Global classification of two-component approximately integrable evolution equations

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Abstract

We globally classify two-component evolution equations, with homogeneous diagonal linear part, admitting infinitely many approximate symmetries. Important ingredients are the symbolic calculus of Gel'fand and Dikiĭ, the Skolem–Mahler–Lech theorem, results on diophantine equations in roots of unity by F. Beukers, and an algorithm of C.J. Smyth.

1 Introduction

A long standing open problem is the classification, up to linear transformations, of two-component integrable equations

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} au_n + F(u, v, u_1, v_1, \ldots) \\ bv_n + G(u, v, u_1, v_1, \ldots) \end{pmatrix} \tag{1}$$

where F, G are purely nonlinear polynomials in variables u_i, v_i, which denote the i-th x-derivatives of $u(x,t), v(x,t)$. Among the many different approaches to recognition and classification of integrable equations, the so called symmetry approach has proven to be particularly successful, see for example [17, 27]. Until recently, all results obtained were for classes of equations at fixed (low) order n. This situation changed dramatically when, by using a symbolic calculus and results from number theory, Sanders and Wang classified scalar evolution equations with respect to symmetries globally, that is, where the order n can be arbitrarily high [22]. Our aim is to obtain a similar result for the class of multi-component equations (1).

In the symmetry approach the existence of infinitely many generalized symmetries is taken as the definition of integrability. A generalized symmetry of equation (1) is a pair of differential polynomials $S = (S_1, S_2)$ such that equation (1) is also satisfied by $\tilde{u} = u + \epsilon S_1$, $\tilde{v} = v + \epsilon S_2$ up to order ϵ^2. This leads to the notion of Lie-derivative: $\mathcal{L}(K)S = 0 \Leftrightarrow S$ is a symmetry of $(u_t, v_t) = K$.

The Lie algebra of pairs of differential polynomials is a graded algebra. The linear part (au_n, bv_n) has total grading 0, the quadratic terms have total grading 1, and so on. Gradings are used to divide the condition for the existence of a symmetry into a number of simpler conditions: $\mathcal{L}(K)S \equiv 0$ modulo quadratic terms, $\mathcal{L}(K)S \equiv 0$ modulo cubic terms, and so on. This has been called the perturbative symmetry approach [15]. In the same spirit the idea of an approximate symmetry was defined [16]. If $\mathcal{L}(K)S \equiv 0$ modulo cubic terms, we say that S is an approximate symmetry of degree 2. And, we call an equation approximately integrable if it has infinitely many approximate symmetries.

We contribute to the above mentioned problem by globally classifying equations (1) that are approximately integrable of degree 2. This is achieved by applying the techniques developed in the special case of so called $\mathcal{B}$-equations, where any approximate symmetry of degree 2 is a genuine symmetry [8]. It extends older results obtained by Beukers, Sanders and Wang [2, 3]. The present article is a revised and extended version of the report [9].

As remarked in [16] the requirement of the existence of approximate symmetries of degree 2 is very restrictive and highly non-trivial. On the other hand an equation may have infinitely many approximate symmetries of degree 2, but fail to have any symmetries. This problem involves conditions of higher grading and is left open.

2 Generalized symmetries

A symmetry-group transforms one solution to an equation to another solution of the same equation. We like to refer to the book of Olver [19] for a good introduction to the subject, numerous examples, applications and references. A completely algebraic description of the notion of infinitesimal symmetry can be found in [11]. That paper gives an overview on the application of number theory in the analysis of integrable evolution equations.

We denote $\mathcal{A} = \mathbb{C}[u, v, u_1, v_1, \ldots]$ and $\mathfrak{g} = \mathcal{A} \times \mathcal{A}$. We will endow $\mathfrak{g}$ with the structure of a Lie algebra. For any $K = (K_1, K_2) \in \mathfrak{g}$ the pair $S = (S_1, S_2) \in \mathfrak{g}$ is a *generalized symmetry* of the two-component evolution equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \tag{2}$$

if the *Lie derivative of S with respect to K*,

$$\mathcal{L}(K)S = \begin{pmatrix} \delta_K(S_1) - \delta_S(K_1) \\ \delta_K(S_2) - \delta_S(K_2) \end{pmatrix}, \tag{3}$$

vanishes. Here δ_Q is the prolongation of the evolutionary vector field with characteristic Q, cf. [19, equation 5.6],

$$\delta_{(Q_1,Q_2)} = \sum_{k=0}^{\infty} D_x^k Q_1 \frac{\partial}{\partial u_k} + D_x^k Q_2 \frac{\partial}{\partial v_k},$$

and the *total derivative* D_x is[1]

$$D_x = \sum_{k=0}^{\infty} u_{k+1}\frac{\partial}{\partial u_k} + v_{k+1}\frac{\partial}{\partial v_k}.$$

The Lie derivative is a representation of $\mathfrak{g}$. This property, with $P, Q \in \mathfrak{g}$,

$$\mathcal{L}(\mathcal{L}(P)Q) = \mathcal{L}(P)\mathcal{L}(Q) - \mathcal{L}(Q)\mathcal{L}(P) \tag{4}$$

corresponds to the Jacobi identity for the Lie bracket $[P, Q] = \mathcal{L}(P)Q$ which is clearly bilinear and antisymmetric, cf. [19, Proposition 5.15]. Another way of expressing (4) is saying that $\mathfrak{g}$ is a $\mathfrak{g}$-module. Another $\mathfrak{g}$-module is given by $\mathcal{A}$, the representation being $\mathcal{L}(K)F = \delta_K(F)$ with $K \in \mathfrak{g}, F \in \mathcal{A}$.

The word 'generalized' stresses the fact that the order of a symmetry can be bigger than one. Generally symmetries come in hierarchies with periodic gaps between their orders. For example, the Korteweg–De Vries equation $u_t = u_3 + uu_1$ possesses odd order symmetries only. Concurrently, the KDV equation has approximately symmetries at any order.

3 Grading

Denote $\sigma_u = (u, 0)$ and $\sigma_v = (0, v)$. If P in some $\mathfrak{g}$-module is an eigenvector of $\mathcal{L}(\sigma_u)$ (or of $\mathcal{L}(\sigma_v)$), the corresponding eigenvalue is called the u- (or v-) *grading* of P. One verifies that $\mathfrak{g}$ can be written as the direct sum $\mathfrak{g} = \bigoplus_{i,j\geq -1} \mathfrak{g}^{i,j}$ where elements of $\mathfrak{g}^{i,j}$ have u-grading i and v-grading j. Similarly $\mathcal{A} = \bigoplus_{i,j\geq 0} \mathcal{A}^{i,j}$.

The crucial property of a graded Lie algebra is that the u- (or v-) grading of $\mathcal{L}(P)Q$ is the sum of the u- (or v-) gradings of P and Q. This follows directly from equation (4). If P has u-grading i and v-grading j we say that $i + j$ is the *total grading* of P. For example, $(u_1v_2, v_3v_4) \in \mathfrak{g}^{0,1}$ has total grading 1. Gradings are used to divide the condition for the existence of a symmetry into a number of simpler conditions.

We study evolution equations of the form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = K^{0,0} + K^{-1,2} + K^{0,1} + K^{1,0} + K^{2,-1} + \cdots, \tag{5}$$

with $K^{0,0} = (au_n, bv_n)$ and symmetries of similar form $S = S^{0,0} + S^{-1,2} + S^{0,1} + S^{1,0} + S^{2,-1} + \cdots$ with $S^{0,0} = (cu_m, dv_m)$.[2] Here the dots may contain terms with total grading > 1. Certainly we have $\mathcal{L}(K^{0,0})S^{0,0} = 0$. The symmetry

[1] In [11, section 4.1] the total derivative was denoted δ_x. This is misleading as $D_x = \delta_{(u_1,v_1)}$. Also δ_Q is the unique $\mathbb{C}$-linear derivation on $\mathcal{A}$ satisfying $\delta_Q(u, v) = Q$ *and* $\delta_Q \circ D_x = D_x \circ \delta_Q$.

[2] We remark that only if $a = b$ then S may also contain terms $S^{\pm 1,\mp 1}$. In this paper we implicitly assume this does not happen.

conditions with total grading 1 are

$$\begin{array}{ll} \mathcal{L}(K^{-1,2})S^{0,0}+\mathcal{L}(K^{0,0})S^{-1,2} & =0, \\ \mathcal{L}(K^{0,0})S^{0,1}+\mathcal{L}(K^{0,1})S^{0,0} & =0, \\ \mathcal{L}(K^{0,0})S^{1,0}+\mathcal{L}(K^{1,0})S^{0,0} & =0, \\ \mathcal{L}(K^{0,0})S^{2,-1}+\mathcal{L}(K^{2,-1})S^{0,0} & =0. \end{array} \tag{6}$$

A criterion which guarantees that S is a symmetry of K if the first few symmetry conditions of total grading $0, 1, \dots$ are fulfilled, was given by Sanders and Wang [22]. In this paper we restrict ourselves to solving equations (6). Thus we classify equations that admit infinitely many approximate symmetries of degree 2, which is a necessary condition for integrability. In the sequel we omit the adjective 'of degree 2'.

4 The Gel'fand–Dikiĭ transformation

Comparing the Leibniz rule and Newton's binomial formula,

$$(uv)_n=\sum_{i=0}^{n}\binom{n}{i}u_i v_{n-i}, \qquad (x+y)^n=\sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i},$$

we see that differentiating a product is quite similar to taking the power of a sum. On the right hand side the index, counting the number of derivatives, gets interchanged with the power, while on the left hand side differentiation becomes multiplying with the sum of symbols. Of course, with expressions containing both indices and powers, one has to be more careful. The Gel'fand–Dikiĭ transformation [7] provides a one to one correspondence between $\mathcal{A}^{i,j}$ and the space $\mathbb{C}^{i,j}$: polynomials in $\mathbb{C}[x_1,\dots,x_i,y_1,\dots,y_j]$ that are symmetric in both the x and the y symbols. One may deduce the general rule from

$$u_1u_2v_3 \trianglerighteq \frac{x_1^1x_2^2+x_2^1x_1^2}{2!}\frac{y_1^3}{1!}=\widehat{u_1u_2v_3},$$

or consult one of the papers [11, 15]. All usual operations from differential algebra translate naturally. In particular,[3]

$$\mathcal{L}(K^{0,0})S^{i,j} \trianglerighteq \begin{pmatrix} \mathcal{G}^{i,j}_{1;n}[a,b] & 0 \\ 0 & \mathcal{G}^{i,j}_{2;n}[a,b] \end{pmatrix}\widehat{S^{i,j}},$$

where the so called $\mathcal{G}$-functions are given by

$$\begin{aligned} \mathcal{G}^{i,j}_{1;n}[a,b](x,y) \quad &= \quad a(x_1^n+\cdots+x_{i+1}^n)+b(y_1^n+\cdots+y_j^n) \\ & \quad\quad -a(x_1+\cdots+x_{i+1}+y_1+\cdots+y_j)^n, \end{aligned}$$

[3]As a correction to [11, Section 4.3], when $(f,g)\in\mathfrak{g}^{i,j}$ then $f\in\mathcal{A}^{i+1,j}$ and $g\in\mathcal{A}^{i,j+1}$. One should think of (f,g) as representing the vector field $f\partial_u+g\partial_v$.

and

$$\mathcal{G}^{i,j}_{2;n}[a,b](x,y) = \mathcal{G}^{j,i}_{1;n}[b,a](y,x). \tag{7}$$

Symbolically we can solve the symmetry conditions of total grading 1, equations (6), as follows. We may write the components of the quadratic parts of S as, with $k = 1, 2$,

$$\widehat{S^{i,j}_k} = \frac{\mathcal{G}^{i,j}_{k;m}[c,d]}{\mathcal{G}^{i,j}_{k;n}[a,b]}\widehat{K^{i,j}_k}. \tag{8}$$

Equation (5) has an approximate symmetry at order m with linear coefficients c, d iff for all $i + j = 1$ and $k = 1, 2$ the right hand side of equation (8) is either polynomial or undefined (0/0).

