# Closed-form expressions for integrals of traveling wave reductions of integrable lattice equations

**Dinh T Tran, Peter H van der Kamp, G R W. Quispel**

Department of Mathematics, La Trobe University, Victoria 3086, Australia

E-mail: `td2tran@students.latrobe.edu.au`

**Abstract.** We give a method to calculate closed-form expressions in terms of multisums of products for integrals of ordinary difference equations which are obtained as traveling wave reductions of integrable partial difference equations. Important ingredients are the staircase method, a non-commutative Vieta formula, and certain splittings of the Lax matrices. The method is applied to all equations of the Adler-Bobenko-Suris classification, with the exception of $Q_4$.

PACS numbers: 02.30.Ik

Submitted to: *J. Phys. A: Math. Gen.*

## 1. Introduction

Two main classes of discrete integrable systems that may be distinguished are ordinary difference equations (OΔEs) and partial difference equations (PΔEs). By imposing periodic initial conditions, a PΔE reduces to an OΔE. The staircase method which was introduced in [7, 8] provides us with a tool to construct integrals of OΔEs, or mappings, derived as reductions of integrable PΔEs. These integrals are obtained by expanding the trace of a monodromy matrix in powers of the spectral parameter.

Recently in [6], elegant and succinct formulas for integrals of sine-Gordon and modified Korteweg-de Vries (mKdV) maps have been given for the first time. The integrals are expressed in terms of multi-sums of products, Theta, which are defined in section 2. These multi-sums of products were discovered by inspection and proved by induction. Properties of these multi-sums of products were used to prove the invariance of the integrals independently of the staircase method and make it possible to prove functional independence and involutivity of the integrals directly.

In a recent paper by Adler, Bobenko and Suris (ABS), multi-linear equations on quad-graphs are classified with respect to consistency around the cube [1]. It is also known that for PΔEs on quad-graphs which satisfy the consistency property, we can obtain a Lax pair algorithmically [2, 3]. Hence, the staircase method can be applied to obtain integrals of traveling wave reductions of the equations in the ABS list

Some questions arise here: is it possible to give similar explicit expressions for these integrals as was done for mKdV and sine-Gordon? If so, is there any method to obtain closed-form expressions for integrals directly, rather than using inspection and induction? This paper answers both these questions in the affirmative. Based on a non-commutative Vieta formula, and certain splittings of the Lax-matrices, we explain how multi-sums of products emerge and we give a method to actually derive closed-form expressions for the integrals.

## 2. Outline of this paper

In this paper, we restrict ourselves to so-called $(1, z_2)$ traveling wave reductions ($z_2 \in \mathbb{N}^+$) which we now explain. With $(l, m) \in \mathbb{Z} \times \mathbb{Z}$, we consider a 2-dimensional PΔE with field variable $v$,

$$f(v_{l,m}, v_{l+1,m}, v_{l,m+1}, v_{l+1,m+1}; \alpha) = 0, \tag{1}$$

and parameters $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Now we introduce a reduction $v_{l,m} = v_n$, where $n = l + mz_2$. Then, $v_{l,m}$ satisfies the periodicity $v_{l,m} = v_{l+z_2,m-1}$, and the PΔE reduces to an OΔE

$$f(v_n, v_{n+1}, v_{n+z_2}, v_{n+z_2+1}; \alpha) = 0. \tag{2}$$

We suppose that our equation (1) arises as the compatibility condition of two linear equations, that is, it has a Lax pair. A Lax pair $L_{l,m}, M_{l,m}$ for a PΔE (1) is a pair of matrices that satisfy, cf [9],

$$L_{l,m}M_{l,m}^{-1} - M_{l+1,m}^{-1}L_{l,m+1} = 0. \tag{3}$$

Similarly, a OΔE has a Lax pair $\mathcal{L}_n, \mathcal{M}_n$ if they are non-singular matrices that satisfy

$$\mathcal{M}_n\mathcal{L}_n - \mathcal{L}_{n+1}\mathcal{M}_n = 0. \tag{4}$$

Using the $(1, z_2)$ traveling wave reduction, the Lax pair for a PΔE reduces to matrices $L_n, M_n$ which satisfy the following equation

$$L_nM_n^{-1} - M_{n+1}^{-1}L_{n+z_2} = 0. \tag{5}$$

The monodromy matrix $\mathcal{L}_n$ for the $(1, z_2)$ reduction is given by, cf [8],

$$\mathcal{L}_n = M_n^{-1}\prod_{i=0}^{\overset{\curvearrowleft}{z_2-1}} L_{i+n} \tag{6}$$

where the inversely ordered product is

$$\prod_{i=1}^{\overset{\curvearrowleft}{b}} L_i := L_bL_{b-1}\dots L_{a+1}L_a. \tag{7}$$

Taking $\mathcal{M}_n = L_n$, we obtain a Lax pair $\mathcal{L}_n, \mathcal{M}_n$ for the reduced OΔE (2) from the Lax pair of the corresponding PΔE (1). Recently, a description of $(z_1, z_2)$-reduction, with general $(z_1, z_2) \in \mathbb{Z} \times N$ is given in [11]. This generalizes the reductions in [8] where the case $(z_1, z_2) \in \mathbb{N}^2$ with $z_1, z_2$ co-prime was considered. In [11], it was proved that for any $(z_1, z_2)$-reduction, there is a $\mathcal{M}$ with whom the monodromy matrix $\mathcal{L}$ forms a Lax pair for the reduction and explicit formulas for $\mathcal{L}, \mathcal{M}$ in terms of the reduced Lax matrices $L, M$ were provided.

It follows from equation (4) that the trace of $\mathcal{L}_n$ is invariant under the map obtained from a OΔE. Since the reduced Lax matrices generally depend on a spectral parameter, integrals for the OΔE (2) are obtained by expanding the trace of the monodromy matrix in powers of the spectral parameter. Therefore, to give explicit expressions for these integrals, we need to expand a product of $L$ matrices. We split the $L$ matrices in the form $L_i = r_i(\lambda X_i + Y_i)$, where $\lambda$ is (a function of) the spectral parameter. Next, we consider the formal expansion of the matrix product in a non-commutative Vieta formula:

$$\prod_{i=a}^{\overset{\curvearrowleft}{b}}(\lambda X_i + Y_i) = \sum_{r=0}^{b-a+1} \lambda^{b-a+1-r}Z_r^{a,b} \tag{8$a$}$$

$$= \sum_{r=0}^{b-a+1} \lambda^r\widetilde{Z}_r^{a,b}, \tag{8$b$}$$

where

$$Z_r^{a,b} = \sum_{a\le i_1<i_2<\cdots<i_r\le b} X_bX_{b-1}\cdots X_{i_r+1}Y_{i_r}X_{i_r-1}\cdots X_{i_1+1}Y_{i_1}X_{i_1-1}\cdots X_a, \tag{9$a$}$$

$$\widetilde{Z}_r^{a,b} = \sum_{a\le i_1<i_2<\cdots<i_r\le b} Y_bY_{b-1}\cdots Y_{i_r+1}X_{i_r}Y_{i_r-1}\cdots Y_{i_1+1}X_{i_1}Y_{i_1-1}\cdots Y_a. \tag{9$b$}$$

This is a generalization of the Vieta expansion with commutative variables:

$$\prod_{i=a}^{\overset{\frown}{b}}(\lambda+f_i)=\sum_{r=0}^{b-a+1}\lambda^{b-a+1-r}s_r^{a,b}, \tag{10}$$

where the multi-sums of products $s_r^{a,b}$ are the elementary symmetric functions,

$$s_r^{a,b}\{f_i\}:=\sum_{a\leq i_1<i_2<\cdots<i_r\leq b}\prod_{j=1}^{r}f_{i_j}. \tag{11}$$

In this paper we will consider two different forms of $X_i$ and $Y_i$ with special properties such that the elements of the matrices $Z_r^{a,b}$ or $\widetilde{Z}_r^{a,b}$ can be expressed in terms of multi-sums of products Theta, respectively Phi.

For the reduced mKdV equation [6]

$$\alpha_1(v_nv_{n+z_2}-v_{n+1}v_{n+z_2+1})+\alpha_2v_nv_{n+1}-\alpha_3v_{n+z_2}v_{n+z_2+1}=0, \tag{12}$$

and the reduced sine-Gordon equation [6]

$$\beta_1(v_nv_{n+z_2+1}-v_{n+1}v_{n+z_2})+\beta_2v_nv_{n+1}v_{n+z_2}v_{n+z_2+1}-\beta_3=0, \tag{13}$$

the $L$ matrix can be written as $L_i=\lambda X_i+Y_i$, where

$$X_i=\begin{pmatrix}0&1\\1&0\end{pmatrix}=:J,\qquad Y_i=\begin{pmatrix}v_i/v_{i+1}&0\\0&v_{i+1}/v_i\end{pmatrix}, \tag{14}$$

and $\lambda$ is the spectral parameter. Substituting (14)in the equations $(9a)$ and $(9b)$, we derive multi-sums of products, Theta, with different arguments,

$$\Theta_{r,\epsilon}^{a,b}\{(\frac{v_i}{v_{i+1}})^{(-1)^i}\}=\sum_{a\leq i_1<i_2<\cdots<i_r\leq b}\prod_{j=1}^{r}\Big((\frac{v_{i_j}}{v_{i_j+1}})^{(-1)^{i_j}}\Big)^{(-1)^{j+\epsilon}}, \tag{15}$$

$$\Theta_{r,\epsilon}^{a,b}\{v_iv_{i+1}\}\qquad=\sum_{a\leq i_1<i_2<\cdots<i_r\leq b}\prod_{j=1}^{r}(v_{i_j}v_{i_j+1})^{(-1)^{j+\epsilon}}, \tag{16}$$

see Lemma 1 and Lemma 4. The later one is the definition of Theta given in [6]. The general definition of Theta,

$$\Theta_{r,\epsilon}^{a,b}\{f_i\}:=\sum_{a\leq i_1<i_2<\cdots<i_r\leq b}\prod_{j=1}^{r}(f_{i_j})^{(-1)^{j+\epsilon}}, \tag{17}$$

where $f$ is a function on $[a,b]\subset\mathbb{Z}$, is obtained by replacing $Y_i$ with

$$D_i=\begin{pmatrix}f_i^{(-1)^i}&0\\0&f_i^{(-1)^{i+1}}\end{pmatrix}. \tag{18}$$

Substituting $X_i=J$ and $Y_i=D_i$ into the Vieta expansion $(9a)$ we obtain

$$Z_r^{a,b}=J^{b-a+1-r}\Theta_r^{a,b}, \tag{19}$$

where

$$\Theta_r^{a,b}=\begin{pmatrix}\Theta_{r,a-1}^{a,b}\{f_i\}&0\\0&\Theta_{r,a}^{a,b}\{f_i\}\end{pmatrix}. \tag{20}$$

