + \{\Psi_r^{2,p-1}, \Psi_{s-1}^{1,p-1}\}_c) + \Psi_s^{2,p-1}\Psi_{r-1}^{1,p-2} - \Psi_r^{2,p-1}\Psi_{s-1}^{1,p-2} \\
&\quad - \Psi_r^{1,p}\Psi_s^{2,p-1} + \Psi_s^{1,p}\Psi_r^{2,p-1}.
\end{aligned}$$

Since $\{\Psi_{r-1}^{1,p-1}, \Psi_s^{2,p-1}\}_c + \{\Psi_{r-1}^{2,p-1}, \Psi_s^{1,p-1}\}_c = 0$ (by (4)), we get $\{\Psi_{r-1}^{1,p-1}, \Psi_s^{2,p-1}\}_c = \{\Psi_s^{1,p-1}, \Psi_{r-1}^{2,p-1}\}_c$. Similarly, we obtain $\{\Psi_r^{2,p-1}, \Psi_{s-1}^{1,p-1}\}_c = \{\Psi_{s-1}^{2,p-1}, \Psi_r^{1,p-1}\}_c$. Therefore, we have

$$\begin{aligned}
\{\Psi_{r-1}^{1,p-1}, \Psi_s^{2,p-1}\}_c + \{\Psi_r^{2,p-1}, \Psi_{s-1}^{1,p-1}\}_c &= \{\Psi_s^{1,p-1}, \Psi_{r-1}^{2,p-1}\}_c + \{\Psi_{s-1}^{2,p-1}, \Psi_r^{1,p-1}\}_c \\
&= \Psi_r^{1,p-1}\Psi_s^{2,p-1} - \Psi_s^{1,p-1}\Psi_r^{2,p-1}.
\end{aligned}$$

Thus, we get $\{\Psi_r^{1,p+1}, \Psi_s^{2,p}\}_c + \{\Psi_r^{2,p}, \Psi_s^{1,p+1}\}_c = 0$.

9. We have

$$\begin{aligned}
LHS &= \{\Psi_r^{1,p}, \Psi_{s-1}^{2,p+1} - c_{p+2}\Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p+1} - c_{p+2}\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}_c \\
&= \{\Psi_r^{1,p}, \Psi_{s-1}^{2,p+1}\}_c + \{\Psi_{r-1}^{2,p+1}, \Psi_s^{1,p}\}_c - \Psi_{s-1}^{2,p}\Psi_r^{1,p-1} + \Psi_{r-1}^{2,p}\Psi_{s-1}^{1,p-1} - c_{p+2}(\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p}\}_c + \{\Psi_{r-1}^{2,p}, \Psi_s^{1,p}\}_c) \\
&= \{\Psi_r^{1,p}, \Psi_{s-1}^{2,p+1}\}_c + \{\Psi_{r-1}^{2,p+1}, \Psi_s^{1,p}\}_c - \Psi_{s-1}^{2,p}\Psi_r^{1,p-1} + \Psi_{r-1}^{2,p}\Psi_{s-1}^{1,p-1} - c_{p+2}(\Psi_s^{1,p}\Psi_r^{2,p} - \Psi_r^{1,p}\Psi_s^{2,p}).
\end{aligned}$$

Now we have

$$\begin{aligned}
&\{\Psi_r^{1,p}, \Psi_{s-1}^{2,p+1}\}_c + \{\Psi_{r-1}^{2,p+1}, \Psi_s^{1,p}\}_c \\
&= \{\Psi_r^{1,p}, \Psi_s^{0,p+1} - c_0\Psi_s^{1,p+1}\}_c + \{\Psi_r^{0,p+1} - c_0\Psi_r^{1,p+1}, \Psi_s^{1,p}\}_c \\
&= \{\Psi_r^{1,p}, \Psi_s^{0,p+1}\}_c + \{\Psi_r^{0,p+1}, \Psi_s^{1,p}\}_c - c_0(\{\Psi_r^{1,p}, \Psi_s^{1,p+1}\}_c + \Psi_r^{1,p+1}, \Psi_s^{1,p}) - \Psi_s^{1,p+1}\{\Psi_r^{1,p}, c_0\}_c \\
&\quad - \Psi_r^{1,p+1}\{c_0, \Psi_s^{1,p}\}_c \\
&= \Psi_s^{1,p+1}\Psi_r^{2,p} - \Psi_r^{1,p+1}\Psi_s^{2,p}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
LHS &= \Psi_s^{1,p+1}\Psi_r^{2,p} - \Psi_r^{1,p+1}\Psi_s^{2,p} - c_{p+2}(\Psi_s^{1,p}\Psi_r^{2,p} - \Psi_r^{1,p}\Psi_s^{2,p}) - \Psi_{s-1}^{2,p}\Psi_r^{1,p-1} + \Psi_{r-1}^{2,p}\Psi_{s-1}^{1,p-1} \\
&= \Psi_r^{2,p}\Psi_{s-1}^{1,p-1} - \Psi_s^{2,p}\Psi_{r-1}^{1,p-1} + \Psi_{r-1}^{2,p}\Psi_s^{1,p-1} - \Psi_{s-1}^{2,p}\Psi_r^{1,p-1} = RHS.
\end{aligned}$$

□

References

[1] Adler V, Bobenko A and Suris Y 2003 Classification of Integrable Equations on quad-graphs. The Consistency Approach *Commun. Math. Phys.* **233** 513-43

[2] Bruschi M, Ragnisco O, Santini P M and Gui-Zhang T 1991 Integrable symplectic maps *Physica D* **49** 273-94

[3] Capel H W, Nijhoff F W and Papageorgiou V G 1991 Completely integrability of Lagrangian mappings and lattices of KdV type *Phys. Lett.* **155A** 377–87

[4] A.Iatrou 2003 Higher dimensional integrable mappings *Physica D* 179 229–253

[5] Van der Kamp P H, Rojas O and Quispel G R W 2007 Closed-form expressions for integrals of sine-Gordon and MKDV maps *J.Phys A.: Math Gen.* **40** 12789–98

[6] Nijhoff F W, Capel H W and Papageorgiou V G 1992 Integrable quantum mappings, *Phys. Rev.* **A 46** 2155–58

[7] Papageorgiou V G, Nijhoff F W and Capel H W 1990 Integrable mappings and nonlinear integrable lattice equations *Phys. Lett. A* **147** 10614

[8] Quispel G R W, Capel H W, Papageorgiou V G and Nijhoff F W 1991 Integrable mappings derived from soliton equations *Physica A* **173** 24366

[9] Roberts J A Gand Quispel G R W, Symplectic structures for difference equations including integrable maps, *in preperation*

[10] Veselov A P 1991 Integrable maps, *Russian Math. Surveys* **46** 1–51

[11] Tran D T, van der Kamp P H and Quispel G R W, Closed-form expressions for integrals of traveling wave reductions of integrable lattice equations 2009 *J. Phys. A: Math. Theor.* **42** 225201

[12] Tran D T, Van der Kamp P H and Quispel G R W, Functional independence of of integrals of sine-Gordon, mKdV and pKdV maps, *in preparation*