5 Nonlinear injectivity

In our classification we distinguish between equations whose symmetries necessarily have non-vanishing linear part and equations that allow purely nonlinear symmetries.

Definition 1 *Let K^0 have total grading 0. We call K^0* nonlinear injective *if $\mathcal{L}(K^0)S = 0$ implies that S has total grading 0. And, we call an equation nonlinear injective if its linear part is nonlinear injective.*

With $K^0 = (au_n, bv_n)$, the k-th component of $\mathcal{L}(K^0)S^{i,j}$, with non-zero $S^{i,j}$, vanishes iff $\mathcal{G}^{i,j}_{k,n}[a,b] = 0$. Solving the later equation with $i+j = 1$ yields $ab = 0$, $n \geq 0$, or $n = 1$, or $(a - 2b)(2a - b) = 0$, $n = 0$. In Table 1 we have displayed all K^0 and corresponding S^1, such that the equation $(u_t, v_t) = K^0 + K^1$, with arbitrary $K^1 \in \mathfrak{g}$ of total grading 1, has purely nonlinear symmetries S^1. For

K^0	$(0, v)$	$(2u, v)$	(au_1, v_1)	(u_1, v_1)	$(0, v_n)$, $n > 1$
S^1	$\mathcal{A}^{2,0} \otimes \mathcal{A}^{1,1}$	$\mathcal{A}^{0,2} \otimes 0$	$\mathcal{A}^{2,0} \otimes \mathcal{A}^{0,2}$	$\mathfrak{g}$	$\mathcal{A}^{2,0} \otimes 0$

Table 1: List of K^0 and S^1 such that $\mathcal{L}(K^0)S^1 = 0$.

the same choices of K^0 and S^1 the linear equations $(u_t, v_t) = K^0$ have symmetries $(cu_m, dv_m) + S^1$ for all $m \in \mathbb{N}$ and $c, d \in \mathbb{C}$. Indeed, every tuple $S \in \mathfrak{g}$ is a symmetry of $(u_t, v_t) = (u_1, v_1)$. In other words, (u_1, v_1) is a symmetry of every equation. Also, every $\mathcal{B}$-equation, that is, an equation $(u_t, v_t) = (au_n, bv_n) + K^1$ with $K^1 \in \mathcal{A}^{0,2}$, admits the zeroth order symmetry $(2u, v)$. Only a subset of the equations $(u_t, v_t) = K^0 + K^1$, with particular $K^1 \in \mathfrak{g}$, has infinitely many symmetries with non-vanishing linear part. These will be classified in section 10.1.

There is a good reason for including such equations: their approximate symmetries may correspond to approximately integrable nonlinear injective equations. One integrable example, equation (30), is given in section 11.

6 Necessary and sufficient conditions

In this section we introduce convenient notation, we give necessary and sufficient conditions for a nonlinear injective equation to be approximately integrable, and we outline how we perform the classification.

The components of equation (5) are

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} au_n + K_1^{1,0} + K_1^{0,1} + K_1^{-1,2} + \cdots \\ bv_n + K_2^{0,1} + K_2^{1,0} + K_2^{2,-1} + \cdots \end{pmatrix}. \tag{9}$$

We denote the symbolic representation of the 6-tuple $K_1^{1,0}$, $K_1^{0,1}$, $K_1^{-1,2}$, $K_2^{0,1}$, $K_2^{1,0}$, $K_2^{2,-1}$ by K^1. And similarly we write $S_1^{1,0}$, …, $S_2^{2,-1} \trianglerighteq S^1$ and $\mathcal{G}_n = \mathcal{G}_{1;n}^{1,0}, \ldots, \mathcal{G}_{2;n}^{2,-1}$. A 6-tuple H is called proper if it consists of polynomials with the right symmetry properties, that is, if $H \in \mathbb{C}^{2,0} \otimes \mathbb{C}^{1,1} \otimes \mathbb{C}^{0,2} \otimes \mathbb{C}^{0,2} \otimes \mathbb{C}^{1,1} \otimes \mathbb{C}^{2,0}$. So, K^1, S^1, and $\mathcal{G}_n[a,b]$ are proper. We will also consider s-tuples, with $s < 6$. It should be clear from the context in which space a proper s-tuple lives. We say that an s-tuple $H = H_1, \ldots, H_s$ divides an s-tuple $P = P_1, \ldots, P_s$ if $H_i \mid P_i$ for all $1 \leq i \leq s$ and we write $P/H = P_1/H_1, \ldots, P_s/H_s$. We are now able to state the following: Equation (9) is nonlinear injective and has an approximate symmetry of order m with linear coefficients c, d iff the 6-tuple $S^1 = \mathcal{G}_m[c,d]K^1/\mathcal{G}_n[a,b]$ is proper.

Let H be a proper tuple. With $m(H)$ we denote the set of all $m \in \mathbb{N}$ such that there exists $c, d \in \mathbb{C}$ for which $H \mid \mathcal{G}_m[c,d]$. We have the following lemma.

Lemma 2 *Equation (9) is nonlinear injective and approximately integrable iff there is a proper 6-tuple H with $m(H)$ infinite, such that $\mathcal{G}_n[a,b]$ divides K^1H.*

Proof: $\Leftarrow$ The fact that $\mathcal{G}_n[a,b]$ divides a proper tuple implies that equation (9) is nonlinear injective. The equation is integrable because for every $m \in m(H)$ there are c, d such that

$$S^1 = \frac{\mathcal{G}_m[c,d]}{H} \frac{K^1 H}{\mathcal{G}_n[a,b]}$$

is proper. $\Rightarrow$ Because equation (9) is nonlinear injective, the tuple $S^1 = \mathcal{G}_m[c,d]K^1/\mathcal{G}_n[a,b]$ is well defined for all m. The integrability implies that S^1 is proper for infinitely many $m \in \mathbb{N}$ and $c, d \in \mathbb{C}$. This only happens when $\mathcal{G}_n = HP$ factorizes such that $P \mid K^1$ and $m(H)$ is infinite. $\square$

According to Lemma 2, to classify approximately integrable nonlinear injective equations it suffices to determine the set $\mathcal{H}$ of all proper 6-tuples H with infinite $m(H)$. This will be done using results from number theory, provided in section 7. In section 8 we determine the proper divisors of infinitely many functions $\mathcal{G}_{k,m}^{i,1-i}$ for possible i,k. And, in section 9 we determine the proper divisors of infinitely many 2-tuples $\mathcal{G}_{1,m}^{i,1-i}$, $\mathcal{G}_{k,m}^{j,1-j}$, where $i \neq j$ if $k=1$. From those result $\mathcal{H}$ can be determined.

We organize $H \in \mathcal{H}$ by the lowest order n at which H divides a $\mathcal{G}_n$-tuple. Let $\mathcal{H}_n$ denote the set of all proper tuples H with infinite $m(H)$ whose smallest element is n. So, we have $\mathcal{H} = \bigcup_{n\in\mathbb{N}} \mathcal{H}_n$. For each $n \in \mathbb{N}$ the set $\mathcal{H}_n$ is related to the set of approximate integrable equations at order n, which are not in a lower order hierarchy.

We would like to provide an explicit but minimal list of approximate integrable equations from which one can derive all approximately integrable equations. The following observation is useful. Let P and Q be proper tuples. From Lemma 2 it follows that if equation (9), with $K^1 = P$, is approximately integrable, then the same equation, but with $K^1 = PQ$, is also approximately integrable. Therefore, the list we provide only contains equations with quadratic parts K^1 of minimal degree, see section 10.

From the results of sections 8, 9 it follows that $\mathcal{H}_n$ is non-empty for all $n \in \mathbb{N}$. That means there are new approximately integrable equations at every order. Therefore, although we classify them completely, we cannot explicitly list them all. In section 10 we provide a complete list of approximately integrable equations of order $n \leq 5$. We decided to go up to order 5, because both the cases $n < 4$, $n > 3$ and n odd, n even are quite distinct.

We explicitly provide the linear parts (cu_m, du_m) of all the symmetries of the equations in our list. This enables one to calculate any symmetry in principle, using the Maple code provided in the Appendix. In this paper we do not explicitly describe all symmetries of all approximately integrable equations that can be obtained from the list. We remark that if one multiplies the quadratic tuple of an equation with a proper tuple, the resulting equation may have more symmetries than the original one. It may also be in a lower hierarchy.

From Lemma 2 we know that if $H \in \mathcal{H}_n$ and $\mathcal{G}_n[a,b] \mid K^1 H$, then equation (9) is approximately integrable with approximate symmetries at (higher) order $m \in m(H)$. The following lemma applies.

Lemma 3 *Suppose $H \in \mathcal{H}_n$ and $\mathcal{G}_n[a,b]$ divides $K^1 H$. Then equation (9) has more symmetries than the ones at order $m \in m(H)$ iff there is a divisor $Q \in \mathcal{H}_{k\leq n}$ of H, with $m(H)$ smaller than and contained in $m(Q)$, such that $\mathcal{G}_n[a,b]$ divides $K^1 Q$.*

Proof: Given a divisor $Q \in \mathcal{H}_k$ of H such that $\mathcal{G}_n[a,b] \mid K^1 Q$, it is clear that equation (9) has a symmetry at every order $m \in m(Q)$ with

$$S^1 = \frac{\mathcal{G}_m[c,d]}{Q} \frac{K^1 Q}{\mathcal{G}_n[a,b]}.$$

To see that the converse holds, let Y denote the set of symmetries, with $m(H)$ smaller than and contained in Y. We need to prove that there is a Q such that $Y = m(Q)$. Take $m \in Y \setminus m(H)$ and write $\mathcal{G}_n = HP$. Since $\mathcal{G}_n \mid K^1 H$ we have $K^1 = PR$. The tuple $S^1 = \mathcal{G}_m K^1 / \mathcal{G}_n = \mathcal{G}_m R / H$ is proper. Since $m \notin m(H)$,

H does not divide $\mathcal{G}_m$. There is a proper divisor Q of H such that $Q \mid \mathcal{G}_m$ and H/Q divides R, that is, $\mathcal{G}_n \mid K^1 Q$. Since $Q \mid H$ the set $m(Q)$ is infinite. $\square$

In fact, one can also start with an equation that is not nonlinear injective, multiply its quadratic tuple, and end up in the hierarchy of an nonlinear injective equation. For example, apart from certain purely nonlinear symmetries, equation 1.1. has approximately symmetries with linear part (cu_m, dv_m) for any $c, d \in \mathbb{C}$ when m is odd. By multiplying its quadratic tuple with the tuple $[0, (f_1x_1 + f_2y_1)/f, (y_1 + y_2)/2, 0, (i_1x_1 + i_2y_1)/i, (x_1 + x_2)/2]$ we obtain the equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} au_1 + f_1u_1v + f_2uv_1 + gvv_1 \\ v_1 + i_1u_1v + i_2uv_1 + juu_1 \end{pmatrix},$$

which has approximate symmetries at all orders $m > 0$ for any $c, d \in \mathbb{C}$, and, it is in the hierarchy of an equation of the form 0.3 iff $f_1 = i_2 = 0$.