The equation (19) can be generalized to $2n \times 2n$ matrices $J$ and $D$, provided they satisfy the following properties

$$J^2 = I, \; D_i J D_i = J, \; \text{and } D_i \text{ is diagonal.}$$

To illustrate the second way of splitting $L$, we write the $L$ matrices of the mKdV and sine-Gordon equations in a different way with $X_i = H$ and $Y_i = s_i A^i_i$, where $H$ and $A^i_j$ are defined as

$$H := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A^i_j = \begin{pmatrix} a_i & a_i b_j \\ 1 & b_j \end{pmatrix}. \tag{21}$$

Then, with special properties of $H$ and $A$ we obtain multi-sums of products called Psi which are introduced below. In particular, if we take $s_i = f_i$, $a_i = 1/f_i$ and $b_i = 0$, then the formula ($9b$) can be evaluated as

$$\widetilde{Z}^{a,b}_r = \begin{pmatrix} \Phi^{a,b-1}_r & \Phi^{a+1,b-1}_{r-1} \\ f_b \Phi^{a,b-2}_r & f_b \Phi^{a+1,b-2}_{r-1} \end{pmatrix}, \tag{22}$$

where multi-sums of products, $\Phi$, are defined as:

$$\Phi^{a,b}_r\{f_i\} := \sum_{a \le i_1, i_1+1 < i_2, i_2+1 < \ldots < i_{r-1}, i_{r-1}+1 < i_r \le b} \prod_{j=1}^{r} f_{i_j}, \tag{23}$$

with $f$ a function on $[a, b] \subset \mathbb{Z}$. More generally, when $H$ and $A$ satisfy

$$H^2 = 0, \; A^n_m A^k_l = \alpha_{k,m} A^n_l, \; H A^n_m H = H, \; \text{and } A^n_m H A^k_l = A^n_l, \tag{24}$$

substituting $X_i = H$ and $Y_i = s_i A^i_i$ in equation ($9b$) yields

$$\widetilde{Z}^{a,b}_r = \Psi^{a+1,b-2}_{r-1} H A^{b-1}_a + \Psi^{a+2,b-1}_{r-1} A^b_{a+1} H + \Psi^{a+2,b-2}_{r-2} H + \Psi^{a+1,b-1}_r A^b_a, \tag{25}$$

where $\Psi$ is

$$\Psi^{a,b}_r := s_{a-1} \Phi^{a,b}_r \{\frac{1}{s_{i+1}\alpha_{i-1,i}\alpha_{i,i+1}}\} \prod_{i=a}^{b+1} (s_i \alpha_{i-1,i}). \tag{26}$$

The elementary symmetric functions satisfy the following recursive formulas (for all $c$ such that $0 \le c \le b - a + 1$),

$$s^{a,b}_r = \sum_{i=m_1}^{m_2} s^{a+c,b}_{r-i} s^{a,a+c-1}_i, \tag{27a}$$

$$s^{a,b}_r = \sum_{i=m_1}^{m_2} s^{b-c+1,b}_i s^{a,b-c}_{r-i}, \tag{27b}$$

where $m_1 = max(0, r + a + c - b - 1)$, $m_2 = min(r, c)$. These recursive formulas are obtained by using the Vieta expansion (10) and writing

$$\prod_{i=a}^{\overset{\curvearrowleft}{b}} z_i = \prod_{i=a+c}^{\overset{\curvearrowleft}{b}} z_i \prod_{i=a}^{\overset{\curvearrowleft}{a+c-1}} z_i, \tag{28}$$

where $z_i = \lambda + f_i$ in this case. Taking $c = r$, these recursive formulas provide us with an efficient way of producing and storing the elementary symmetric functions.

We will obtain similar recursive formulas for $Z_r^{a,b}$ and $\widetilde{Z}_r^{a,b}$ by using Vieta expansion and the matrix analogue of (28). Recursive formulas for Theta and Phi are derived from those for $Z_r^{a,b}$ and $\widetilde{Z}_r^{a,b}$, respectively. The recursivity of these multi-sums of products gives us a convenient way of computing these multi-sums of products.

The rest of this paper is organized as follows. In section 3, we explain how Theta arises from equation ($9a$) using properties of $J$ (14) and $D_i$ (18). Then, the recursive formulas for Theta are obtained . We give two closed-form expressions for integrals of the mKdV equation in terms of Theta, one of which coincides with the form discovered in [6], using the two Vieta formulas ($9a$), ($9b$). In section 4, we show how the multi-sums of products, Phi, emerge as entries of matrix coefficients. The recursive formulas for Phi are given and we prove equation (25). As a first application, we express integrals of mKdV equation in term of Psi. Next, we present a general formula for integrals for all equations whose reduced Lax matrices can be written in the form

$$r_i(\lambda H + s_i A_i^i). \tag{29}$$

We explicitly write the reduced Lax matrices for the equation in the ABS clasification in the form (29), with the exception of $Q_4$. So we obtain close-form expressions for integrals of these equations from the general formula. The final section compares two sets of integrals expressed in terms of Theta and Psi, respectively. This section also discusses possible future work.

## 3. Multi-sums of products, $\Theta$

In this section, we first present properties of the multi-sums of products, Theta. Then, an application of Theta is given. We derive two closed-form expressions for integrals of mKdV.

### 3.1. Properties of the multi-sums of products, $\Theta$

The integrals in terms of the multi-sums of products (17) with $f_i = v_i v_{i+1}$, first introduced in [6], were discovered by inspection and proved by induction. We found that the multi-sums of products can actually be derived from the (Vieta like) formula ($9a$), ($9b$) and special properties of the reduced Lax matrices. Also we show that the following properties, which were used in the proofs in [6, identity (3),(4)],

$$\Theta_{n,\epsilon}^{a,b} = \Theta_{n,\epsilon}^{a+1,b} + f_a^{(-1)^{1+\epsilon}} \Theta_{n-1,\epsilon\pm1}^{a+1,b}, \tag{30}$$

$$\Theta_{n,\epsilon}^{a,b} = \Theta_{n,\epsilon}^{a,b-1} + f_b^{(-1)^{n+\epsilon}} \Theta_{n-1,\epsilon}^{a,b-1}, \tag{31}$$

are special cases of the more general identities ($33a$) and ($33b$) given below. These identities ($33a$), ($33b$) are similar to the recursive formulas for the symmetric functions ($27a$) and ($27b$) and can be used to efficiently compute the multi-sums of products, Theta. They also play an important role in calculating gradients for proving the functional independence and the involutivity of integrals [13, 14].

**Lemma 1.** *Let* $J$ *and* $D_i$ *be as in (14) and(18) respectively. The multi-sums of products* $\Theta_{r,\epsilon}^{a,b}$ *defined by (17) are the entries in the matrix coefficients in the expansion*

$$\prod_{i=a}^{\overset{\curvearrowleft}{b}}(\lambda J + D_i) = \sum_{r=0}^{b-a+1} (\lambda J)^{b-a+1-r}\Theta_r^{a,b} \tag{32}$$

*where* $\Theta_r^{a,b}$ *is defined in (20).*

*Proof.* We use Vieta expansion ($8a$), ($9a$) to expand the left hand side of equation (32)

$$\prod_{i=a}^{\overset{\curvearrowleft}{b}}(\lambda J + D_i) = \sum_{r=0}^{b-a+1} \lambda^{b-a+1-r} Z_r^{a,b}.$$

Using properties of $J$ and $D_i$ such as $D_i J^k = J^k D_i^{(-1)^k}$ and $J^2 = I$, we have

$$\begin{aligned}
Z_r^{a,b} &= \sum_{a\le i_1<i_2\ldots<i_r\le b} JJ\ldots JD_{i_r}J\ldots JD_{i_{r-1}}J\ldots JD_{i_1}J\ldots J\\
&= \sum_{a\le i_1<i_2\ldots<i_r\le b} J^{b-i_r}D_{i_r}J^{i_r-i_{r-1}-1}D_{i_{r-1}}\ldots J^{i_2-i_1-1}D_{i_1}J^{i_1-a}\\
&= \sum_{a\le i_1<i_2\ldots<i_r\le b} J^{b-i_r}D_{i_r}J^{i_r-i_{r-1}-1}D_{i_{r-1}}\ldots J^{i_3-i_2-1}D_{i_2}J^{i_2-a-1}D_{i_1}^{(-1)^{i_1-a}}\\
&= \sum_{a\le i_1<i_2\ldots<i_r\le b} J^{b-i_r}D_{i_r}J^{i_r-i_{r-1}-1}D_{i_{r-1}}\ldots J^{i_3-a-2}D_{i_2}^{(-1)^{i_2-a-1}}D_{i_1}^{(-1)^{i_1-a}}\\
&\;\vdots\\
&= \sum_{a\le i_1<i_2\ldots<i_r\le b} J^{b-a+1-r}D_{i_r}^{(-1)^{i_r-(r-1)-a}}D_{i_{r-1}}^{(-1)^{i_{r-1}-(r-2)-a}}\ldots D_{i_1}^{(-1)^{i_1-a}}.
\end{aligned}$$

Now since

$$D_i^{(-1)^k} = \begin{pmatrix} f_i^{(-1)^{i+k}} & 0\\ 0 & f_i^{(-1)^{i+k+1}}\end{pmatrix},$$

we have

$$\begin{aligned}
\Theta_r^{a,b} &= \sum_{a\le i_1<i_2\ldots<i_r\le b}\\
&\begin{pmatrix} f_{i_r}^{(-1)^{-r-a+1}}f_{i_{r-1}}^{(-1)^{-(r-1)-a+1}}\ldots f_{i_1}^{(-1)^{-1-a+1}} & 0\\ 0 & f_{i_r}^{(-1)^{-r-a+2}}f_{i_{r-1}}^{(-1)^{-(r-1)-a+2}}\ldots f_{i_1}^{(-1)^{-1-a+2}}\end{pmatrix}\\
&= \begin{pmatrix}\Theta_{r,a-1}^{a,b} & 0\\ 0 & \Theta_{r,a}^{a,b}\end{pmatrix}.
\end{aligned}$$

□

Using Lemma 1 and the matrix analogue of (28) we obtain the following recursive formulas for Theta.

