Integrable Evolution Equations: a Diophantine Approach

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VRIJE UNIVERSITEIT

Integrable Evolution Equations: a Diophantine Approach

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geboren te Medemblik

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under the auspices of:

s STIELTJES INSTITUTE
R MATHEMATICS

To clearly state a problem
and to actually solve it
are of equal importance.

Acknowledgements

A good four years ago I applied for a PhD position in pure mathematics. In my letter of application I wrote that I would like to stay at the boundary of physics and mathematics. It turned out to be a good thing that J.A. Sanders, who had a specific project in mind, gave me a phone call. This was exactly what I needed; being graduated in physics I did not quite know what mathematics was about. I remember that Jan Sanders told me mathematics is that which is invariant under the notation. Meanwhile he convinced me of the significance of 'good' notation. He lended me his books and pointed out to me the important problems. Jan, thank you for letting me jump onto your running train.

At the very last moment I asked J. Hulshof to be my promotor. Joost, thank you for your support. This thesis is based on the symbolic calculus of Gel'fand and Dikiĭ. J.P. Wang was the first to use this calculus systematically in the classification of integrable equations. Jing Ping, thank you for being my room-mate and for answering all sorts of questions. I would have been less proud of this thesis if it had been written without the fruitful ideas of F. Beukers. Frits, thank you for all the hours you spend pointing out how to prove Propositions 7.15, 7.17, 7.19, 7.21, E.3, E.6 and Theorems E.1, E.4. I had a good time with A.V. Mikhailov when I was in Cambridge attending the workshop 'What is integrability?' he organised. Sasha, thank you for reflecting on the object of my research. To know that some people do read your scientific work is very stimulating. In particular I would like to thank Thanasis Fokas and Jaap Top for their interest in the result of my research.

After all, especially with Conjecture 8.4 in mind, I can say that instead of staying at the boundary of physics and mathematics, with this thesis I crossed the border going from physics to mathematics.

I very much appreciated all the help in writing this thesis. I also thank Jan en Riet te Riele, Lidewij van de Berg, Emiel van der Plas, Rick Bethlem and Alistair Vardy for their remarks on earlier drafts of this book.

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