7 Results from number theory

Generally speaking, progress in classifying global classes of evolution equations has been going hand in hand with applying new results or techniques from number theory. For the classification of scalar equations [22] the new result was obtained by F. Beukers, who applied sophisticated techniques from diophantine approximation theory [1]. The Skolem–Mahler–Lech theorem first appeared in the literature in connection with symmetries of evolution equations in [2]. Beukers, Sanders and Wang used a partial corollary of the full theorem, stated below, to conjecture that there are only finitely many integrable equations (9) with $K^1 = [0, 0, 1, 0, 0, 0]$. This conjecture became a theorem in [3], where a recent algorithm of C.J. Smyth [4], that solves equations $f(x, y) = 0$ for roots of unity x, y, was used to produce an exhaustive list of the integrable cases. And, the classification of $\mathcal{B}$-equations was due to results on diophantine equations in roots of unity, again proved by Beukers.

However, as it turns out, we do not need a new result from number theory to globally classify two component evolution equations, with homogeneous diagonal linear part, admitting infinitely many approximate symmetries.

7.1 The Skolem–Mahler–Lech theorem

A sequence $\{U_m, m \in \mathbb{N}\}$ satisfies an order n linear recurrence relation if there exist $s_1, \ldots, s_n$ such that

$$U_{m+n} = s_1 U_{m+n-1} + \cdots s_n U_m.$$

The general solution can be expressed in terms of a generalized power sum

$$U_m = \sum_{i=1}^{k} A_i(m) \alpha_i^m,$$

such that the roots α_i are distinct and non-zero, and the coefficients $A_i(m)$ are polynomial in m. By definition the degree of U_m is $d = \sum_{i=1}^{k} d_i$, where d_i is the degree of $A_i(m)$. It can be shown that the order of the sequence equals $n = k + d$ [20].[4]

A generalized power sum vanishes identically, $U_m = 0$ for all m, precisely when all its coefficients vanish as polynomials in m, $A_i(m) = 0$ for all i. We prove this by induction on the degree. For $d = 0$ the statement is plain, the functions $h \to \alpha_i^h$ are linearly independent for distinct α_i. Let $S : f(m) \to f(m+1)$ be the shift operator. Suppose $d > 0$. Then for some i we have $d_i > 0$. The generalized power sum $V_m = (S-\alpha_i)U_m$ has degree $d-1$. By the induction hypothesis we have, in particular, $\alpha_i(S-1)A_i(h) = 0$. Since $\alpha_i \neq 0$ this implies $d_i = 0$ and hence we are done.

Theorem 4 (Skolem–Mahler–Lech) *The zero set of a linear recurrence sequence $\{m \in \mathbb{N} : U_m = 0\}$ is the union of a finite set and finitely many complete arithmetic progressions $\{r + gh : h \in \mathbb{N}\}$.*

The theorem was first proved by Skolem for the rational numbers [24], by Mahler for algebraic numbers [13], and by Lech for arbitrary fields of characteristic zero [12]. The proofs rely on p-adic analysis and consist of showing the existence of a difference $g \in \mathbb{N}$ such that every partial sum, with $0 \leq f < g$,

$$U_{f+gh} = \sum_{i=1}^{k} (A_k(f+gh)\alpha_i^f)(\alpha_i^g)^h \tag{10}$$

either has finitely many solutions h or vanishes identically. We refer to [18, 26], and references in there, for sensible sketches of a proof.

If (10) vanishes identically the sum on the right breaks up into disjoint pieces $I \subset \{1, \ldots, m\}$ each of which vanishes because the roots α_i^g, $i \in I$, coincide and the sum of their coefficients $\sum_{i \in I} A_i(f+gh)\alpha_i^f$ vanishes identically as a function of the variable h. Since $A_i(f+gh)$ does not vanish identically, each piece contains at least two terms. We may infer that if the diophantine equation $U_m = 0$ has infinitely many solutions m, at least there are two distinct roots α_i, α_j such that the ratio α_i/α_j is a root of unity. In practise, much more will be inferred. For example, when $k = 3$ the triple $\alpha_1/\alpha_2, \alpha_2/\alpha_3, \alpha_1/\alpha_3$ consists of roots of unity.

7.2 Diophantine equations in roots of unity

The following theorems were proved by F. Beukers for the classification of $\mathcal{B}$-equations [8, Theorem 22,25]. They are of crucial importance for the classification problem considered in this paper. We formulate the results a little sharper than in [8]. We do not assume that $\mu, \nu \neq -1$. In certain cases this follows from [8, Proposition 24], in others one has to rely on the following.

[4] In [20] one should replace equation 2.1.2 by equation 1.3 from [21].

Proposition 5 *If ν is a root of unity such that*

$$(1+\nu^m)2^{m-1} = (1-\nu)^m,$$

then $\nu = -1$ and m is even.

Proof: This can be seen as follows. By Galois theory we may assume that $\nu = e^{2\pi i/n}$. Using that $2|x|/\pi \le |1-e^{ix}|$ when $0 \le x \le \pi$, and $|1+e^{ix}| \le 2$ we have

$$\frac{1}{2^{m-1}} = \left|\frac{1+\nu^m}{(1-\nu)^m}\right| \le \frac{2}{(4/n)^m}.$$

It follows that $n \le 2$ and one verifies that $\nu \ne 1$ and $\nu \ne -1$ when m is odd. □

Also we do not in general need $\nu \ne \mu$ and $\nu \ne 1/\mu$. We note that in [8, Theorem 25] it was mistakenly supposed that $\mu^n \ne -1$. This should have been $\mu^n \ne \mp 1$ depending on the sign in [8, equation (10)].

Theorem 6 (Beukers) *Take $m > 1$ integer. Let μ, ν be distinct roots of unity, both not equal to 1, such that $\nu \ne \mu^{-1}$ when m is odd. Then*

$$(1-\nu^m)(1-\mu)^m = (1-\mu^m)(1-\nu)^m \tag{11}$$

implies $\mu^m = \nu^m = 1$.

Theorem 7 (Beukers) *Take $m > 1$ integer. Let μ, ν be distinct roots of unity, not both equal to 1, such that $\nu \ne \mu^{-1}$ when m is even. Then*

$$(1+\nu^m)(1-\mu)^m = (1+\mu^m)(1-\nu)^m \tag{12}$$

implies $\mu^m = \nu^m = -1$.

Theorem 8 (Beukers) *Take $m > 1$ integer. Let μ, ν be roots of unity with $\mu \ne 1$. Then*

$$(1+\nu^m)(1-\mu)^m = (1-\mu^m)(1-\nu)^m \tag{13}$$

implies $\mu^m = -\nu^m = 1$.

Whereas the Skolem–Mahler–Lech theorem implies that certain ratios are roots of unity for the equation to have infinitely many solutions, the above theorems tell us precisely what the solutions are. In particular, they imply that the zero sets consist of arithmetic progressions only.

8 Homogeneous quadratic parts

In this section we determine the proper divisors of infinitely many 1-tuples $\mathcal{G}_m = \mathcal{G}_{k,m}^{i,1-i}$ for all possible choices of i, k. We use the same notation for s-tuples as for 6-tuples. Thus, $m(H)$ denote the set of orders m such that there exist $c, d \in \mathbb{C}$ for which the s-tuple H divides the s-tuple $\mathcal{G}_m[c,d]$. And, $\mathcal{H}_n$

denotes the set of proper s-tuples H with infinite $m(H)$ whose smallest element is n.

Due to equation (7) we may take $k = 1$; equations of the form $(u_t, v_t) = (au_n, bv_n + K)$ are related, by the linear transformation $u \leftrightarrow v$, to equations of the form $(u_t, v_t) = (au_n + K, bv_n)$. We start with the simplest case $i = 1$.

8.1 Classifying approximately integrable scalar equations

The Lie derivative of the quadratic part S^1 of a possible scalar symmetry with respect to the linear part $K^0 = u_n$ of a scalar equation $u_t = K^0 + K^1 + \cdots$ is symbolically given by $\mathcal{L}(K^0)S^1 \trianglerighteq G_n^1 \widehat{S^1}$ with $\mathcal{G}$-function

$$\mathcal{G}_n^1(x, y) = x^n + y^n - (x + y)^n = \mathcal{G}_{1;n}^{1,0}[a,b](x, y)/a.$$

Thus the case $i = k = 1$ is equivalent to the scalar problem, which is easily seen by taking $v = 0$. The function is also proportional to $\mathcal{G}_{k,n}^{i,1-i}[a,a]$ so the results apply to the case $a = b$ as well.

In the classification of scalar equations [22] a different route was taken than the one we take. Sanders and Wang performed the classification with respect to the existence of symmetries. They showed in particular that any scalar equation admitting a generalized symmetry admits infinitely many generalized symmetries, which confirms the first part of the conjecture of Fokas [5]:

> If a scalar equation possesses at least one time-independent non-Lie point symmetry, then it possesses infinitely many. Similarly for N-component equations one needs N symmetries.

We note that the conjecture of Fokas does not hold inside the class of $\mathcal{B}$-equations [10]. In their classification Sanders and Wang relied on the following 'hard to obtain' result from number theory, proved in [1].

Theorem 9 (Beukers) *Let $r \in \mathbb{C}$ such that $r(r + 1)(r^2 + r + 1) \neq 0$. Then at most one integer $m > 1$ exists such that $\mathcal{G}_m^1(1, r) = 0$.*

In contrast, we classify the equations with respect to (approximate) integrability. Thus, we only need the following 'easy to obtain' result. For obvious reasons we do not include the constant divisors in $\mathcal{H}_0$ in our lists.

Proposition 10 *The proper divisors of infinitely many $\mathcal{G}_{1;m}^{1,0}[c,d](1, y)$ are products of*

1. $y \in \mathcal{H}_2$, $m > 1$
2. $(1 + y) \in \mathcal{H}_3$, $m \equiv 1 \mod 2$
3. $1 + y + y^2 \in \mathcal{H}_5$, $m \equiv 1, 5 \mod 6$
4. $(1 + y + y^2)^2 \in \mathcal{H}_7$, $m \equiv 1 \mod 6$

Proof: According to the Skolem–Mahler–Lech theorem if the diophantine equation $\mathcal{G}^1_m(1,r)=0$ has infinitely many solutions m, then $r=0,-1$ or r and $r+1$ are both roots of unity, in which case r is a primitive 3-rd root of unity. The orders are found by substituting the values for r. We have $\mathcal{G}^1_m(1,0)=0$ for all m, $\mathcal{G}^1_{f+2h}(1,-1)=1+(-1)^f=0$ when $f=1$, and, with $1+r+r^2=0$, $\mathcal{G}^1_{f+6h}(1,r)=1+r^f-(1+r)^f=0$ when $f=1$ or $f=5$. Finally, by solving the simultaneous equations $\mathcal{G}^1_m(1,r)=\partial_r\mathcal{G}^1_m(1,r)=0$ we find that r is a double zero when both r and $1+r$ are $(m-1)$-st roots of unity. $\square$

The reader should compare Proposition 10 with Theorem 9. As a particular corollary of Proposition 10 we have the following. Equation (9) with $a=b$ and $n=2,3,5,7$ is approximately integrable.

8.2 $\mathcal{B}$-equations

The case $i=-2$ has been globally classified with respect to integrability in [8]. This class of equations is particularly nice because any approximate symmetry is a symmetry. We go through the main ideas and formulate the results slightly different from [8], minimizing the role of biunit coordinates. This makes the argument cleaner and sets the stage for the main results of this paper.