**Proposition 2.** *For any* $0 \le c \le b-a+1$ *we have*

$$\Theta^{a,b}_{r,\epsilon} = \sum_{i=m_1}^{m_2} \Theta^{a,a+c-1}_{i,\epsilon}\Theta^{a+c,b}_{r-i,\epsilon+i}, \tag{33$a$}$$

$$\Theta^{a,b}_{r,\epsilon} = \sum_{i=m_1}^{m_2} \Theta^{a,b-c}_{r-i,\epsilon}\Theta^{b-c+1,b}_{i,r+\epsilon+i}, \tag{33$b$}$$

*where* $m_1 = max(0, r+a+c-b-1)$ *and* $m_2 = min(r, c)$.

*Proof.* It is easy to see that (33$b$) follows from (33$a$) by substituting $c = b-a-c+1$ in (33$a$), changing variable $i = r-j$ and using the fact that $-j \le min(x, y) \Longrightarrow j \ge max(-x, -y)$.

By using the Vieta expansion (8$a$) and the product structure (28), the recursive formulas (27$a$) and (27$b$) still hold if we replace $s$ by $Z$. Therefore, using Lemma 1 we have

$$J^{b-a+1-r}\Theta^{a,b}_{r} = \sum_{i=m_1}^{m_2} J^{b-(a+c)+1-(r-i)}\Theta^{a+c,b}_{r-i}J^{c-i}\Theta^{a,a+c-1}_{i}. \tag{34}$$

If $b-a+1-r$ is even and using equation (20), we have

$$\begin{aligned}\begin{pmatrix} \Theta^{a,b}_{r,a-1} & 0 \\ 0 & \Theta^{a,b}_{r,a} \end{pmatrix} &= \sum_{m_1 \le i \le m_2,\ c-i \text{ even}} \begin{pmatrix} \Theta^{a+c,b}_{r-i,a+c-1} & 0 \\ 0 & \Theta^{a+c,b}_{r-i,a+c} \end{pmatrix}\begin{pmatrix} \Theta^{a,a+c-1}_{i,a-1} & 0 \\ 0 & \Theta^{a,a+c-1}_{i,a} \end{pmatrix} \\ &+ \sum_{m_1 \le i \le m_2,\ c-i \text{ odd}} \begin{pmatrix} 0 & \Theta^{a+c,b}_{r-i,a+c} \\ \Theta^{a+c,b}_{r-i,a+c-1} & \end{pmatrix}\begin{pmatrix} 0 & \Theta^{a,a+c-1}_{i,a} \\ \Theta^{a,a+c-1}_{i,a-1} & 0 \end{pmatrix}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\Theta^{a,b}_{r,a-1} &= \sum_{i=m_1}^{m_2} \Theta^{a,a+c-1}_{i,a-1}\Theta^{a+c,b}_{r-i,a+i-1}, \\ \Theta^{a,b}_{r,a} &= \sum_{i=m_1}^{m_2} \Theta^{a,a+c-1}_{i,a}\Theta^{a+c,b}_{r-i,a+i}.\end{aligned}$$

Since $\{a-1, a\} = \{0, 1\}$ (mod 2), we obtain

$$\Theta^{a,b}_{r,\epsilon} = \sum_{i=m_1}^{m_2} \Theta^{a,a+c-1}_{i,\epsilon}\Theta^{a+c,b}_{r-i,\epsilon+i}.$$

The proof for odd $b-a+1-r$ is similar. □

Note that from the definition of Theta (17), or from equation (32), we have the following properties

- $\Theta^{a,b}_{r,\epsilon} = 0$ if $r < 0$ or $r > b-a+1$,
- $\Theta^{a,b}_{0,\epsilon} = 1$,
- $\Theta^{a,a}_{1,\epsilon} = f_a^{(-1)^{1+\epsilon}}$.

If we consider these properties as initial values of Theta, then we can efficiently calculate Theta from the recursive formulae (33$a$) by taking $c = b-a$.

### *3.2. Applications of $\Theta$ to the mKdV equation*

The $(1,z_2)$-reductions of the Lax matrices for mKdV are given by

$$L_n = \begin{pmatrix} \frac{v_n}{v_{n+1}} & k \\ k & v_{n+1}/v_n \end{pmatrix} \quad \text{and} \quad M_n^{-1} = \begin{pmatrix} \alpha_3 \frac{v_{n+z_2}}{v_n} & \alpha_1 k \\ \alpha_1 k & \alpha_2 \frac{v_n}{v_{n+z_2}} \end{pmatrix}. \tag{35}$$

Indeed, we have $L_n M_n^{-1} - M_{n+1}^{-1} L_{n+z_2} = \mathcal{F}_{\mathrm{mKdV}} \cdot N_n$, where $\mathcal{F}_{\mathrm{mKdV}}$ is the left hand side of (12), and

$$N_n = \begin{pmatrix} 0 & \frac{k}{v_{n+1}v_{n+z_2}} \\ \frac{-k}{v_n v_{n+z_2+1}} & 0 \end{pmatrix}.$$

Using Lemma 1 with $f_i = (v_i/v_{i+1})^{(-1)^i}$, $a = 0, b = z_2 - 1$, and taking the trace of the monodromy matrix (6), we obtain the following result.

**Proposition 3.** *The closed-form expressions for integrals of the mKdV equation are given as follows*

$$I_r = \alpha_1(\Theta_{r,0}^{0,z_2-1} + \Theta_{r,1}^{0,z_2-1}) + \alpha_2 \frac{v_0}{v_{z_2}} \Theta_{r-1,0}^{0,z_2-1} + \alpha_3 \frac{v_{z_2}}{v_0} \Theta_{r-1,1}^{0,z_2-1}, \tag{36}$$

*where $z_2 - r$ is odd.*

However, in [6], integrals of the mKdV equation are expressed in terms of Theta with different arguments, namely $f_i = v_i v_{i+1}$. To derive the later close-form expressions for integrals of the mKdV equation, the following Lemma was used [6].

**Lemma 4.** *We have*

$$\prod_{i=a}^{\overset{\curvearrowleft}{b-1}} L_i = \sum_{i=0}^{b-a} \widetilde{Z}_r^{a,b-1} \lambda^r, \tag{37}$$

*where $f_i = v_i v_{i+1}$, and*

$$\widetilde{Z}_r^{a,b-1} = \begin{pmatrix} \frac{v_a}{v_b} \Theta_{r,0}^{a,b-1} & 0 \\ 0 & \frac{v_b}{v_a} \Theta_{r,1}^{a,b-1} \end{pmatrix}$$

*when $r$ is even, and*

$$\widetilde{Z}_r^{a,b-1} = \begin{pmatrix} 0 & \frac{1}{v_a v_b} \Theta_{r,1}^{a,b-1} \\ v_a v_b \Theta_{r,0}^{a,b-1} & 0 \end{pmatrix}$$

*when $r$ is odd.*

Whereas for Lemma 1 we used the Vieta expansion $(9a)$, this Lemma is proved using the Vieta expansion $(9b)$, see Appendix A.

Multiplying both sides of equation (37) by $M_0^{-1}$ and taking the trace we obtain the following result, which is Theorem 3 in [6].

**Proposition 5.** *The trace of the monodromy matrix of the mKdV equation is*

$$\mathrm{Tr}(\mathcal{L}_0) = \sum_{i=0}^{\lfloor (z_2+1)/2 \rfloor} k^{2r} I_r,$$

*where*

$$I_r = \alpha_1 (v_0 v_{z_2} \Theta^{0,z_2-1}_{2r-1,0} + \frac{1}{v_0 v_{z_2}} \Theta^{0,z_2-1}_{2r-1,1}) + \alpha_2 \Theta^{0,z_2-1}_{2r,1} + \alpha_3 \Theta^{0,z_2-1}_{2r,0}, \tag{38}$$

*with* $f_i = v_i v_{i+1}$.

It follows that the coefficients $I_r$ form a set of integrals for the mKdV equation. Their invariance can also be proven directly. By using the properties (30) and (31) of multi-sums of products it was shown in [6] that

$$S(I_r) - I_r = \mathcal{F}_{\mathrm{mKdV}} \cdot \Lambda_r, \tag{39}$$

where $S$ is the shift operator $S(v_i) = v_{i+1}$, and

$$\Lambda_r = \frac{1}{v_0 v_1 v_{z_2} v_{z_2+1}} \Theta^{1,z_2-1}_{2r-1,1} - \Theta^{1,z_2-1}_{2r-1,0} \tag{40}$$

is called an integrating factor.

It is important to note that the set of integrals expressed in Theta with $f_i = (v_i/v_{i+1})^{(-1)^i}$ is the same as the one expressed in Theta with $f_i = v_i v_{i+1}$ due to the following identities

$$\Theta^{a,b}_{r,a+\epsilon}\{(\frac{v_i}{v_{i+1}})^{(-1)^i}\} = \left(\frac{v_a}{v_{b+1}}\right)^{(-1)^{\epsilon+1}} \Theta^{a,b}_{b-a+1-r,\epsilon+1}\{v_i v_{i+1}\},$$

if $b-a+1-r$ is even, and

$$\Theta^{a,b}_{r,a+\epsilon}\{(\frac{v_i}{v_{i+1}})^{(-1)^i}\} = (v_a v_{b+1})^{(-1)^{\epsilon+1}} \Theta^{a,b}_{b-a+1-r,\epsilon+1}\{v_i v_{i+1}\}$$

if $b-a+1-r$ is odd.

## 4. Multi-sums of products, $\Phi$

Writing the reduced Lax matrices as linear combinations of rank one 2×2 matrices, gives rise to closed-form expressions in terms of multi-sums of products Phi, defined in equation (23). In this section, we first give recursive formulas for Phi. Then, by using the non-commutative Vieta expansion, we give the formula (25) for products of $L_i = r_i(\lambda H + s_i A^i_i)$ matrices in terms of Psi (26) where $A^i_j$ is defined in (21). This formula is first applied to the mKdV equation. At the end we give the analogous results for nearly all equations in the Adler-Bobenko-Suris list [1].

### *4.1. Recursive formulas for the multi-sums of products, $\Phi$*

The Lemma below explains how Phi is derived from the Vieta expansion $(8b)$ and $(9b)$.

**Lemma 6.** *Let $H$ and $F_i$ be defined by*

$$H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; F_i = \begin{pmatrix} 1 & 0 \\ f_i & 0 \end{pmatrix}$$

*and let $f$ be a function on $[a, b+1] \subset \mathbb{Z}$. The multi-sums of products, $\Phi$, defined by (23) appear in the entries of the matrix coefficients in the expansion*

$$\prod_{i=a}^{\overset{\curvearrowleft}{b+1}} (\lambda H + F_i) = \sum_{r=0}^{b-a+2} \lambda^r W_r^{a,b+1},$$

*where*

$$W_r^{a,b+1} = \begin{pmatrix} \Phi_r^{a,b} & \Phi_{r-1}^{a+1,b} \\ f_{b+1}\Phi_r^{a,b-1} & f_{b+1}\Phi_{r-1}^{a+1,b-1} \end{pmatrix}.$$

*Proof.* Using the properties $H^2 = 0,\; F_iF_j = F_i,\; F_iHF_j = f_jF_i,\; HF_iH = f_iH$ and the non-commutative Vieta formula $(9b)$, we have

$$\begin{aligned}
W_r^{a,b+1} = &\sum_{a+2\le i_1, i_1+1<i_2, i_2+1<\ldots<i_{r-2}\le b-1} HF_b \ldots F_{i_{r-2}+1}HF_{i_{r-2}-1} \ldots F_{i_1+1}HF_{i_1-1} \ldots F_{a+1}H \\
&+ \sum_{a+1\le i_1, i_1+1<i_2, i_2+1<\ldots<i_{r-1}\le b-1} HF_b \ldots F_{i_{r-1}+1}HF_{i_{r-1}-1} \ldots F_{i_1+1}HF_{i_1-1} \ldots F_a \\
&+ \sum_{a+2\le i_1, i_1+1<i_2, i_2+1<\ldots<i_{r-1}\le b} F_{b+1} \ldots F_{i_{r-1}+1}HF_{i_{r-1}-1} \ldots F_{i_1+1}HF_{i_1-1} \ldots F_{a+1}H \\
&+ \sum_{a+1\le i_1, i_1+1<i_2, i_2+1<\ldots<i_r\le b} F_{b+1} \ldots F_{i_r+1}HF_{i_r-1} \ldots F_{i_1+1}HF_{i_1-1} \ldots F_a \\
= &\, f_b\Phi_{r-2}^{a+1,b-2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + f_b\Phi_{r-1}^{a,b-2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \Phi_{r-1}^{a+1,b-1}\begin{pmatrix} 0 & 1 \\ 0 & f_{b+1} \end{pmatrix} + \Phi_r^{a,b-1}\begin{pmatrix} 1 & 0 \\ f_{b+1} & 0 \end{pmatrix} \\
= &\begin{pmatrix} f_b\Phi_{r-1}^{a,b-2} + \Phi_r^{a,b-1} & f_b\Phi_{r-2}^{a+1,b-2} + \Phi_{r-1}^{a+1,b-1} \\ f_{b+1}\Phi_r^{a,b-1} & f_{b+1}\Phi_{r-1}^{a+1,b-1} \end{pmatrix}.
\end{aligned}$$

From the definition of Phi, we have the following properties

$$\Phi_r^{a,b} = \Phi_r^{a,b-1} + f_b\Phi_{r-1}^{a,b-2}, \tag{41}$$

$$\Phi_r^{a,b} = f_a\Phi_{r-1}^{a+2,b} + \Phi_r^{a+1,b}. \tag{42}$$

Therefore, we obtain

$$W_r^{a,b+1} = \begin{pmatrix} \Phi_r^{a,b} & \Phi_{r-1}^{a+1,b} \\ f_{b+1}\Phi_r^{a,b-1} & f_{b+1}\Phi_{r-1}^{a+1,b-1} \end{pmatrix}.$$

□

Using this Lemma, we derive recursive formulas for Phi.

**Proposition 7.** *With $a-1 \le c \le b+1$ we have*

$$\Phi_r^{a,b} = \sum_{i=0}^{r} (\Phi_{r-i}^{a,c-1}\Phi_i^{c+1,b} + \Phi_{r-i-1}^{a,c-2}\Phi_1^{c,c}\Phi_i^{c+2,b}) \tag{43$a$}$$

$$= \sum_{i=0}^{r} (\Phi_{r-i}^{a,c-1}\Phi_1^{c+1,c+1}\Phi_{i-1}^{c+3,b} + \Phi_{r-i}^{a,c}\Phi_i^{c+2,b}). \tag{43$b$}$$

We note that properties (41) and (42) are special cases of these recursive formulas (43$a$) and (43$b$) where $c = b$ and $c = a-1$, respectively.

*Proof.* We use recursive formula 33$a$ with $W$ replacing $s$. We have

$$\begin{aligned} W_r^{a,b+1} &= \sum_{i=0}^{r} W_i^{c+1,b+1} W_{r-i}^{a,c} \\ &= \sum_{i=0}^{r} \begin{pmatrix} \Phi_i^{c+1,b} & \Phi_{i-1}^{c+2,b} \\ f_{b+1}\Phi_i^{c+1,b-1} & f_{b+1}\Phi_{i-1}^{c+2,b-1} \end{pmatrix} \begin{pmatrix} \Phi_{r-i}^{a,c-1} & \Phi_{r-i-1}^{a+1,c-1} \\ f_c\Phi_{r-i}^{a,c-2} & f_c\Phi_{r-i-1}^{a+1,c-2} \end{pmatrix}. \end{aligned}$$

Equating the entry in the first column and first row, we get

$$\Phi_r^{a,b} = \sum_{i=0}^{r} (\Phi_{r-i}^{a,c-1}\Phi_i^{c+1,b} + \Phi_{r-i-1}^{a,c-2} f_c \Phi_i^{c+2,b}),$$

which is the first recursive formula (43$a$). The second recursive formula (43$b$) is obtained from the first one by using the following

$$\begin{aligned} \Phi_i^{c+1,b} &= \Phi_i^{c+2,b} + \Phi_1^{c+1,c+1}\Phi_{i-1}^{c+3,b}, \\ \Phi_{r-i-1}^{a,c-2}\Phi_1^{c,c} &= \Phi_{r-i}^{a,c} - \Phi_{r-i}^{a,c-1}. \end{aligned}$$

□

Note that from the definition of Phi (23), we have the following properties

- $\Phi_r^{a,b} = 0$ when $r < 0$ or $r > \lfloor (b-a+1)/2 \rfloor$,
- $\Phi_0^{a,b} = 1$,
- $\Phi_1^{a,a} = f_a$.

Once again, these properties can be considered as initial values to calculate Phi using the recursive formulas for Phi provided.

### *4.2. Product of L matrices and multi-sum of products, Ψ*

Let $A_j^i$ and $H$ be matrices that are defined in (21). Now using the Vieta-expansion (8$b$,9$b$) and properties of $H$ and $A$, we obtain the following lemma.

**Lemma 8.** *Let $L_i = r_i(\lambda H + s_i A_i^i)$. We have*

$$\prod_{i=a}^{\overset{\curvearrowleft}{b}} L_i = \Big( \sum_{r=0}^{b-a+1} X_r^{a,b} \lambda^r \Big) \prod_{i=a}^{b} r_i,$$

*with*

$$X_r^{a,b} = \Psi_{r-1}^{a+1,b-2} H A_a^{b-1} + \Psi_{r-1}^{a+2,b-1} A_{a+1}^b H + \Psi_{r-2}^{a+2,b-2} H + \Psi_r^{a+1,b-1} A_a^b, \tag{44}$$

*where* $c_i = s_i(a_{i-1} + b_i)$ *and*

$$\Psi_r^{a,b} := s_{a-1} \Phi_r^{a,b} \{ \frac{s_{i+1}}{c_i c_{i+1}} \} \prod_{i=a}^{b+1} c_i. \tag{45}$$

The general case (25) is obtained in the same way.

*Proof.* For brevity of notation, we write $B_i := s_i A_i^i$. Using the properties of $H$ and $A_m^n$ we have

$$B_k B_{k-1} \dots B_l = s_k A_k^k \dots s_l A_l^l = s_l A_l^k \prod_{r=l+1}^{k} c_r$$

so when $l < i < k$ we have

$$\begin{aligned} B_k B_{k-1} \dots B_{i+1} H B_{i-1} \dots B_l &= s_{i+1} \Big( A_{i+1}^k \prod_{r=i+2}^{k} c_r \Big) H s_l \Big( A_l^{i-1} \prod_{r=l+1}^{i-1} c_r \Big) \\ &= \frac{s_{i+1}}{c_i c_{i+1}} s_l A_l^k \prod_{r=l+1}^{k} c_r = s_l f_i A_l^k \prod_{r=l+1}^{k} c_r. \end{aligned}$$

Therefore, we obtain

$$B_k \dots B_{i_r+1} H B_{i_r-1} \dots B_{i_1+1} H B_{i_1-1} \dots B_l = s_l A_l^k \prod_{j=1}^{r} f_{i_j} \prod_{r=l+1}^{k} c_r$$

when $l < i_1, i_1 + 1 < i_2, \dots < i_{i_r} < k$, and it equals 0 when $i_j = i_{j-1} + 1$ for some $j$. Now applying these formulas and the formula $(9b)$ we have

$$\begin{aligned} X_r^{a,b} = & \sum_{a+1 \le i_1, i_1+1 < i_2, \dots < i_{r-1} \le b-2} H B_{b-1} \dots B_{i_{r-1}+1} H B_{i_{r-1}-1} \dots B_{i_1+1} H B_{i_1-1} \dots B_a \\ & + \sum_{a+2 \le i_1, i_1+1 < i_2, \dots < i_{r-1} \le b-1} B_b \dots B_{i_{r-1}+1} H B_{i_{r-1}-1} \dots B_{i_1+1} H B_{i_1-1} \dots B_{a+1} H \\ & + \sum_{a+2 \le i_1, i_1+1 < i_2, \dots < i_{r-2} \le b-2} H B_{b-1} \dots B_{i_{r-2}+1} H B_{i_{r-2}-1} \dots B_{i_1+1} H B_{i_1-1} \dots B_{a+1} H \\ & + \sum_{a+1 \le i_1, i_1+1 < i_2, \dots < i_r \le b-1} B_b \dots B_{i_r+1} H B_{i_r-1} \dots B_{i_1+1} H B_{i_1-1} \dots B_a \\ = & \Big( \frac{s_a}{c_b} \Phi_{r-1}^{a+1,b-2} H A_a^{b-1} + \frac{s_{a+1}}{c_{a+1}} \Phi_{r-1}^{a+2,b-1} A_{a+1}^b H + \frac{s_{a+1}}{c_{a+1} c_b} \Phi_{r-2}^{a+2,b-2} H + s_a \Phi_r^{a+1,b-1} A_a^b \Big) \prod_{i=a+1}^{b} c_i \\ = & \Psi_{r-1}^{a+1,b-2} H A_a^{b-1} + \Psi_{r-1}^{a+2,b-1} A_{a+1}^b H + \Psi_{r-2}^{a+2,b-2} H + \Psi_r^{a+1,b-1} A_a^b. \end{aligned}$$

□

Note that if $r > \lfloor (b-a+1)/2 \rfloor$ we have $X_r^{a,b} = 0$. We also note that Lemma 6 is a special case of this Lemma with $a_i = 1/f_i$ and $b_i = 0$ and $s_i = f_i$.