Proposition 11 *All proper divisors H of $\mathcal{G}^{-1,2}_{1;m}[c,d](1,y)$ with $m(H)$ infinite can be obtained from the following list.*

1. $1+y\in\mathcal{H}_1$, $m\equiv 1 \mod 2$, $d\neq 0$
2. $(1+y)^n\in\mathcal{H}_n$, $m\geq n$, $d=0$
3. $(y-r)(ry-1)\in\mathcal{H}_2$, $r\neq -1$, $m\geq 1$
4. $(y-r)^2(ry-1)^2\in\mathcal{H}_n$, $r\neq -1$, $n>3$ *the smallest integer such that* $r^{n-1}=1$, $m\equiv 1 \mod n-1$
5. $(y-r)(yr-1)(y-\bar r)(y\bar r-1)\in\mathcal{H}_n$, $r=\nu(\mu-1)/(\nu-1)$, μ,ν *roots of unity such that* $(\mu-1)(\nu-1)(\mu-\nu)(\mu\nu-1)\neq 0$, $n>3$ *the smallest integer such that* $\mu^n=\nu^n=1$, $m\equiv 0 \mod n$
6. $\{r^n=-1\}\in\mathcal{H}_n$, $m\equiv n \mod 2n$, $c=0$

Unless stated otherwise, the coefficients of the linear part of the symmetries satisfy $c/d=(1+r^m)/(1+r)^m$.

Proof: We study the zeros of the function

$$\mathcal{G}^{-1,2}_{1;n}[a,b](1,r)=b(1+r^n)-a(1+r)^n.$$

Take $b\neq 0$. Then $r\neq -1$ is a zero when

$$\frac{a}{b}=\frac{1+r^n}{(1+r)^n}, \tag{14}$$

in which case $1/r$ is a zero as well. The point $r=-1$ is a zero when n is odd, where it has multiplicity 1, or when $b=0$, where the multiplicity is n. The other multiple zeros are obtained from setting the r-derivatives of the function to zero, see also [2]. Taking $r \neq -1$ and solving the simultaneous equations $\mathcal{G}_{1;n}^{-1,2}(1,r) = \partial_r \mathcal{G}_{1;n}^{-1,2}(1,r) = 0$ yields $r^{n-1}=1$, while $\partial_r \mathcal{G}_{1;n}^{-1,2}(1,r) = \partial_r^2 \mathcal{G}_{1;n}^{-1,2}(1,r) = 0$ yields $r=-1$. Therefore, all multiple zeros $r \neq -1$ are double zeros. We have $a/b = 1/(1+r)^{n-1}$ and $1/r$ is a double zero as well. Moreover, r and $1/r$ are also double zeros of $\mathcal{G}_{1;m}^{-1,2}[1,(1+r)^{m-1}]$ with $m \equiv 1 \mod n-1$. There are no other double zeros since the equations $|r|=|s|$ and $|1+r|=|1+s|$ imply that $r=s$ or $r=\bar{s}$. To classify higher degree divisors we have to find all $r,s \in \mathbb{C}$, with $(1+r)(1+s)(r-s)(rs-1) \neq 0$ such that the diophantine equation

$$\begin{aligned} U_n(r,s) &= \mathcal{G}_{1,n}^{-1,2}[1+r^n,(1+r)^n](1,s) \\ &= (1+r)^n + ((1+r)s)^n - (1+s)^n - ((1+s)r)^n = 0 \end{aligned}$$

has infinitely many solutions n. According to the Skolem–Mahler–Lech theorem either $rs=0$ or one of the pairs

$$\frac{1+r}{1+s}, \frac{(1+s)r}{(1+r)s} \quad \text{or} \quad \frac{1+r}{r(1+s)}, \frac{1+s}{s(1+r)} \quad \text{or} \quad r,s \tag{15}$$

consists of roots of unity. When $rs=0$ we have $a=b$ which we exclude. Suppose the first pair of (15) consists of roots of unity, let $\mu = (1+r)/(1+s)$ and $\nu = (1+1/s)/(1+1/r)$. We may write $r = \mathcal{M}(\mu,\nu)$, where

$$\mathcal{M}(\mu,\nu) = \nu \frac{\mu-1}{\nu-1},$$

and find that $s = \mathcal{M}(1/\mu, 1/\nu) = \bar{r}$. In terms of roots of unity μ,ν we have

$$U_n(r,s) = \left(\frac{1-\mu\nu}{\mu(1-\nu)^2}\right)^n \left((1-\mu)^n(1-\nu^n) - (1-\nu)^n(1-\mu^n)\right).$$

Note that $(\mu-1)(\nu-1)(\mu-\nu)(\mu\nu-1) \neq 0$ because $(r-s)(rs-1) \neq 0$. Hence, using Theorem 6, we obtain $\mu^n = \nu^n = 1$, which implies that $\mu^m = \nu^m = 1$ when $m \equiv 0 \mod n$. Next, suppose the second pair of (15) consists of roots of unity. By a transformation $r \to 1/r$ we get the first pair. Since $\mathcal{M}(a,b)^{-1} = \mathcal{M}(1/b,1/a)$ we get the same solutions, but with $s = 1/\bar{r}$. Finally, when r,s are roots of unity $U_n(r,s)=0$ can be written in terms of $\mu=-r, \nu=-s$,

$$U_n(r,s) = (1-\mu)^n(1-(-\nu)^n) - (1-\nu)^n(1-(-\mu)^n).$$

When n is odd Theorem 6 applies and when n is even Theorem 7 applies. □

Consider the set of points

$$\{r \in \mathbb{C} : r = \mathcal{M}(\mu,\nu), \mu^n = \nu^n = 1, (\mu-1)(\nu-1)(\mu-\nu)(\mu\nu-1) \neq 0\}. \tag{16}$$

To illustrate where these points lie in the complex plane we use biunit coordinates. Suppose ψ, ϕ are such that $|\psi| = |\phi| = 1$ and r is the unique intersection point of the lines $\psi\mathbb{R}$ and $\phi\mathbb{R} - 1$. Then $r = \mathcal{R}(\psi, \phi)$, with

$$\mathcal{R}(\psi, \phi) = \psi^2 \frac{\phi^2 - 1}{\psi^2 - \phi^2},$$

and (ψ, ϕ) are called the biunit coordinates of r. Denote further

$$\mathcal{R}(A, B) = \{r \in \mathbb{C} : r = \mathcal{R}(a, b), a \in A, b \in B, a^2 \neq b^2\},$$

and

$$\Phi_n = \{r \in \mathbb{C} : r^n = 1, r^2 \neq 1\}.$$

Using the algebraic relation $\mathcal{M}(\phi^2, \psi^2/\phi^2) = \mathcal{R}(\psi, \phi)$ one verifies that the set (16) is equal to $\{r \in \mathcal{R}(\Phi_{2n}, \Phi_{2n}) : |r| \neq 1\}$. For $n = 7$ the upper half of this set is plotted in Figure 1.

Figure 1: The corner points r, with $|r| \neq 1$ and $r \neq 0$ satisfy $U_m(r, \bar{r}) = 0$ when $m \equiv 0 \bmod 7$. The two circles are $|r| = 1$ and $|r + 1| = 1$.

8.3 Quadratic terms bilinear in u-, and v-derivatives

This section deals with the case $i = 0$.

Proposition 12 *If H is a proper divisor of $\mathcal{G}^{0,1}_{1;m}[c,d](1,y)$ with $m(H)$ infinite then H is a product of the following polynomials.*

1. $y \in \mathcal{H}_1$, $m > 0$, $c \neq 0$
2. $y \in \mathcal{H}_n$, $m \geq n$, $c = 0$
3. $(y - r) \in \mathcal{H}_2$, $r \neq 0$, $m > 1$
4. $(y-r)(y(1+r)+r) \in \mathcal{H}_3$, $m \equiv 1 \mod 2$. *When* $(1+r)^{2n} = 1$, $r \neq 0$, *we have extra symmetry at* $m \equiv 0 \mod 2n$ *with* $d = 0$.
5. $(y - r)^2 \in \mathcal{H}_{2n}$, $r \neq 0$, *n the smallest integer such that* $(1 + r)^{2n-1} = 1$, $m \equiv 1 \mod 2n - 1$
6. $(y - r)^2(y(1 + r) + r)^2 \in \mathcal{H}_{2n+1}$, $n > 1$ *the smallest integer such that* $(1+r)^{2n} = 1$, $m \equiv 1 \mod 2n$
7. $(y-r)(y(1+r)+r)(y-\bar{r})(y(1+\bar{r})+\bar{r}) \in \mathcal{H}_n$, $n > 3$ *odd*, $r = (\mu-\nu)/(\nu-1)$, $(\mu - 1)(\nu - 1)(\mu - \nu)(\mu\nu - 1) \neq 0$, *$n$ the smallest integer such that* $\mu^n = \nu^n = 1$, $m \equiv n \mod 2n$
8. $(y - r)(y - \bar{r}) \in \mathcal{H}_n$, $n > 2$ *even*, $r = (\mu - \nu)/(\nu - 1)$, $(\mu - 1)(\nu - 1)(\mu - \nu)(\mu\nu - 1) \neq 0$, *$n$ the smallest integer such that* $\mu^n = \nu^n = 1$, $m \equiv 0 \mod n$
9. $(y-r)(y(1+\bar{r})+\bar{r}) \in \mathcal{H}_n$, $n > 2$ *even*, $r = (\nu-\mu)/(\mu-1)$, $(\mu-\nu)(\mu\nu-1) \neq 0$, *$n$ the smallest integer for which* $\mu^n = \nu^n = -1$, $m \equiv n \mod 2n$

Unless stated otherwise, the coefficients of the linear part of the symmetries satisfy $c/d = r^m/((1 + r)^m - 1)$.

Proof: We are after the zeros of infinitely many

$$\mathcal{G}^{0,1}_{1;n}[a,b](1, y) = a - a(1 + y)^n + by^n.$$

Take $a \neq 0$. Then $r \neq 0$ is a zero of precisely when

$$\frac{b}{a} = \frac{(1 + r)^n - 1}{r^n}. \tag{17}$$

When n is odd $-r/(1 + r)$ is a zero as well. The point $r = 0$ is a zero for all a, b, n. It has multiplicity 1, except when $a = 0$ where the multiplicity is n. One can show that the multiple zeros $r \neq 0$ of $\mathcal{G}^{0,1}_{1;n}[a,b]$ are the double zeros $\{r \neq 0 : (1 + r)^{n-1} = 1\}$, with $a/b = r^{n-1}$. When r is a double zero the only other double zero is $\bar{r} = -r/(1 + r)$ when n is odd. Higher degree divisors are given by distinct non-zero $r,s\in\mathbb{C}$, with $r+rs+s\neq 0$ when n is odd, such that the diophantine equation

$$\begin{aligned} U_n(r,s) &= \mathcal{G}^{0,1}_{1,n}[r^n,(1+r)^n-1](1,s) \\ &= r^n-r^n(1+s)^n+s^n(1+r)^n-s^n=0 \end{aligned}$$

has infinitely many solutions n. We exclude the cases $r=-1$, $s=-1$, as this corresponds to equations with $a=b$. Then, according to the the Skolem–Mahler–Lech theorem, at least one of the pairs

$$\frac{r}{s},\frac{r(1+s)}{s(1+r)} \quad \text{or} \quad \frac{s}{r(1+s)},\frac{s}{r}(1+r) \quad \text{or} \quad 1+r,1+s \tag{18}$$

consists of roots of unity. Suppose the first pair consist of roots of unity. Let $\mu=r/s$ and $\nu=r(1+s)/s/(1+r)$. Then $(\mu-1)(\nu-1)(\mu-\nu)\neq 0$, $\mu\nu\neq 1$ when n odd, $r=\mathcal{N}(\mu,\nu)$ and $s=\mathcal{N}(1/\mu,1/\nu)=\bar{r}$ with

$$\mathcal{N}(\mu,\nu)=\frac{\mu-\nu}{\nu-1}.$$

When $\mu\nu=1$ and n even we have $\bar{r}=-r/(1+r)$. In terms of μ,ν we get

$$U_m(r,s)=\left(\frac{\nu-\mu}{\mu(\nu-1)^2}\right)^m((1-\mu)^m(1-\nu^m)-(1-\nu)^m(1-\mu^m)),$$

which implies, using Theorem 6, that $\mu^n=\nu^n=1$ and $m\equiv 0 \bmod n$. In bi-unit coordinates we have $r\in\mathcal{R}(\Phi_{2n},\Phi_{2n})$ such that $|r+1|\neq 1$ when n odd.