### *4.3. Application of $\Psi$ to the mKdV equation*

In this section, we present closed-form expressions for integrals of mKdV in terms of the multi-sums of products Psi.

The reduced Lax pair for the mKdV equation given by (35) gives rise to the multi-sums of products, Theta. Now we use a gauge transformation to obtain a new reduced Lax pair which gives the multi-sums of products, Psi. Recall that for a P$\Delta$E (1) with a Lax pair $(L_{l,m}, M_{l,m})$, a gauge matrix $G_{l,m}$ gives us a new Lax pair for the equation, that is

$$\widetilde{L}_{l,m} = G_{l+1,m} L_{l,m} G_{l,m}^{-1}, \ \widetilde{M}_{l,m} = G_{l,m+1} M_{l,m} G_{l,m}^{-1}.$$

These matrices reduce to matrices $\widetilde{L}, \widetilde{M}$ of the corresponding O$\Delta$E.

Using the gauge matrix

$$G_{l,m} = \begin{pmatrix} 0 & 1/v_{l,m} \\ -1/k & 0 \end{pmatrix},$$

we have a new reduced Lax pair for the mKdV equation

$$\widetilde{L}_n = \begin{pmatrix} \frac{v_{n+1}}{v_n} & \frac{-k^2}{v_n} \\ -v_{n+1} & 1 \end{pmatrix} \qquad \text{and} \qquad \widetilde{M}_n^{-1} = \begin{pmatrix} \alpha_2 \frac{v_n}{v_{n+z_2}} & -\alpha_1 \frac{k^2}{v_{n+z_2}} \\ -\alpha_1 v_n & \alpha_3 \end{pmatrix}. \tag{46}$$

We note that the trace of the monodromy matrix is invariant under gauge transformations.

Now we write the reduced Lax pair for the mKdV equation as follows

$$\widetilde{L}_i = r_i(s_i A_i^i + \lambda H), \ (\widetilde{M}_i)^{-1} = \breve{r}_i(\breve{s}_i \breve{A}_i^i + \breve{\lambda} H), \tag{47}$$

where

$$\breve{A}_j^i = \begin{pmatrix} \breve{a}_i & \breve{a}_i \breve{b}_j \\ 1 & \breve{b}_j \end{pmatrix},$$

and

$$\begin{array}{lllll} \lambda = 1 - k^2, & r_i = \frac{1}{v_i}, & s_i = -v_i v_{i+1}, & a_i = -\frac{1}{v_i}, & b_i = -\frac{1}{v_{i+1}}, \\ \breve{\lambda} = \lambda + \frac{\alpha_2\alpha_3 - \alpha_1^2}{\alpha_1}, & \breve{r}_i = \frac{1}{v_{i+z_2}}, & \breve{s}_i = -\alpha_1 v_i v_{i+z_2}, & \breve{a}_i = \frac{-\alpha_2}{\alpha_1 v_{i+z_2}}, & \breve{b}_i = \frac{-\alpha_3}{\alpha_1 v_i}. \end{array}$$

Using Lemma 8 we expand the trace of the monodromy matrix in powers of $\lambda$. After multiplying with $\widetilde{M}_0^{-1}$ and taking the trace, we get

$$\mathrm{Tr}\widetilde{\mathcal{L}}_0 = \sum_{i=0}^{z_2+1} \lambda^i \widetilde{I}_i,$$

where

$$\widetilde{I}_r = \Big(\alpha_1 \Psi_{r-1}^{1,z_2-2} + (\alpha_2 v_0 + \frac{\alpha_1 v_0 v_{z_2}}{v_1})\Psi_{r-1}^{1,z_2-3} + (\alpha_3 v_{z_2} + \frac{\alpha_1 v_0 v_{z_2}}{v_{z_2-1}})\Psi_{r-1}^{2,z_2-2} - \alpha_1 v_0 v_{z_2} \Psi_{r-2}^{2,z_2-3}$$
$$-(\frac{\alpha_2 v_0}{v_{z_2-1}} + \frac{\alpha_3 v_{z_2}}{v_1} + \frac{\alpha_1 v_0 v_{z_2}}{v_1 v_{z_2-1}} + \alpha_1)\Psi_r^{1,z_2-2}\Big) \prod_{i=0}^{z_2} v_i^{-1}, \tag{48}$$

with

$$\Psi_r^{a,b} = (-v_{a-1}v_a)\Phi_r^{a,b}\{\frac{v_{i-1}v_{i+2}}{(v_{i-1}+v_{i+1})(v_i+v_{i+2})}\}\prod_{i=a}^{b+1}\frac{v_i(v_{i-1}+v_{i+1})}{v_{i-1}} \tag{49}$$

Note that if $r > \lfloor (z_2+1)/2 \rfloor$, then $\widetilde{I}_r = 0$. It is clear by the staircase method that $\widetilde{I}_r$ is an integral of the corresponding mKdV map. However, using properties of Psi one can also show that $\widetilde{I}_r$ is invariant under the mKdV map. By doing so, we obtain an integrating factor [6] which we do not get from the staircase.

**Proposition 9.** *For $0 \le r \le \lfloor (z_2+1)/2 \rfloor$, $\widetilde{I}_r$ is an integral of the mKdV map with integrating factor*

$$\Lambda_r = \frac{1}{v_0 v_1 v_{z_2} v_{z_2+1}}(\Psi_r^{1,z_2-1} + v_1 v_{z_2}\Psi_{r-1}^{2,z_2-2} - v_1\Psi_r^{2,z_2-1} - v_{z_2}\Psi_r^{1,z_2-2})\prod_{i=1}^{z_2} v_i^{-1},$$

*where $\Psi$ is given by (49).*

The proof of this Proposition is given in the Appendix B.

### *4.4. General closed-form expressions for integrals of all equations in the ABS list, except $Q_4$*

In this section, we give similar results as we did for the mKdV equation for almost all equations in the ABS list. We present a general formula for closed-form expressions for integrals of all equations whose reduced Lax pairs can be written in the form (47). Then, we write the reduced Lax matrices of ABS equations with the exception of $Q_4$ in this form, so that we can apply the general expressions to obtain integrals.

*4.4.1. General closed-form expressions for integrals* Assume that the reduced Lax matrices are written in the form (47), with $\breve{\lambda} = g\lambda + h$. We now expand the trace of the monodromy matrix $\mathcal{L}_0$ in terms of powers of $\lambda$.

**Theorem 10.** *Let $\mathcal{L}_0$ be given by (6), where $L, M^{-1}$ are matrices which can be written in the form (47). Then, we have*

$$\mathrm{Tr}(\mathcal{L}_0) = \sum_{i=0}^{z_2+1}\lambda^r I_r,$$

*where $I_r$ is expressed in terms of $\Psi$ (45)*

$$\begin{aligned} I_r = (&g\Psi_{r-1}^{1,z_2-2} + h\Psi_r^{1,z_2-2} + \breve{s}_0(\breve{a}_0+b_0)\Psi_{r-1}^{1,z_2-3} + \breve{s}_0(\breve{b}_0+a_{z_2-1})\Psi_{r-1}^{2,z_2-2} + \breve{s}_0\Psi_{r-2}^{2,z_2-3} \\ &+\breve{s}_0(\breve{a}_0+b_0)(a_{z_2-1}+\breve{b}_0)\Psi_r^{1,z_2-2})\breve{r}_0\prod_{i=0}^{z_2-1} r_i. \end{aligned} \tag{50}$$

*Proof.* We use Lemma 8 to expand the monodromy matrix. We have

$$\mathcal{L}_0 = (\breve{s}_0\breve{A}_0^0 + (g\lambda+h)H)(\sum_{r=0}^{z_2}\lambda^r X_r^{0,z_2-1})\breve{r}_0\prod_{i=0}^{z_2-1} r_i = (\sum_{r=0}^{z_2+1}\lambda^r W_r)\breve{r}_0\prod_{i=0}^{z_2-1} r_i, \tag{51}$$

where

$$W_r = (\breve{s}_0\breve{A}_0^0 + hH)X_r^{0,z_2-1} + gHX_{r-1}^{0,z_2-1}.$$

Using properties of H and A, we have

$$\begin{aligned}
\mathrm{Tr}(HX_r^{0,z_2-1}) &= \Psi_r^{1,z_2-2},\\
\mathrm{Tr}(HX_{r-1}^{0,z_2-1}) &= \Psi_{r-1}^{1,z_2-2},\\
\mathrm{Tr}(\breve{A}_0^0X_r^{0,z_2-1}) &= (\breve{a}_0 + b_0)\Psi_{r-1}^{1,z_2-3} + (\breve{b}_0 + a_{z_2-1})\Psi_{r-1}^{2,z_2-2} + \Psi_{r-2}^{2,z_2-3}\\
&\qquad + (\breve{a}_0 + b_0)(a_{z_2-1} + \breve{b}_0)\Psi_r^{1,z_2-2},
\end{aligned}$$

which we use to evaluate the trace of (51). □

Now from this theorem, if the $I_r$ do not depend on $\lambda$, then from the staircase method we have $I_r$ is an invariant. Hence, we have the following corollary.

**Corollary 11.** *Suppose that* $a_{i-1}+b_i, \breve{a}_0+b_0, a_{z_2-1}+\breve{b}_0, r_i, s_i, \breve{r}_i, \breve{s}_i, g$ *and* $h$ *do not depend on the spectral parameter* $k$*. Then* $I_r$ *given by (50) is an integral.*

Here we give a direct proof that $I_r$ is an integral of the equation derived from the Lax equation $L_0M_0^{-1} = M_1^{-1}L_{z_2}$.

*Proof.* Since $L_0M_0^{-1} = M_1^{-1}L_{z_2}$, we have

$$r_0\breve{r}_0(s_0A_0^0 + \lambda H)(\breve{s}_0\breve{A}_0^0 + (g\lambda + h)H) = \breve{r}_1r_{z_2}(\breve{s}_1\breve{A}_1^1 + (g\lambda + h)H)(s_{z_2}A_{z_2}^{z_2} + \lambda H).$$

If $a_i, \breve{a}_i, b_i, \breve{b}_i$ do not depend on $k$ (the cases $H_1$ and $H_3$), equating coefficients of $\lambda$ in both sides we obtain $E_i = 0,\ i = 1, 2, ..., 6$, with

$$\begin{aligned}
E_1 :&= r_0\breve{r}_0\breve{s}_0 - \breve{r}_1r_{z_2}gs_{z_2},\\
E_2 :&= r_0\breve{r}_0gs_0 - \breve{r}_1r_{z_2}\breve{s}_1,\\
E_3 :&= r_0\breve{r}_0(\breve{s}_0\breve{b}_0 + gs_0a_0) - \breve{r}_1r_{z_2}(\breve{s}_1\breve{a}_1 + gs_{z_2}b_{z_2}),\\
E_4 :&= \breve{r}_0r_0s_0\breve{s}_0(b_0 + \breve{a}_0) - r_d\breve{r}_1\breve{s}_1s_{z_2}(a_{z_2} + \breve{b}_1),\\
E_5 :&= r_0\breve{r}_0s_0\Big(\breve{s}_0(b_0 + \breve{a}_0)\breve{b}_0 + h\Big) - \breve{r}_1r_{z_2}\breve{s}_1s_{z_2}b_{z_2}(a_{z_2} + \breve{b}_1),\\
E_6 :&= \breve{r}_0r_0s_0a_0\breve{s}_0(b_0 + \breve{a}_0) - \breve{r}_1r_{z_2}s_{z_2}\Big(\breve{s}_1(a_{z_2} + \breve{b}_1)\breve{a}_1 + h\Big).
\end{aligned}$$

For the case where $a_i, b_i, \breve{a}_i, \breve{b}_i$ depend on $k$, we were able to check that the identities $E_i = 0$ hold for the cases $Q_1$ and $Q_3^0$. For the rest of the equations, calculations get complicated as we have to deal with square roots.

Using the fourth and fifth identities, we have $h = \breve{s}_0(b_0 + \breve{a}_0)(b_{z_2} - b_0)$. Similarly, using the fourth and sixth identities, we have $h = \breve{s}_1(a_0 - \breve{a}_1)(a_{z_2} + \breve{b}_1)$. We have

$$\begin{aligned}
S(I_r) = \Big(&g\Psi_{r-1}^{2,z_2-1} + h\Psi_r^{2,z_2-1} + \breve{s}_1(\breve{a}_1 + b_1)\Psi_{r-1}^{2,z_2-2} + \breve{s}_1(\breve{b}_1 + a_{z_2})\Psi_{r-1}^{3,z_2-1}\\
&+ \breve{s}_1\Psi_{r-2}^{3,z_2-2} + s_1(\breve{a}_1 + b_1)(a_{z_2} + \breve{b}_1)\Psi_r^{2,z_2-1}\Big)\breve{r}_1\prod_{i=1}^{z_2} r_i.
\end{aligned}$$

Now we write $S(I_r) - I_r = (A+B)\prod_{i=1}^{z_2-1} r_i$ where

$$\begin{aligned} A: &= \breve{r}_1 r_{z_2}\Big(g\Psi_{r-1}^{2,z_2-1} + \breve{s}_1(\breve{a}_1 + b_1)\Psi_{r-1}^{2,z_2-2} + \breve{s}_1\Psi_{r-2}^{3,z_2-2}\Big) \\ &\quad - r_0\breve{r}_0\Big(g\Psi_{r-1}^{1,z_2-2} + \breve{s}_0(\breve{b}_0 + a_{z_2-1})\Psi_{r-1}^{2,z_2-2} + \breve{s}_0\Psi_{r-2}^{2,z_2-3}\Big) \\ B: &= \breve{r}_1 r_{z_2}\Big((h + \breve{s}_1(\breve{a}_1 + b_1)(a_{z_2} + \breve{b}_1))\Psi_r^{2,z_2-1} + \breve{s}_1(\breve{b}_1 + a_{z_2})\Psi_{r-1}^{3,z_2-1}\Big) \\ &\quad - \breve{r}_0 r_0\Big((h + \breve{s}_0(\breve{a}_0 + b_0)(a_{z_2-1} + \breve{b}_0))\Psi_r^{1,z_2-2} + \breve{s}_0(\breve{a}_0 + b_0)\Psi_{r-1}^{1,z_2-3}\Big). \end{aligned}$$

From the properties of Phi (41) and (42), we obtain the following properties of Psi

$$\Psi_r^{n,m} = s_{m+1}(a_m + b_{m+1})\Psi_r^{n,m-1} + s_{m+1}\Psi_{r-1}^{n,m-2}, \tag{52}$$

$$\Psi_r^{n,m} = s_{n-1}(a_{n-1} + b_n)\Psi_r^{n+1,m} + s_{n-1}\Psi_{r-1}^{n+2,m}. \tag{53}$$

where $n \le m$ and $0 \le r$. Using these properties (52) and (53), we get

$$\begin{aligned} A &= \Big(\breve{r}_1 r_{z_2}\Big(gs_{z_2}(a_{z_2-1} + b_{z_2}) + \breve{s}_1(\breve{a}_1 + b_1)\Big) - \breve{r}_0 r_0\Big(gs_0(a_0 + b_1) + \breve{s}_0(\breve{b}_0 + a_{z_2-1})\Big)\Big) \\ &\qquad \Psi_{r-1}^{2,z_2-2} + \Big(\breve{r}_1 r_{z_2} g s_{z_2} - r_0\breve{r}_0\breve{s}_0\Big)\Psi_{r-2}^{2,z_2-3} + \Big(\breve{r}_1 r_{z_2}\breve{s}_1 - r_0\breve{r}_0 g s_0\Big)\Psi_{r-2}^{3,z_2-2} \\ &= (a_{z_2-1}\Psi_{r-1}^{2,z_2-2} + \Psi_{r-2}^{2,z_2-3})(\breve{r}_1 r_{z_2} g s_{z_2} - r_0\breve{r}_0\breve{s}_0) + (b_1\Psi_{r-1}^{2,z_2-2} + \Psi_{r-2}^{3,z_2-2}). \\ &\qquad (\breve{r}_1 r_{z_2}\breve{s}_1 - \breve{r}_0 r_0 g s_0) + (\breve{r}_1 r_{z_2}(g s_{z_2} a_{z_2} + \breve{s}_1\breve{a}_1) - r_0\breve{r}_0(g s_0 a_0 + \breve{s}_0\breve{b}_0))\Psi_{r-1}^{2,z_2-2} \\ &= -(a_{z_2-1}\Psi_{r-1}^{2,z_2-2} + \Psi_{r-2}^{2,z_2-3})E_1 - (b_1\Psi_{r-1}^{2,z_2-2} + \Psi_{r-2}^{3,z_2-2})E_2 - \Psi_{r-1}^{2,z_2-2}E_3 \\ &= 0. \end{aligned}$$

We also have

$$\begin{aligned} B &= \breve{r}_1 r_{z_2}(h + \breve{s}_1(a_{z_2} + \breve{b}_1)(a_0 - \breve{a}_0))\Psi_r^{2,z_2-1} - \breve{r}_0 r_0(h + s_0(\breve{a}_0 + b_0)(\breve{b}_0 - b_{z_2}))\Psi_r^{1,z_2-2} \\ &\quad -\tfrac{E_4}{s_0 s_{z_2}}\Psi_r^{1,z_2-1} \\ &= 0. \end{aligned}$$

That proves our statement. □

From this proof, one can derive an integrating factor by dividing $E_1, E_2, E_3, E_4$ and $h + \breve{s}_1(a_{z_2} + \breve{b}_1)(a_0 - \breve{a}_0), h + s_0(\breve{a}_0 + b_0)(\breve{b}_0 - b_{z_2})$ by the corresponding equation.

*4.4.2. Application to equations in the ABS list* In Table 1, we give a table for writing the reduced Lax matrices for equations $H_1, H_2, H_3$ and $Q_1, Q_2, Q_3$, see [1], in the form (47). All the reduced Lax pairs given satisfy the conditions in Corollary 11, so that we obtain closed-form expressions for the integrals from the formula (50).