Next, suppose that the second pair of (18) consists of roots of unity, $\mu=-r/s/(1+r)$, $\nu=-(1+s)r/s$. We have $(\mu-1)(\nu-1)(\mu-\nu)(\mu\nu-1)\neq 0$ when $r+rs+s\neq 0$, that is, when n odd. When $r+rs+s=0$ and n even we get $(1+r)^n=1$, which corresponds to $b=0$. Otherwise, $r=\mathcal{K}(\mu,\nu)=(\nu-\mu)/(\mu-1)$ and $s=-\bar{r}/(1+\bar{r})$. In terms of μ,ν we have

$$U_n(r,s)=\left(\frac{\nu-\mu}{\mu(\mu-1)(\nu-1)}\right)^n((1-\nu)^n(1+(-\mu)^n)-(1-\mu)^n(1+(-\nu)^n))$$

When n is odd Theorem 6 implies $\mu^n=\nu^n=1$, while for n even Theorem 7 yields $\mu^n=\nu^n=-1$. The biunit coordinate description can be found as follows. Solve the simultaneous equations $\mathcal{K}(\mu,\nu)=\mathcal{R}(\psi,\phi)$, $\mathcal{K}(1/\mu,1/\nu)=\mathcal{R}(1/\psi,1/\phi)$ to find that $\mu=\psi^2/\phi^2,\nu=\psi^2$. For odd n we don't find new values for r, but for n even we get $r\in\mathcal{R}(\Phi_{4n}\setminus\Phi_{2n},\Phi_{2n})$ such that $|r+1|\neq 1$. Finally, suppose that the last pair of (18) consists of roots of unity. Then $\mu=1+r$ and $\nu=1+s$ satisfy equation (11). According to Theorem 6 we have $(1+r)^n=(1+s)^n=1$, that is, the second eigenvalue equals 0. □

Actually, when n is odd the two cases $i=-1,i=0$ are related. We have

$$\mathcal{G}^{0,1}_{1;n}[a,b](1,r)=\mathcal{G}^{-1,2}_{1;n}[b,a](1,-1-r). \tag{19}$$

Indeed, at odd order n the zero $r=-1$ of $\mathcal{G}_{1,n}^{-1,2}$ translates into the zero $r=0$ of $\mathcal{G}_{1,n}^{0,1}$. Also the image of the unit circle $|z|=1$ under $f_3 : r \to -1-r$ is the unit circle $|z+1|=1$, relating the double zeros of the two $\mathcal{G}$-functions. The symmetry $f_2 : r \to 1/r$ is translated into $f_4 = f_3 \circ f_2 \circ f_3 : r \to -r/(1+r)$. And we note that set $\mathcal{R}(\Phi_n, \Phi_n)$ is invariant under the group of an-harmonic ratios, generated by f_2 and f_3, cf. [14]. Using the above, for odd n one may obtain Propostion 12 from Proposition 11 and vise versa.

Summarizing this section, it implies that equations with homogeneous quadratic parts are approximately integrable when $n < 4$. At any order $n \geq 4$ a finite number of new approximately integrable equations has been found.

9 Non-homogeneous quadratic parts

This section deals with equations whose quadratic part is not homogeneous, that is, $K^1 = K_1^{i,1-i}, K_k^{j,1-j}$ with $i \neq j$ when $k=1$. We provide the corresponding sets $\mathcal{H}_n$ of 2-tuples. This time we do find conditions on the ratio a/b for low orders $n < 4$.

When $i=1$ the first part of the condition $H \in \mathcal{H}_n$, H_1 being a divisor of infinitely many $\mathcal{G}_{1;n}^{1,0}$, does not give conditions on a/b, see Proposition 10. In this case the $\mathcal{H}_n$ are obtained from the classification of H_2 dividing infinitely many $\mathcal{G}_{k;m}^{j,1-j}$, which was obtained in the previous section. A similar remark can be made when $(j,k)=(0,2)$. Due to equation (7) there are four cases left to consider, with $k=1$: $(i,j)=(-1,0)$; and with $k=2$: $(i,j)=(0,1)$, $(i,j)=(-1,2)$, $(i,j)=(-1,1)$.

There are certain divisors of infinitely many $\mathcal{G}_m[c,d]$-functions for any value of c/d. These will be called trivial divisors. Apart from the constant divisors we have

$$\begin{array}{ll} (1+y) \mid \mathcal{G}_{1;2m+1}^{-1,2}(1,y) & \quad y \mid \mathcal{G}_{1;m}^{-1,2}(1,y) \\ (x+1) \mid \mathcal{G}_{2;2m+1}^{2,-1}(x,1) & \quad x \mid \mathcal{G}_{2;m}^{1,0}(x,1) \end{array}$$

We may take H_1 (or H_2) to be trivial. Then $H \in \mathcal{H}$ if H_2 (H_1) is one of the divisors of infinitely many $\mathcal{G}$-functions presented in the previous section. In the sequel we assume that neither H_1 nor H_2 is trivial. Also we will assume that $ab(a-b) \neq 0$.

Proposition 13 *We list the non-trivial divisors H of the 2-tuple $\mathcal{G}_{1;m}^{-1,2}[c,d](1,y)$, $\mathcal{G}_{2;m}^{1,0}[c,d](x,1)$ with $m(H)$ infinite. Firstly suppose n is odd and $P(y)$ divides $\mathcal{G}_{1;m}^{-1,2}[c,d](1,y)$ with infinite $m(P)$ whose smallest element is n, cf. Proposition 11. Then $P(y), P(-1-x) \in \mathcal{H}_n$. Secondly, when n is even we have:*

1. $(y-r)(ry-1), x+1 \in \mathcal{H}_2$, $r \in \Phi_3'$, $m \equiv 2,4 \mod 6$
2. $(y-r)^2(ry-1)^2, x+1 \in \mathcal{H}_4$, $r \in \Phi_3'$, $m \equiv 4 \mod 6$
3. $(y-r)(ry-1), \bar{r}x+\bar{r}+1 \in \mathcal{H}_n$, $r = -\nu(\mu-1)/\mu/(\nu-1)$, $\mu \neq 1$, *n the lowest integer such that* $\mu^n = -\nu^n = 1$, $m \equiv n \mod 2n$

The linear coefficients of the symmetries satisfy $c/d = (1+r^m)/(1+r)^m$.

Proof: When the order of the equation n is odd, no new conditions on the linear part are obtained since the relations (7) and (19) imply that

$$\mathcal{G}^{-1,2}_{1;m}[c,d](1,-1-r) = \mathcal{G}^{1,0}_{2;m}[c,d](r,1).$$

For even n, there should be $r \in \mathbb{C}$ with $s \neq 0$ such that

$$\mathcal{G}^{-1,2}_{1;m}[c,d](1,r) = \mathcal{G}^{1,0}_{2;m}[c,d](s,1) = 0,$$

or, equivalently,

$$U_m(r,s) = s^m + (rs)^m + (1+r)^m - ((1+r)(1+s))^m = 0 \tag{20}$$

for infinitely many m including n. Then, using the Skolem-Mahler-Lech theorem, we may infer that either $rs(1+r)(1+s) = 0$ or at least one of the pairs

$$r, 1+s \qquad \frac{rs}{1+r}, \frac{s}{(1+s)(1+r)}, \qquad \frac{rs}{(1+r)(1+s)}, \frac{s}{1+r} \tag{21}$$

consists of roots of unity. When $r(r+1) = 0$ we have $a = b$ or $b = 0$, which we excluded. When $s = -1$ we are left with the equation $U_m = (-1)^m + (-r)^m + (1+r)^m = 0$. Applying the Skolem–Mahler–Lech theorem we see that both r and $1+r$ are roots of unity and hence, that r is a primitive third root of unity. One verifies that $U_{i+3k} = \left((-1)^i + (-r)^i + (1+r)^i\right)(-1)^k = 0$ when i equals 1 or 2. Also, if -1 is a zero of $\mathcal{G}^{1,0}_{2;m}[a,b]$, then $a/b = (-1)^{m+1}$.

Suppose the first pair of (21) consists of roots of unity. Writing equation (20) in terms of $\mu = 1+s$, $\nu = -r$, we get equation (13). Theorem 8 then implies $\mu^m = -\nu^m = 1$, which corresponds to the case $a = 0$, which we excluded. Suppose the second pair of (21) consists of roots of unity. Then $\mu = s/(1+s)/(1+r)$ and $\nu = -rs/(1+r)$ are roots of unity, and we get $r = \mathcal{K}(u,v) = -\nu(\mu-1)/\mu/(\nu-1)$, $s = -(1+\bar{r})/\bar{r}$, and

$$U_n = \left(\frac{\mu-\nu}{(\mu-1)(\nu-1)\mu}\right)^n ((1+(-\nu)^n)(1-\mu)^n - (1-\mu^n)(1-nu)^n).$$

When n is even, Theorem 8 yields $\mu^n = -\nu^n = 1$ or $\mu = 1$. But, when $\mu = 1$ we have $s = -(1+r)/r$ and $U_{2n} = 2(1+r)^{2n} = 0$ iff $r = -1$, which we excluded. Using $\mathcal{K}(1/\phi^2, \psi^2/\phi^2) = \mathcal{R}(\psi,\phi)$ we may write $r \in \mathcal{R}(\Phi_{4n} \setminus \Phi_{2n}, \Phi_{2n})$.

The third pair of (21) is obtained from the second by $f_2 : r \to 1/r$. Under this transformation we have $\mathcal{R}(\psi,\phi) \to \mathcal{R}(\psi^{-1}, \phi\psi^{-1})$. Hence we get the solutions $r \in \mathcal{R}(\Phi_{4n} \setminus \Phi_{2n}, \Phi_{4n} \setminus \Phi_{2n})$ and $s = -1-\bar{r}$. Or one can express $U_m = 0$ in terms of $\mu = rs/(1+r)/(1+s)$, $\nu = -(1+r)/s$ to find these values. Another way of describing the last item would be: *3.* $[(y-r)(ry-1), x+\bar{r}+1] \in \mathcal{H}_n$, *$r = \mu(\nu-1)/(\mu-1)$, $\mu \neq 1$, n the lowest integer such that $\mu^n = -\nu^n = 1$, $m \equiv n \mod 2n$* $\square$

In the remaining cases the diophantine equation we obtain from the zeros of the $\mathcal{G}$-functions will be of the form

$$(1 + aA^m)(1 + bB^m) + cC^m = 0. \tag{22}$$

Lemma 14 *Suppose that the diophantine equation (22), with $ABC \neq 0$, has infinitely many solutions. Then A, B, and C are roots of unity.*

Proof: As a corollary to the Skolem–Mahler–Lech theorem 7.1, three of the numbers $1, A, B, AB, C$ have a root of unity as a ratio and the same is true for the remaining two. Therefore at least one of the pairs C, A; C, B; $C/A, B$; $C/B, A$ consists of roots of unity. When C and A are roots of unity, their powers yield a finite number of values. Moreover, for the infinite number of solutions we have $(1 + aA^m) \neq 0$. Hence, for these infinite number of solutions $(1 + bB^m)$ has only finitely many values. This only happens when B is a root of unity. The other cases lead to the same result, e.g. when C/A and B are roots of unity we divide the equation by A^m and find that A is a root of unity. □

Suppose that the triple $\zeta, \eta, f(\zeta, \eta)$ consist of roots of unity. Then we can apply the algorithm of Smyth, cf. [4], to solve the equation $f(\zeta, \eta)^{-1} = f(\zeta^{-1}, \eta^{-1})$ for roots of unity. In particular, a finite number of values will be obtained. We denote the set of all primitive n-th roots of unity by Φ'_n.