| E | $\lambda$ | $g$<br>$h$ | $a_i$<br>$\breve{a}_i$ | $b_i$<br>$\breve{b}_i$ | $r_i$<br>$\breve{r}_i$ | $s_i$<br>$\breve{s}_i$ |
|---|---|---|---|---|---|---|
| $H_1$ | $p-k$ | $1$<br>$q-p$ | $v_i$<br>$v_{i+z_2}$ | $-v_{i+1}$<br>$-v_i$ | $1$<br>$1$ | $1$<br>$1$ |
| $H_2$ | $p-k$ | $1$<br>$q-p$ | $p-k+v_i$<br>$q-k+v_{i+z_2}$ | $-(p-k+v_{i+1})$<br>$-(q-k+v_i)$ | $2\sqrt{p+v_i+v_{i+1}}$<br>$2\sqrt{q+v_i+v_{i+z_2}}$ | $2/r_i^2$<br>$2/\breve{r}_i^2$ |
| $H_3$ | $\frac{k^2-p^2}{k^2}$ | $q^2/p^2$<br>$\frac{p^2-q^2}{p^2}$ | $v_i/p$<br>$v_{i+z_2}/p$ | $-v_{i+1}/p$<br>$-v_i/q$ | $\frac{\sqrt{\delta p+v_iv_{i+1}}}{p}$<br>$\frac{\sqrt{\delta q+v_iv_{i+z_2}}}{q}$ | $1/r_i^2$<br>$1/\breve{r}_i^2$ |
| $Q_1$ | $\frac{p-k}{k}$ | $q/p$<br>$\frac{q-p}{p}$ | $\frac{pv_{i+1}/k-v_{i+1}+v_i}{p}$<br>$\frac{qv_i/k-v_i+v_{i+z_2}}{q}$ | $-\frac{pv_i/k-v_i+v_{i+1}}{p}$<br>$-\frac{qv_{i+z_2}/k-v_{i+z_2}+v_i}{q}$ | $\frac{v_i-v_{i+1}-\delta}{p}$<br>$\frac{v_i-v_{i+z_2}+\delta q}{q}$ | $\frac{p^2}{(v_i-v_{i+1}+p\delta)(v_i-v_{i+1}-p\delta)}$<br>$\frac{q^2}{(v_i-v_{i+z_2}-\delta q)(v_i-v_{i+z_2}+\delta q)}$ |
| $Q_2$ | $\frac{p-k}{k}$ | $q/p$<br>$\frac{q-p}{p}$ | $\frac{(k-p)(\delta pk-v_{i+1})+kv_i}{kp}$<br>$\frac{(k-q)(\delta qk-v_i)+kv_{i+z_2}}{kq}$ | $-\frac{(k-p)(\delta pk-v_i)+kv_{i+1}}{kp}$<br>$-\frac{(k-q)(\delta qk-v_{i+z_2})+kv_i}{kq}$ | $\frac{\sqrt{\delta p^2(\delta p^2-2v_i-2v_{i+1})+(v_i-v_{i+1})^2}}{p}$<br>$\frac{\sqrt{\delta q^2(\delta q^2-2v_i-2v_{i+z_2})+(v_i-v_{i+z_2})^2}}{q}$ | $1/r_i^2$<br>$1/\breve{r}_i^2$ |
| $Q_3^0$ | $\frac{k^2-p^2}{k^2-1}$ | $\frac{q^2-1}{p^2-1}$<br>$\frac{p^2-q^2}{p^{-}1}$ | $\frac{(p^2-k^2)v_{i+1}+p(k^2-1)v_i}{(k^2-1)(p^2-1)}$<br>$\frac{(q^2-k^2)v_i+q(k^2-1)v_{i+z_2}}{(k^2-1)(q^2-1)}$ | $-\frac{(p^2-k^2)v_i+p(k^2-1)v_{i+1}}{(k^2-1)(p^2-1)}$<br>$-\frac{(q^2-k^2)v_{i+z_2}+q(k^2-1)v_i}{(k^2-1)(q^2-1)}$ | $\frac{pv_{i+1}-v_i}{p^2-1}$<br>$\frac{qu_i-v_{i+z_2}}{q^2-1}$ | $\frac{(p^2-1)^2}{(pv_{i+1}-v_i)(pv_i-v_{i+1})}$<br>$\frac{(q^2-1)^2}{(qv_{i+z_2}-v_i)(qv_i-v_{i+z_2})}$ |
| $Q_3$ | $\frac{p^2-k^2}{k^2-1}$ | $\frac{q^2-1}{p^2-1}$<br>$\frac{q^2-p^2}{p^{-}1}$ | $\frac{p(k^2-1)v_i+(p^2-k^2)v_{i+1}}{(k^2-1)(p^2-1)}$<br>$\frac{(q^2-k^2)v_i+q(k^2-1)v_{i+z_2}}{(k^2-1)(q^2-1)}$ | $-\frac{p(k^2-1)v_{i+1}+(p^2-k^2)v_i}{(k^2-1)(p^2-1)}$<br>$-\frac{(q^2-k^2)v_{i+z_2}+q(k^2-1)v_i}{(k^2-1)(q^2-1)}$ | $\frac{\sqrt{\delta^2(1-p^2)^2+4pv_iv_{i+1}(1-p^2)+4p^2(v_i^2-v_{i+1}^2)}}{(p^2-1)4p}$<br>$\frac{\sqrt{\delta^2(1-q^2)^2+4qv_iv_{i+z_2}(1-q^2)+4q^2(v_i^2-v_{i+z_2}^2)}}{4q(q^2-1)}$ | $\frac{1}{4pr_i^2}$<br>$\frac{1}{4q\breve{r}_i^2}$ |

**Table 1.** Reduced Lax pairs for equations in the ABS list

## 5. Discussion

### 5.1. Comparing the two sets of integrals of the mKdV equation

We have obtained two sets of integrals for the mKdV equation in equation (38) and (48), which are expressed in terms of multi-sums of products, Theta and Psi, respectively. This is because we can expand the trace of the monodromy matrix either in powers of $k$ or in powers of $\lambda = 1 - k^2$. Therefore, we have

$$\mathrm{Tr}(\mathcal{L}_0) = \sum_{i=0}^{\lfloor((z_2+1)/2)\rfloor} k^{2i} I_i = \sum_{i=0}^{\lfloor(z_2+1)/2\rfloor} (1-k^2)^i \widetilde{I}_i.$$

Equating the coefficient of $k^{2r}$, we have

$$I_r = \sum_{i=r}^{\lfloor(z_2+1)/2\rfloor} (-1)^r \binom{i}{i-r} \widetilde{I}_i.$$

In particular we have

$$I_{\lfloor(z_2+1)/2\rfloor} = (-1)^{\lfloor(z_2+1)/2\rfloor} \widetilde{I}_{\lfloor(z_2+1)/2\rfloor}$$

and

$$I_0 = \alpha_2 + \alpha_3 = \sum_{i=0}^{\lfloor(z_2+1)/2\rfloor} \widetilde{I}_i.$$

The later equation means that the set of non-constant integrals $\{\widetilde{I}_r\}$ is not functionally independent. Explicitly taking $z_2 = 3$, we have

$$\begin{aligned}
I_0 &= \alpha_2 + \alpha_3,\\
I_1 &= \alpha_1\Big(\frac{v_0}{v_2} + \frac{v_0v_3}{v_1v_2} + \frac{v_3}{v_1} + \frac{v_2}{v_0} + \frac{v_1v_2}{v_0v_3} + \frac{v_1}{v_3}\Big) + \alpha_2\Big(\frac{v_1}{v_3} + \frac{v_0v_1}{v_2v_3} + \frac{v_0}{v_2}\Big)\\
&\quad + \alpha_3\Big(\frac{v_3}{v_1} + \frac{v_2v_3}{v_0v_1} + \frac{v_2}{v_0}\Big),\\
I_2 &= 2\alpha_1,
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{I}_0 &= \alpha_1\Big(2 + \frac{v_0}{v_2} + \frac{v_0v_3}{v_1v_2} + \frac{v_3}{v_1} + \frac{v_2}{v_0} + \frac{v_1v_2}{v_0v_3} + \frac{v_1}{v_3}\Big) + \alpha_2\Big(1 + \frac{v_1}{v_3} + \frac{v_0v_1}{v_2v_3} + \frac{v_0}{v_2}\Big)\\
&\quad + \alpha_3\Big(1 + \frac{v_3}{v_1} + \frac{v_2v_3}{v_0v_1} + \frac{v_2}{v_0}\Big),\\
\widetilde{I}_1 &= -\alpha_1\Big(4 + \frac{v_0}{v_2} + \frac{v_0v_3}{v_1v_2} + \frac{v_3}{v_1} + \frac{v_2}{v_0} + \frac{v_1v_2}{v_0v_3} + \frac{v_1}{v_3}\Big) - \alpha_2\Big(\frac{v_1}{v_3} + \frac{v_0v_1}{v_2v_3} + \frac{v_0}{v_2}\Big)\\
&\quad - \alpha_3\Big(\frac{v_3}{v_1} + \frac{v_2v_3}{v_0v_1} + \frac{v_2}{v_0}\Big),\\
\widetilde{I}_2 &= 2\alpha_1.
\end{aligned}$$

It seems that the set of non-constant integrals $I_r$ expressed in Theta is simpler and shorter than the integrals $\widetilde{I}_r$ expressed in Psi. However, it is interesting to know that with our Maple programme, it took much more time to calculate the integrals in terms of Theta than in terms of Phi.

Similarly, for the sine-Gordon mapping the set of non-constant integrals expressed in terms of Psi is not functionally independent and also longer than the one expressed in terms of Theta.

### *5.2. Future work*

We have presented a tool to obtain closed-form expressions for integrals in terms of multi-sums of products, Theta and Psi. This is a first step to prove the integrability of a discrete map in the Liouville-Arnold sense [10], [12] (a map has sufficiently many functionally independent integrals in involution). The recursive formulas for the multi-sums of products make it possible for us to prove functional independence and involutivity which we hope to publish elsewhere [13, 14].

We have given closed-form expressions for integrals of all equations in the ABS list but $Q_4$. It would be worth studying this exceptional case, as $Q_4$ is the most general equation in the ABS list.

In this paper, we have considered $(1, z_2)$ traveling wave reductions. It would be interesting to study more general traveling wave reductions, cf [8, 11].

Another direction of interest is studying more general equations, and systems of equations, which are not necessary on defined on elementary squares, cf [4, 5].

## Acknowledgments

This research has been funded by the Australian Research Council through the Centre of Excellence for Mathematics and Statistics of Complex Systems. Dinh T. Tran acknowledges the support of two scholarships, one from La Trobe University and the other from the Endeavour IPRS programme. Thanks to Prof. Willy Hereman for a very useful collaboration on the topic of Lax pairs for equations on the ABS list.

## Appendix A: Proving Lemma 4

Here we give a proof of Lemma 4.

*Proof.* We write $L_i = kJ + Y_i$ as in (14). So we have

$$Y_{i+k}Y_{i+k-1}\dots Y_i = \begin{pmatrix} \frac{v_i}{v_{i+k+1}} & 0 \\ 0 & \frac{v_{i+k+1}}{v_i} \end{pmatrix}.$$

Using the Vieta formula $(9b)$ and the above formula, we obtain

$$\begin{aligned} \widetilde{Z}_r^{a,b-1} &= \textstyle\sum_{a\leq i_1<i_2<\ldots<i_r\leq b-1} Y_{b-1}\dots Y_{i_r+1}JY_{i_r-1}\dots Y_{i_{r-1}+1}J\dots Y_{i_1+1}JY_{i_1-1}\dots Y_a \\ &= \textstyle\sum_{a\leq i_1<i_2<\ldots<i_r\leq b-1} \\ &\quad \begin{pmatrix} \frac{v_{i_r+1}}{v_b} & 0 \\ 0 & \frac{v_b}{v_{i_{i_r+1}}} \end{pmatrix} J \begin{pmatrix} \frac{v_{i_{r-1}+1}}{v_{i_r}} & 0 \\ 0 & \frac{v_{i_r}}{v_{i_{r-1}+1}} \end{pmatrix} J \dots \begin{pmatrix} \frac{v_{i_1+1}}{v_{i_2}} & 0 \\ 0 & \frac{v_{i_2}}{v_{i_1+1}} \end{pmatrix} J \begin{pmatrix} \frac{v_a}{v_{i_1}} & 0 \\ 0 & \frac{v_{i_1}}{v_a} \end{pmatrix}. \end{aligned}$$

If $r$ is even and using the properties $JY = Y^{-1}J$ and $J^2 = I$, then we have

$$\begin{aligned}
\widetilde{Z}_r^{a,b-1} &= \textstyle\sum_{a\le i_1<i_2<\ldots<i_r\le b-1} \\
&\quad \begin{pmatrix} \frac{v_{i_r+1}}{v_b} & 0 \\ 0 & \frac{v_b}{v_{i_{i_r+1}}} \end{pmatrix} \begin{pmatrix} \frac{v_{i_r}}{v_{i_{r-1}+1}} & 0 \\ 0 & \frac{v_{i_{r-1}+1}}{v_{i_r}} \end{pmatrix} \cdots \begin{pmatrix} \frac{v_{i_2}}{v_{i_1+1}} & 0 \\ 0 & \frac{v_{i_1+1}}{v_{i_2}} \end{pmatrix} \begin{pmatrix} \frac{v_a}{v_{i_1}} & 0 \\ 0 & \frac{v_{i_1}}{v_a} \end{pmatrix} \\
&= \textstyle\sum_{a\le i_1<i_2<\ldots<i_r\le b-1} \\
&\quad \begin{pmatrix} \frac{v_a}{v_b}\frac{v_{i_r}v_{i_r+1}}{v_{i_{r-1}}v_{i_{r-1}+1}}\frac{v_{i_{r-2}}v_{i_{r-2}+1}}{v_{i_{r-3}}v_{i_{r-3}+1}}\cdots\frac{v_{i_2}v_{i_2+1}}{v_{i_1}v_{i_1+1}} & 0 \\ 0 & \frac{v_b}{v_a}\frac{v_{i_{r-1}}v_{i_{r-1}+1}}{v_{i_r}v_{i_r+1}}\frac{v_{i_{r-3}}v_{i_{r-3}+1}}{v_{i_{r-2}}v_{i_{r-2}+1}}\cdots\frac{v_{i_1}v_{i_1+1}}{v_{i_2}v_{i_2+1}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{v_a}{v_b}\Theta_{r,0}^{a,b-1} & 0 \\ 0 & \frac{v_b}{v_a}\Theta_{r,1}^{a,b-1} \end{pmatrix}.
\end{aligned}$$

Similarly, if $r$ is odd, then we have

$$\begin{aligned}
\widetilde{Z}_r^{a,b-1} &= \textstyle\sum_{a\le i_1<i_2<\ldots<i_r\le b-1} \\
&\qquad J\begin{pmatrix} \frac{v_b}{v_{i_r+1}} & 0 \\ 0 & \frac{v_{i_{i_r+1}}}{v_b} \end{pmatrix} \begin{pmatrix} \frac{v_{i_{r-1}+1}}{v_{i_r}} & 0 \\ 0 & \frac{v_{i_r}}{v_{i_{r-1}+1}} \end{pmatrix} \cdots \begin{pmatrix} \frac{v_{i_2}}{v_{i_1+1}} & 0 \\ 0 & \frac{v_{i_1+1}}{v_{i_2}} \end{pmatrix} \begin{pmatrix} \frac{v_a}{v_{i_1}} & 0 \\ 0 & \frac{v_{i_1}}{v_a} \end{pmatrix} \\
&= J\begin{pmatrix} v_a v_b\Theta_{r,0}^{a,b-1} & 0 \\ 0 & \frac{1}{v_a v_b}\Theta_{r,1}^{a,b-1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{v_a v_b}\Theta_{r,1}^{a,b-1} \\ v_a v_b\Theta_{r,0}^{a,b-1} & 0 \end{pmatrix}.
\end{aligned}$$

□

## Appendix B: Direct proof of invariance of integrals of the mKdV equation in terms of Psi (Proposition 9)

Here we give a proof of Proposition 9.

*Proof.* Applying a shift operator on the integral $\widetilde{I}_r$, we obtain

$$\begin{aligned}
S(\widetilde{I}_r) = \Big(&\alpha_1\Psi_{r-1}^{2,z_2-1} + (\alpha_2 v_1 + \frac{\alpha_1 v_1 v_{z_2+1}}{v_2})\Psi_{r-1}^{2,z_2-2} + (\alpha_3 v_{z_2+1} + \frac{\alpha_1 v_1 v_{z_2+1}}{v_{z_2}})\Psi_{r-1}^{3,z_2-1} \\
&-\alpha_1 v_1 v_{z_2+1}\Psi_{r-2}^{3,z_2-2} - (\frac{\alpha_2 v_1}{v_{z_2}} + \frac{\alpha_3 v_{z_2+1}}{v_2} + \frac{\alpha_1 v_1 v_{z_2+1}}{v_2 v_{z_2}} + \alpha_1)\Psi_r^{2,z_2-1}\Big)\prod_{i=1}^{z_2+1} v_i^{-1}.
\end{aligned}$$

Now we write $S(\widetilde{I}_r) - \widetilde{I}_r = (A+B)\prod_{i=1}^{z_2} v_i^{-1}$ where

$$\begin{aligned}
A &= \Big(\alpha_1\Psi_{r-1}^{2,z_2-1} + (\alpha_2 v_1 + \tfrac{\alpha_1 v_1 v_{z_2+1}}{v_2})\Psi_{r-1}^{2,z_2-2} - \alpha_1 v_1 v_{z_2+1}\Psi_{r-2}^{3,z_2-2}\Big)v_{z_2+1}^{-1} \\
&\quad -\Big(\alpha_1\Psi_{r-1}^{1,z_2-2} + (\alpha_3 v_{z_2} + \tfrac{\alpha_1 v_0 v_{z_2}}{v_{z_2-1}})\Psi_{r-1}^{2,z_2-2} - \alpha_1 v_0 v_{z_2}\Psi_{r-2}^{2,z_2-3}\Big)v_0^{-1} \\
B &= \Big((\alpha_3 v_{z_2+1} + \tfrac{\alpha_1 v_1 v_{z_2+1}}{v_{z_2}})\Psi_{r-1}^{3,z_2-1} - (\tfrac{\alpha_2 v_1}{v_{z_2}} + \tfrac{\alpha_3 v_{z_2+1}}{v_2} + \tfrac{\alpha_1 v_1 v_{z_2+1}}{v_2 v_{z_2}} + \alpha_1)\Psi_r^{2,z_2-1}\Big)v_{z_2+1}^{-1} \\
&\quad -\Big((\alpha_2 v_0 + \tfrac{\alpha_1 v_0 v_{z_2}}{v_1})\Psi_{r-1}^{1,z_2-3} - (\tfrac{\alpha_2 v_0}{v_{z_2-1}} + \tfrac{\alpha_3 v_{z_2}}{v_1} + \tfrac{\alpha_1 v_0 v_{z_2}}{v_1 v_{z_2-1}} + \alpha_1)\Psi_r^{1,z_2-2}\Big)v_0^{-1}.
\end{aligned}$$

Using properties (52) and (53), we have

$$\begin{aligned}\Psi_r^{2,z_2-1} &= v_{z_2}v_{z_2+1}(\frac{1}{v_{z_2-1}}+\frac{1}{v_{z_2+1}})\Psi_{r-1}^{2,z_2-2}-v_{z_2}v_{z_2+1}\Psi_{r-2}^{2,z_2-3}\\ \Psi_{r-1}^{1,z_2-2} &= v_0v_1(\frac{1}{v_0}+\frac{1}{v_2})\Psi_{r-1}^{2,z_2-2}-v_0v_1\Psi_{r-2}^{3,z_2-2}.\end{aligned}$$

Substituting these formulas into A, we obtain

$$A=\frac{\Psi_{r-1}^{2,z_2-2}}{v_0v_{z_2+1}}.\mathcal{F}_{\mathrm{mKdV}}.$$

Applying the properties (52) and (53), we have

$$\begin{aligned}\Psi_r^{3,z_2-1} &= (\frac{1}{v_0}+\frac{1}{v_2})\Psi_r^{2,z_2-1}-\frac{\Psi_r^{1,z_2-1}}{v_0v_1}\\ \Psi_{r-1}^{1,z_2-3} &= (\frac{1}{v_{z_2-1}}+\frac{1}{v_{z_2+1}})\Psi_r^{1,z_2-2}-\frac{\Psi_r^{1,z_2-1}}{v_{z_2}v_{z_2+1}}.\end{aligned}$$

Substituting these formulas into B, we get

$$B=\frac{1}{v_0v_1v_{z_2}v_{z_2+1}}(\Psi_r^{1,z_2-1}-v_1\Psi_r^{2,z_2-1}-v_{z_2}\Psi_r^{1,z_2-2})\mathcal{F}_{\mathrm{mKdV}}.$$

This proves the statement. □

## References

[1] Adler V, Bobenko A and Suris Y 2003 Classification of Integrable Equations on quad-graphs. The Consistency Approach *Commun. Math. Phys.* **233** 513-43

[2] Nijhoff F N 2002 Lax pair for the Adler (lattice Krichever-Novikov) system *Phys. Lett. A* **297** 49–58.

[3] Nijhoff F N and Walker A J 2001 The discrete and continuous Painlevé hierarchy and the Garnier systems *Glasgow Math. J.A* **43** 109–23.

[4] Van der Kamp P H and Quispel G R W Initial value problems for lattice equations, discrete traveling waves *in preperation.*

[5] Van der Kamp P H and Quispel G R W The staircase method *in preperation.*

[6] Van der Kamp P H, Rojas O and Quispel G R W 2007 Closed-form expressions for integrals of sine-Gordon and MKDV maps *J.Phys A.: Math Gen.* **40** 12789–98.

[7] Papageorgiou V G, Nijhoff F W and Capel H W 1990 Integrable mappings and nonlinear integrable lattice equations *Phys. Lett. A* **147** 10614

[8] Quispel G R W, Capel H W, Papageorgiou V G and Nijhoff F W 1991 Integrable mappings derived from soliton equations *Physica A* **173** 24366

[9] Quispel G R W, Capel H W and Roberts J A G 2005 Duality for discrete integrable systems, *J.Phys. A: Math. Gen.* **38** 3965–80.

[10] Roberts J A G and Quispel G R W 2008 Symplectic structures for difference equations including integrable maps *in preperation.*

[11] Rojas O, Van der Kamp P H and Quispel G R W Lax representations for integrable maps, *in preperation.*

[12] Veselov A P 1991 Integrable maps, *Russian Math. Surveys* **46** 1–51.

[13] Tran D T, Van der Kamp P H and Quispel G R W Involutivity of integrals of sine-Gordon, mKdV and pKdV maps, *in preperation.*

[14] Tran D T, Van der Kamp P H and Quispel G R W Functional independence of of integrals of sine-Gordon, mKdV and pKdV maps, *in preperation.*