Proposition 15 *We list the non-trivial divisors H of the tuple $\mathcal{G}^{-1,2}_{1;m}[c,d](1, y)$, $\mathcal{G}^{0,1}_{1;m}(1, y)$ with $m(H)$ infinite.*

1. $y^2 + y + 1, y - r + 1 \in \mathcal{H}_2$, $r = 0$, $m \equiv 1, 2 \mod 3$
2. $(y^2 + y + 1)^2, y - r + 1 \in \mathcal{H}_4$, $r = 0$, $m \equiv 1 \mod 3$
3. $(y - r^2)(y - \bar{r}^2)$, $(y - r + 1)(y - \bar{r} + 1) \in \mathcal{H}_3$, $r \in \Phi'_{10}$, $m \equiv 1, 3, 7, 9 \mod 10$
4. $(y - r^2)^2(y - \bar{r}^2)^2$, $(y - r + 1)^2(y - \bar{r} + 1)^2 \in \mathcal{H}_{11}$, $r \in \Phi'_{10}$, $m \equiv 1 \mod 10$
5. $(y - r)(y - \bar{r})$, $(y - r + 1)(y - \bar{r} + 1) \in \mathcal{H}_5$, $r \in \Phi'_{12}$, $m \equiv 1, 5, 7, 11 \mod 12$
6. $(y - r)^2(y - \bar{r})^2$, $(y - r + 1)^2(y - \bar{r} + 1)^2 \in \mathcal{H}_{13}$, $r \in \Phi'_{12}$, $m \equiv 1 \mod 12$

The linear coefficients of the symmetries satisfy $c/d = (r - 1)^m/(r^m - 1)$.

Proof: We have $\mathcal{G}^{-1,2}_{1;n}[a, b](1, r) = \mathcal{G}^{0,1}_{1;n}[a, b](1, s) = 0$ when

$$U_n(r, s) = (1 + r^n)(1 - (1 + s)^n) + (s(1 + r))^n = 0. \tag{23}$$

We want to classify all $r, s \in \mathbb{C}$, with $rs(1 + r) \neq 0$, such that equation (23) has infinitely many solutions. According to Lemma 22 we have $s = -1$, or $r, 1 + s, s(1 + r)$ consists of roots of unity. When $s = -1$ we obtain that r is a third root of unity and $U_{i+3k} = 0$ iff $i = 1, 2$. When $x = r, y = 1+s, f = s(1+r)$ consists of roots of unity, then x, y are cyclotomic points on the curve

$$1 + (xy - 2(x - y))(xy - 1) + (x - y)^2 = 0,$$

and can be found algorithmically. They are $x \in \Phi_3'$, $y \in \Phi_6'$; $y \in \Phi_{10}'$, $x = y^2$ or $x = \bar{y}^2$; $y \in \Phi_{12}'$, $x = y$ or $x = \bar{y}$. The first case only happens when $a = b$. In the second case we have $f = y^4$ or $f = y^2$ and we find $U_{i+10k} = 0$ iff $i \in \{1, 3, 7, 9\}$. Note that with $s = y - 1$, $|y| = 1$ we have $-s/(1+s) = \bar{s}$. The last case gives $f = y^4$ or $f = y^3$ and $U_{i+12k} = 0$ iff $i \in \{1, 5, 7, 11\}$.

The multiplicity of the zeros is obtained from Proposition 11 and 12. We have $r \in \Phi_3'$ is a double zero of $\mathcal{G}_{1;m}^{-1,2}$ when $m \equiv 1 \bmod 3$. When $y \in \Phi_{10}'$ we have that both $r = y^2$ and $r = \bar{y}^2$ are in Φ_5'. They are double zeros of $\mathcal{G}_{1;m}^{-1,2}$ for $m \equiv 1 \bmod 5$. Also, we have that $s = y - 1$ and $\bar{s}$ are double zeros of $\mathcal{G}_{1;m}^{0,1}$ for $m \equiv 1 \bmod 10$. A similar argument shows the multiplicity in the last item. □

Proposition 16 *We list the non-trivial divisors H of the tuple $\mathcal{G}_{1;m}^{-1,2}[c,d](1, y)$, $\mathcal{G}_{2;m}^{2,-1}(x, 1)$ with $m(H)$ infinite:*

1. $(y - r)(y - \bar{r}), (x - r)(x - \bar{r}) \in \mathcal{H}_2$, $r \in \Phi_3'$, $m \equiv 1, 2 \mod 3$
2. $(y - r)^2(y - \bar{r})^2, (x - r)^2(x - \bar{r})^2 \in \mathcal{H}_4$, $r \in \Phi_3'$, $m \equiv 1 \mod 3$
3. $(y - r^2)(y - \bar{r}^2)$, $(x - r)(x - \bar{r}) \in \mathcal{H}_3$, $r \in \Phi_5'$, $m \equiv 1, 3, 7, 9 \mod 10$
4. $(y - r^2)^2(y - \bar{r}^2)^2$, $(x - r)^2(x - \bar{r})^2 \in \mathcal{H}_{11}$, $r \in \Phi_5'$, $m \equiv 1 \mod 10$
5. $(y + r)(y + \bar{r})$, $(x - r)(x - \bar{r}) \in \mathcal{H}_4$, $r \in \Phi_{12}'$, $m \equiv 1, 4, 5, 7, 8, 11 \mod 12$
6. $(y + r)^2(y + \bar{r})^2$, $(x - r)^2(x - \bar{r})^2 \in \mathcal{H}_{13}$, $r \in \Phi_{12}'$, $m \equiv 1 \mod 12$

The linear coefficients of the symmetries satisfy $c/d = (r + 1)^m/(r^m + 1)$.

Proof: We have $\mathcal{G}_{1;n}^{-1,2}[a, b](1, r) = \mathcal{G}_{2;n}^{2,-1}[a, b](s, 1) = 0$ when

$$U_n(r, s) = (1 + r^n)(1 + s^n) - ((1 + s)(1 + r))^n = 0. \tag{24}$$

We want to classify all $r, s \in \mathbb{C}$, with $rs(s + 1)(1 + r) \neq 0$, such that equation (24) has infinitely many solutions. According to Lemma 22, the points r, s, and $(1 + s)(1 + r)$ are roots of unity. Hence r, s are cyclotomic points on the curve

$$1 + (rs + 1)(rs + 2(r + s)) + (r + s)^2 = 0.$$

Smyths algorithm yields: $r, s \in \Phi_3'$; $s \in \Phi_5'$, $r = s^2$ or $r = \bar{s}^2$; $s \in \Phi_{12}'$, $r = -s$ or $r = -\bar{s}$. Substitution these into the equation (24), we obtained, by performing some Groebner basis calculations, the solutions $n \equiv 1, 2 \mod 3$, $n \equiv 1, 3, 7, 9 \mod 10$, and $n \equiv 1, 4, 5, 7, 8, 11 \mod 12$ respectively. The multiplicities are determined using Proposition 11, and using relation (7). □

Proposition 17 *We list the non-trivial divisors H of the tuple $\mathcal{G}_{1;m}^{0,1}[c,d](1, y)$, $\mathcal{G}_{2;m}^{1,0}(x, 1)$ with $m(H)$ infinite:*

1. $y + 1 - r, x + 1 - r \in \mathcal{H}_2$, $r = 0$, $m > 1$

2. $(y+1+r^2)(y+1+\bar r^2)$, $(x+1-r)(x+1-\bar r) \in \mathcal{H}_3$, $r \in \Phi'_{10}$, $m \equiv 1,3,7,9 \mod 10$

3. $(y+1+r^2)^2(y+1+\bar r^2)^2$, $(x+1-r)^2(x+1-\bar r)^2 \in \mathcal{H}_{11}$, $r \in \Phi'_{10}$, $m \equiv 1 \mod 10$

4. $(y+1+r)$, $(x+1-r) \in \mathcal{H}_2$, $r \in \Phi'_{12}$, $m \equiv 1,2,5,7,10,11 \mod 12$

5. $(y+1+r)(y+1+\bar r)$, $(x+1-r)(x+1-\bar r) \in \mathcal{H}_5$, $r \in \Phi'_{12}$, $m \equiv 1,5,7,11 \mod 12$

6. $(y+1+r)^2(y+1+\bar r)^2$, $(x+1-r)^2(x+1-\bar r)^2 \in \mathcal{H}_{13}$, $r \in \Phi'_{12}$, $m \equiv 1 \mod 12$

The linear coefficients of the symmetries satisfy $c/d = (r^m-1)/(r-1)^m$.

Proof: Similar to the above, $\mathcal{G}^{0,1}_{1;n}[a,b](1,r) = \mathcal{G}^{1,0}_{2;n}[a,b](s,1) = 0$ when

$$(1-(1+r)^n)(1-(1+s)^n) - (rs)^n = 0. \tag{25}$$

We want to classify all $r,s \in \mathbb{C}$, with $rs \neq 0$, such that equation (25) has infinitely many solutions. If one of r,s equals -1, the other is a third root of unity. When $r = s = -1$ we have $a/b = -(-1)^n$, otherwise $a = b$. Suppose that $(1+r)(1+s) \neq 0$. According to Lemma 22 $1+r$, $1+s$, and sr are roots of unity. Then $x = 1+r, y = 1+s$ are cyclotomic points on the curve

$$1 + (xy+1)(xy-2(x+y)) + (x+y)^2 = 0.$$

They are: $x,y \in \Phi'_6$; $y \in \Phi'_{10}$, $x = -y^2$ or $x = -\bar y^2$; $y \in \Phi'_{12}$, $x = -y$ or $x = -\bar y$. The first are zeros only when $a = b$ and the others yield the results. $\square$

10 List of approximately integrable two component evolution equations of order $n < 6$

We list approximately integrable equations whose quadratic tuple has minimal degree. However, we have left some arbitrary constants in the equations. This organizes the quadratic part of the equations and it may remind the reader of the fact that the quadratic tuple of the equations can be multiplied by arbitrary proper tuples.

The list divides naturally into two parts. Any nonlinear equation (9) of order $n < 6$, which is not nonlinear injective and admits infinitely many approximate symmetries with non-vanishing linear terms, can be obtained from an equation listed in section 10.1. And, any nonlinear injective approximately integrable equation (9) of order $n < 6$, can be obtained from an equation listed in section 10.2.

The classification is performed up to the transformation $u \to v$. If an equation, whose quadratic tuple has minimal degree, is approximately integrable of order $n < 6$ and not in our list, then the equation with u and v interchanged is in the list. To make it easier to identify equations in the list, we have scaled the coefficient b of v_n to 1. This we could do, since when $b = 0$ we assume $a \neq 0$ and we apply $u \leftrightarrow v$.

We gave each approximately integrable equation a unique label $n.h$ where n is the order, and h is a counter. And, we expressed the linear coefficients c/d of the approximate symmetries in terms of integer sequences, or in its power sum solution if that displays well.

10.1 Equations that are not nonlinear injective

For each K^0 in Table 1 we have determined the highest degree r-tuple H, with $m(H)$ infinite, which divides the r-tuple consisting of the non-zero components of its $\mathcal{G}_n$-tuple. The quadratic tuple K^1 has only r non-zero components $K^1 = \mathcal{G}_n/H$, unless the remaining components of the $\mathcal{G}_m$-tuples at infinitely $m \in m(H)$ vanish.

The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} eu^2 + fuv + gv^2 \\ v + hv^2 + ju^2 \end{pmatrix} \qquad 0.1$$

has approximate symmetries at order $m = 1$, for all $c/d \in \mathbb{C}$, and at any order m, with $c = 0$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 2u + eu^2 + fuv \\ v + hv^2 + iuv + ju^2 \end{pmatrix} \qquad 0.2$$

has approximate symmetries at any order $m \in \mathbb{N}$, for all $c, d \in \mathbb{C}$. The equation, with $a \neq 1$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} au_1 + fuv + gv^2 \\ v_1 + iuv + ju^2 \end{pmatrix} \qquad 1.1$$

has approximate symmetries at odd orders $m \equiv 1 \mod 2$ for all $c, d \in \mathbb{C}$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} eu^2 + fuv + gv^2 \\ v_2 + hv^2 + iuv + ju^2 \end{pmatrix} \qquad 2.1$$

has approximate symmetries at orders $m \equiv 2 \mod 4$ with $c = 0$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} eu^2 + fuv + gv^2 \\ v_3 + hv^2 + iuv + ju^2 \end{pmatrix} \qquad 3.1$$

has approximate symmetries at orders $m \equiv 3 \mod 6$ with $c = 0$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} eu^2 + fuv + gv^2 \\ v_4 + h(4vv_2 + 3v_1^2) + iuv + ju^2 \end{pmatrix} \qquad 4.1$$

has approximate symmetries at orders $m \equiv 4 \mod 8$ with $c = 0$. And the equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} eu^2 + fuv + gv^2 \\ v_5 + hv^2 + iuv + ju^2 \end{pmatrix} \qquad 5.1$$

has approximate symmetries at orders $m \equiv 5, 25 \mod 30$.

10.2 Equations that are nonlinear injective

For each $0 < n < 6$ we have gone through Proposition 10 to Proposition 17. For each $H \in \mathcal{H}_n$, we determined the coefficient a of the linear part of the equation. Then we constructed the highest degree 6-tuple P that divides $\mathcal{G}_n[a,1]$ such that $m(P)$ is infinite. And we take $\mathcal{G}_n/P$ to be the quadratic tuple of the equation. We remark that by following this procedure, the list can be extended to any higher order $n > 5$, in principle.

The equation, with $a(2a-1)(a-2) \neq 0$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} au + eu^2 + fuv + gv^2 \\ v + hv^2 + iuv + ju^2 \end{pmatrix} \qquad 0.3$$

has approximate symmetries at any order $m \in \mathbb{N}$, for all $c, d \in \mathbb{C}$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_2 + eu^2 + fuv + gv^2 \\ v_2 + hv^2 + iuv + ju^2 \end{pmatrix} \qquad 2.2$$

has approximate symmetries at all orders $m > 1$ with $c = d = 1$. The equation, with $(r^2+1)(r+1) \neq -1$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \frac{1+r^2}{(1+r)^2} u_2 + eu^2 + f((1+r^2)u_1 v - ruv_1) + gv^2 \\ v_2 + hv^2 + i((1+r^2)uv_1 + ru_1 v) + j(2ru_2 u + (1+r)^2 u_1^2) \end{pmatrix} \qquad 2.3$$

has approximate symmetries at all orders $m > 1$ with $c/d = (1+r^m)/(1+r)^m$. The equation, with $r(r+2) \neq 0$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \frac{r}{2+r} u_2 + eu^2 + fuv + g(rv_1^2 - 2vv_2) \\ v_2 + hv^2 + i(u_1 v + (2+r)uv_1) + j(2uu_2 + (2+r)u_1^2) \end{pmatrix} \qquad 2.4$$

has approximate symmetries at all orders $m > 1$ with $c/d = r^m/((1+r)^m - 1)$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} -u_2 + eu^2 + fuv + gv^2 \\ v_2 + hv^2 + iuv + ju^2 \end{pmatrix} \qquad 2.5$$

has approximate symmetries at order $m \equiv 1, 2 \mod 3$, with $c/d = -(-1)^m$. The equation, with $\iota^2 = -1$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (-1+2\iota)u_2 + eu^2 + f(5u_1 v + (3+\iota)uv_1) + gv^2 \\ v_2 + hv^2 + iuv + j(4uu_2 + (1+\iota)u_1^2) \end{pmatrix} \qquad 2.6$$

has approximate symmetries at order $m \equiv 2 \mod 4$, with $c/d = -1 + (-1)^{(m-2)/4} 2^{m/2} \iota$. The equation, with $\gamma^2 = 3$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \iota(2+\gamma)u_2 + eu^2 + fuv + g(4vv_2 + (2+\gamma-\iota)v_1^2) \\ v_2 + hv^2 + iuv + j(4uu_2 + (2+\iota-\gamma)u_1^2) \end{pmatrix} \qquad 2.7$$

has approximate symmetries at orders $m \equiv q \mod 12$, with $q \in \{1, 2, 5, 7, 10, 11\}$. Define integers P_k by $P_1 = 1$, $P_2 = 2$, and

$$P_k = \begin{cases} P_{k-1} + P_{k-3} & k \equiv 1 \mod 3 \\ P_{k-1} + P_{k-2} & k \equiv 0, 2 \mod 3 \end{cases} . \qquad (26)$$

When $q=2$ or $q=10$ the coefficients of the linear part of the approximate symmetries of equation 2.7 are given by $c/d=(-1)^{(m-q)/12}\iota(P_{3m/2-1}+P_{3m/2-2}\gamma)$, or else by

$$\mp(-1)^{(m-q)/12}c/d=\left\{\begin{array}{ll} P_{(3m-5)/2}+P_{(3m-7)/2}\gamma & q=6\pm 5 \\ P_{(3m+1)/2}+P_{(3m-1)/2}\gamma & q=6\pm 1\end{array}\right. . \tag{27}$$

The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix}=\begin{pmatrix} u_3+eu^2+fuv+gv^2 \\ v_3+hv^2+iuv+ju^2 \end{pmatrix} \qquad 3.2$$

has approximate symmetries at odd orders with $c=d=1$. The equation, with $r^3\neq -1$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix}=\begin{pmatrix} \frac{1+r+r^2}{(1+r)^2}u_3+eu^2+f(2u_2v+2u_1v_1+(r^3-2r-1)uv_2)+gv^2 \\ v_3+hv^2+iuv+j(2uu_2-(r^3-2r-1)u_1^2) \end{pmatrix} \qquad 3.3$$

has approximate symmetries at odd orders m with $c/d=(1+r^m)/(1+r)^m$. Let ϕ denote the golden ratio or its conjugate, that is, $\phi(\phi-1)=1$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix}=\begin{pmatrix} -(2+3\phi)u_3+eu^2+fuv+gv^2 \\ v_3+hv^2+iuv+ju^2 \end{pmatrix} \qquad 3.4$$

has approximate symmetries at order $m\equiv q \mod 10$, $q\in\{1,3,7,9\}$, with

$$c/d=\left\{\begin{array}{ll} F_{m-2}+F_{m-1}\phi & q=5\pm 4 \\ -F_m-F_{m+1}\phi & q=5\pm 2\end{array}\right. ,$$

where the F_k are the Fibonacci numbers $F_0=0$, $F_1=1$, $F_k=F_{k-1}+F_{k-2}$ [25, A000045]. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix}=\begin{pmatrix} -3u_4+(e(4u_2u+3u_1^2)+f(6u_3v+9u_2v_1+6u_1v_2 \\ \qquad\qquad +2uv_3)+gv^2 \\ v_4+h(4v_2v+3v_1^2)+i(2u_3v+2u_2v_1+3u_1v_2+2uv_3) \\ \qquad\qquad +j(4u_4u+4u_1u_3+3u_2^2) \end{pmatrix} \qquad 4.2$$

has approximate symmetries at order $m\equiv 0 \mod 4$ with $c/d=1+(-1)^{m/4}2^{m/2}$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix}=\begin{pmatrix} -u_4+e(4u_2u+3u_1^2)+f(uv_2+u_1v_1+2u_2v)+gv^2 \\ v_4+h(4v_2v+3v_1^2)+i(2uv_2+u_1v_1+u_2v)+ju^2 \end{pmatrix} \qquad 4.3$$

has approximate symmetries at order $m\equiv 1 \mod 3$ with $c/d=-(-1)^m$. The equation, with $\zeta(\zeta+1)=1$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix}=\begin{pmatrix} 3(1+2\zeta)u_4+e(4uu_2+3u_1^2)+f(6u_1v+(1-4\zeta)uv_1) \\ \qquad\qquad +g(14v_4v+(4-\zeta)(12v_3v_1+9v_2^2)) \\ v_4+h(4v_2v+3v_1^2)+i(7u_3v+(2+3\zeta)(2u_2v_1+2uv_3 \\ \qquad\qquad +3u_1v_2))+j(14u_4u+(2+3\zeta)(4u_3u_1+3u_2^2)) \end{pmatrix} \qquad 4.4$$

has approximate symmetries at order $m \equiv 1 \mod 3$, with $c/d = (-3)^{(m-1)/2}$ when $m \equiv 1 \mod 6$, and with $c/d = -(1+2\zeta)(-3)^{(m-2)/2}$ when $m \equiv 4 \mod 6$. The equation, with $\epsilon = \pm 1$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \epsilon/5u_4 + e(4uu_2 + 3u_1^2) + f(2u_1v + (1-2\epsilon)uv_1) \\ \hfill +g(4v_3v_1 + 3v_2^2 + (1-5\epsilon)v_4v) \\ v_4 + h(4v_2v + 3v_1^2) + i(10(2u_2v_1 + 3u_1v_2 + 2uv_3) \\ \hfill +(5-\epsilon)u_3v) + j(5(4u_3u_1 + 3u_2^2) + (5-\epsilon)u_4u) \end{pmatrix} \qquad 4.5$$

has approximate symmetries at order $m \equiv 0 \mod 4$, with $c/d = -\epsilon/((-4)^{m/4} - 1)$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \iota/3u_4 + e(4uu_2 + 3u_1^2) + f(2u_1v + (1-2\iota)uv_1) \\ \hfill +g(10vv_4 + (1-3\iota)(4v_1v_3 + 3v_2^2)) \\ v_4 + h(4v_2v + 3v_1^2) + i(5u_3v + 3(3+\iota)(2u_2v_1 + 3u_1v_2 \\ \hfill +2uv_3)) + j(10u_4v + 3(3+\iota)(3u_2^2 + 4u_3u_1)) \end{pmatrix} \qquad 4.6$$

has approximate symmetries at orders $m = 4+k8$, $k \in \mathbb{N}$, with $c/d = (-1)^k 2^{2k}\iota/(2A_{k+1}^2 + 1)$, where the integers A_i are the NSW numbers defined by $A_0 = -1$,$A_1 = 1$, $A_i = 6A_{i-1} - A_{n-2}$ [25, A002315]. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \sqrt{2}\iota u_4 + e(4uu_2 + 3u_1^2) + f(2u_1v + (1-\sqrt{2}\iota)uv_1) \\ \hfill +g(2(4v_1v_3 + 3v_2^2) + (2+\sqrt{2}\iota)v_4v) \\ v_4 + h(4v_2v + 3v_1^2) + i((1-\sqrt{2}\iota)u_3v + 2(2u_2v_1 + 3u_1v_2 \\ \hfill +2uv_3)) + j(3u_2^2 + 4u_3u_1 + (1-\sqrt{2}\iota)u_4v) \end{pmatrix} \qquad 4.7$$

has approximate symmetries at orders $m = 4+k8$, $k \in \mathbb{N}$, with $c/d = (-1)^k 2^{6k}\sqrt{2}\iota/B_{k+1}$, where the integers B_i are defined by $B_0 = -1$,$B_1 = 1$, $B_i = 34B_{i-1} - B_{n-2}$ [25, A046176]. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (7+4\gamma)u_4 + e(4uu_2 + 3u_1^2) + f(2u_1v_2 + 3u_2v_1 + 2u_3v \\ \hfill +(2\gamma-3)uv_3) + g(6vv_2 + (6+\gamma)v_1^2) \\ v_4 + h(4v_2v + 3v_1^2) + i((3+2\gamma)u_3v - 2uv_3 - 3u_1v_2 \\ \hfill -2u_2v_1) + j(6uu_2 + (6-\gamma)u_1^2) \end{pmatrix} \qquad 4.8$$

has approximate symmetries at order $m \equiv q \mod 12$, $q \in \{1, 4, 5, 7, 8, 11\}$. When $q = 6 \pm 2$ the coefficients of the linear part of the approximate symmetries are given by $c/d = \mp(-1)^{(m-q)/12}(P_{3m/2-1} + P_{3m/2-2}\gamma)$, where the integers P_k are defined by the recursive formula (26). When q is odd, c/d is given by equation (27). The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_5 + eu^2 + fuv + gv^2 \\ v_5 + hv^2 + iuv + ju^2 \end{pmatrix} \qquad 5.2$$

has approximate symmetries at orders $m \equiv 1, 5 \mod 6$ with $c = d = 1$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}u_5 + eu^2 + f(u_4v + 2u_3v_1 + 2u_2v_2 + u_1v_3 + uv_4) + gv^2 \\ v_5 + hv^2 + iuv + j(2uu_4 + 6u_1u_3 + 5u_2^2) \end{pmatrix} \quad 5.3$$

has approximate symmetries at orders $m \equiv 1, 5 \mod 12$, with $c/d = (-1)^{(m-1)/4}2^{(1-m)/2}$. The equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} -(4+5\phi)u_5 + eu^2 + f((4-\phi)uv_4 + 11(u_1v_3 + 2u_2v_2 \\ \qquad\qquad +2u_3v_1 + u_4v)) + gv^2 \\ v_5 + hv^2 + iuv + j(2\phi uu_4 - 2u_1u_3 + (2\phi-1)u_2^2) \end{pmatrix} \quad 5.4$$

has approximate symmetries at orders $m \equiv 5, 25 \mod 30$, with $c/d = (-1)^{m/5}(1 + F_{m-1} + F_m\phi)$, where F_m denotes the m-th Fibonacci number. And, the equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (26+15\gamma)u_5 + eu^2 + f(u_2v + u_1v_1 + (\gamma-1)uv_2) \\ \qquad +g(4vv_2 + (3+\gamma)v_1^2) \\ v_5 + hv^2 + i(u_1v_1 + uv_2 - (1+\gamma)u_2v) \\ \qquad +j(4uu_2 + (3-\gamma)u_1^2 \end{pmatrix} \quad 5.5$$

has approximate symmetries at orders $m \equiv 1, 5, 7, 11 \mod 12$, with c/d given by equation (27).

11 Concluding remarks

In the formal symmetry approach [17], as well as in the computer-assisted schemes [6, 27], not knowing the ratios of eigenvalues strongly complicates the classification of integrable equations. There the ratios are obtained, if possible at all, at the very last stage of the calculations. We hope that the a priori knowledge provided in this article will be an impetus to complete the classification.

Usually, in classification programs one considers *homogeneous* equations. A 2-component equation $(u_t, v_t) = K$ is homogeneous of weighting λ if K is an eigenvector of $\mathcal{L}(\sigma_x + \lambda_1\sigma_u + \lambda_2\sigma_v)$, where $\sigma_x = (xu_1, xv_1)$ counts the number of derivatives. We have compactly provided a list of nonhomogeneous equations. A complete list of homogeneous equations can be obtained from our list by multiplying the symbolic quadratic parts with appropriate tuples of polynomials. And Lemma 3 can be used to determine all symmetries of those equations.

A classification of second order integrable 2-component evolution equations has been given in [23]. The lemmas 6.3, 6.4, 6.5, and 6.6, proved in there, are special cases ($n = 2$) of Proposition 13, 16, 17, and 15, respectively. As it turns out, the linear and quadratic parts of the integrable equations listed in [23] can all be obtained from equations 2.3 and 2.5.

A classification of third order 2-component evolution equations with weighting $(2, 2)$ and symmetries of order 5, 7, or 9, is given in [6]. Of the five equations listed [6, Theorem 3.3], two are nonlinear injective and have diagonalizable linear part:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_3 + uu_1 + vv_1 \\ -2v_3 - uv_1 \end{pmatrix} \tag{28}$$

and

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 4u_3 + 3v_3 + 4uu_1 + vu_1 + 2uv_1 \\ 3u_3 + v_3 - 2vv_1 - 4vu_1 - 2uv_1 \end{pmatrix}. \tag{29}$$

When put in Jordan form the ratio of the coefficients of the linear part of equation (29) becomes $a/b = -3\phi - 2$, where ϕ is the golden ratio. Our diophantine approach perfectly explains the 'unusual' symmetry pattern. We remark that the conjugate of ϕ gives rise to another equation with $a/b = -3(1-\phi) - 2 = 1/(-3\phi - 2)$, which can be obtained by interchanging u and v. A similar remark holds for all equations derived from the 'symmetric' Propositions 16 and 17. For example, by interchanging u and v in equation 4.8 we get an equation with $a/b = 1/(7+4\gamma) = 7 - 4\gamma$.

According to [6, Theorem 3.3] we have the following. *A non-decouplable fifth order two component equation in the KDV weighting, possessing a generalized symmetry of order 7 can be reduced by a linear change of variables to a symmetry of lower order equations, or to the Zhou-Jiang-Jiang equation,*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_5 - 5(2uu_3 + 5u_1u_2) + 15(2vv_3 + 3v_1v_2) + 20u_1u^2 \\ \hfill -30(u_1v^2 + 2uvv_1) \\ -9v_5 + 5(2u_3v + 7u_2v_1 + 9u_1v_2 + 6uv_3) - 10(2uu_1v \\ \hfill +2u^2v_1 + 3v^2v_1) \end{pmatrix}.$$

However, since the ratio of coefficients of the linear part $a/b = -1/9$ does not appear in our list, this equation has an approximate symmetry at lower order. In fact, the equation has a genuine symmetry at third order. The Zhou-Jiang-Jiang equation is in the hierarchy of the equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} -3vv_1 \\ v_3 - u_1v - 2uv_1 \end{pmatrix}, \tag{30}$$

which is linearly equivalent to a third order equation that appears in the same paper [6, Equation (17)], cf. [27, Sections 3.2.1, 4.2.6]. The special value of the ratio $a/b = -1/2$ in equation (28) also does not appear in our list and is due to higher grading constraints.

We conclude with a more philosophical remark. The concept of generalized symmetry really is about local symmetry. The (inverse) Gel'fand-Dikiĭ transformation translates polynomials in the symbols x, y into local differential functions, that is expressions in u, v and their derivatives. A question arises: can we also translate rational functions in the symbols x, y? The answer is yes. One could think of non-local variables u_i, v_i with $0 > i \in \mathbb{Z}$. Here a negative index indicates integration and D_x would be such that $D_x(u_{i-1}) = u_i$ for all i. We can expand rational functions in Taylor-series, which are transformed into non-local differential sums. For example, consider the rational function $\widehat{F} = 1/(x_1 + x_2)$. Its symmetric Taylor-series is

$$\widehat{F} = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k(\frac{x_1^k}{x_2^{k+1}} + \frac{x_2^k}{x_1^{k+1}})$$

which is transformed into the non-local object

$$F = \sum_{k=0}^{\infty}(-1)^k u_k u_{-k-1},$$

and we have $D_x F = u^2$. In this non-local setting every equation has a symmetry at any order. For example, the equation, with $a \neq 1$,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} au_1 + (1-a)uv \\ v_1 + (1-a)uv \end{pmatrix},$$

and its approximate symmetries (but the ones in $\mathcal{A}^{2,0} \otimes \mathcal{A}^{0,2}$), are in the approximate hierarchy of the zeroth order non-local equation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u + uv_{-1} \\ v - u_{-1}v \end{pmatrix}.$$

Still, there would be a quest for equations, or symmetries, that are in certain sense close to local.

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A Maple-code

We felt it appropriate to include some Maple-code here. This allows one to perform a computation like $\mathcal{G}_m[c,d]K^1/\mathcal{G}_n[a,b]$ and translate the result into an approximate symmetry of an evolution equation. Also an implementation of the Lie-bracket is provided.

```
g:=proc(k,n,i,j,a,b,x,y) if k=2 then return(g(1,n,j,i,b,a,y,x)) fi:
 a*(add(x[q]^n,q=1..i+1)-(add(x[q],q=1..i+1)+add(y[q],q=1..j))^n)
 +b*add(y[q]^n,q=1..j) end:
```

```
TRANS:=proc(P,d) local R,e,i,Q: R:=0: Q:=expand(P): if type(Q,`+`)
 then Q:=convert(Q,list) else Q:=[Q] fi: for e in Q do for i to d
 do e:=e*u[degree(e,x[i])]/x[i]^degree(e,x[i]) od: for i to 2-d do
 e:=e*v[degree(e,y[i])]/y[i]^degree(e,y[i]) od: R:=R+e od: R end:

Prod:=proc(A,B) [seq(A[i]*B[i],i=1..6)] end:

div:=proc(a,b) if factor(a)=0 and factor(b)=0 then return(nnli)
 fi: factor(a/b) end:

Div:=proc(A,B) [seq(div(A[i],B[i]),i=1..6)] end:

G:=proc(a,b,n) [seq(g(1,n,1-k,k,a,b,x,y),k=0..2),
 seq(g(2,n,k,1-k,a,b,x,y),k=0..2)] end:

TRA:=proc(K) [add(TRANS(K[i],3-i),i=1..3),
 add(TRANS(K[i],i-4),i=4..6)]: end:
```

And here is an implementation of the Lie-bracket:

```
VAR:=proc(P) local R: R:=NULL: for e in indets(P) do
 if evalb(op(0,e) in {u,v}) then R:=R,e fi od: {R} end:

DD:=proc(P,n) local R,i,e,Q: R:=P: for i to n do Q:=0: for e in VAR(R)
 do Q:=Q+op(0,e)[op(1,e)+1]*diff(R,e) od: R:=diff(R,x)+Q od: R end:

FR:=proc(x,A,B) local e,R: R:=0: for e in VAR(A) do
 if op(0,e)=x then R:=R+diff(A,e)*DD(B,op(1,e)) fi od: R end:

MFR:=proc(A,B) local R,i,j,Q,U: U:=[u,v]: R:=[]: for i to 2 do
 Q:=0: for j to 2 do Q:=Q+FR(U[j],A[i],B[j]) od: R:=[op(R),Q] od:
 R end:

LIE:=proc(K,S) RETURN(expand(MFR(S,K)-MFR(K,S))) end:
```

To calculate the first approximate symmetry of equation 0.3, we proceed as follows. The linear part of equation 0.3 is `eq0:=[a*u[n],v[n]]`, with `n:=0`. The constant tuple is set `K:=[e,f,g,h,i,j]`, which is translated into the quadratic part of the equation `eq1:=TRA(K)`. The first approximate symmetry appears at order `m:=1`. It has has linear part `sy0:=[c*u[m],d*u[m]]` and quadratic part `sy1:=TRA(Div(Prod(K,G(c,d,m)),G(a,1,n)))`, that is

$$\begin{pmatrix} cu_1 \\ dv_1 \end{pmatrix} + (c-d)\begin{pmatrix} v_1(2gv/(a-2)-fu) \\ u_1(iv/a+2ju/(2a-1)) \end{pmatrix}.$$

To check that this is an approximate symmetry we verify that `LIE(eq0,sy1)+ LIE(eq1,sy0)` yields `[0,0]`.

One can do the same calculation, but with `a=0`, `i=0` from the start. The program will tell you that the linear part is not nonlinear injective by introducing a constant `nnli` in the approximate symmetry, cf. section 5. The above procedure can be used to calculate any approximate symmetry of any equation in our list, which is useful to further classify with respect to integrability.

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