Initial value problems for lattice equations.

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Abstract

We describe how to pose straight band initial value problems for lattice equations defined on arbitrary stencils. In finitely many directions we arrive at discrete Goursat problems, and in the remaining directions we find Cauchy problems. Next we consider (s_1, s_2)-periodic initial value problems. In the Goursat directions the periodic solutions are generated by correspondences. In the Cauchy directions, assuming the equation to be multi-linear, the periodic solution can be obtained uniquely by iteration of a particularly simple mapping, whose dimension is a piece-wise linear function of s_1, s_2.

1 Introduction

For a given nonlinear partial differential equation (PDE), say, in two independent variables

$$E(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \tag{1}$$

we can consider its traveling wave solution

$$u(x,t) = f(\xi), \quad \xi = x + \beta t. \tag{2}$$

Then, the PDE (1) for the function $u(x,t)$ reduces to an ordinary differential equation (ODE) for the function $f(\xi)$. Note that the traveling wave solution for the PDE satisfies a periodicity condition,

$$u(x,t) = u(x + s_1, t + s_2),$$

with $s_1/s_2 = -\beta$.

Here, we study periodic solutions of partial difference equations (PΔEs)

$$F(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+2,m}, u_{l+1,m+1}, u_{l,m+2}, \dots) = 0. \tag{3}$$

Because the lattice parameters l, m take values in $\mathbb{Z}$, we need *integer* parameters $\mathbf{s} = (s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}$ to impose the $\mathbf{s}$-periodicity condition,

$$u_{l,m} = u_{l+s_1,m+s_2}.$$

In contrast to the continuous case, unless s_1 and s_2 are co-prime, the PΔE reduces to a *system* of ordinary difference equations (OΔEs). In fact, we get a system of r OΔEs for r functions, where r is the greatest common divisor of $|s_1|$ and $|s_2|$. The traveling wave interpretation is given in section 4, where the discrete analogue of (2) is given:

$$u_{l,m} = v_n^p, \quad n = n_{l,m} \in \mathbb{Z},\ p = p_{l,m} \in \mathbb{N}_r$$

for certain functions n, p which depend on the choice of $\mathbf{s}$. Here n plays the role of β in (2) and p distinguishes the r different functions.

Periodic reductions, for lattice equations defined on a square,

$$F(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+1,m+1}) = 0, \tag{4}$$

were performed in [11]. The authors considered the cases $s_2 = s_1$ and $s_2 = s_1+1$. Not long after, in [15], it was realized that such reductions provide traveling wave solutions. The authors showed how to perform $\mathbf{s}$-reduction with $\mathbf{s} = (z_2, -z_1)$, taking z_1, z_2 to be co-prime. They provided a particular nice way of posing initial value problems for lattice equations defined on a square (4). Born out of the third concluding remark in that paper, [15], a completely general description of $\mathbf{s}$-reduction, with $\mathbf{s} \in \mathbb{Z}\times\mathbb{Z}$ was given recently in [13]. At present, we display a more geometric understanding. And, we show how to pose initial value problems for general lattice equations (3).

We call an initial value problem *well-posed* if for generic initial values a solution exists and is unique. If a generic initial value problem yields a finite number of values at any given lattice point, we call it *nearly-well-posed*.

We will prove the following. Let a multi-linear function F be defined on a finite set S of lattice points, not lying on the same line. Then for the equation $F = 0$ on the lattice, there are finitely many directions of $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}\times\mathbb{Z}$ for which the equation admits a nearly-well-posed $\mathbf{s}$-periodic initial value problem. These directions coincide with the directions of the line-pieces in the boundary of the convex hull of S. If the direction of $\mathbf{s}$ is not one of these, then $F = 0$ admits a well-posed $\mathbf{s}$-periodic initial value problem equivalent to a finite dimensional mapping the dimension of which is a piece-wise linear function of s_1, s_2.

In the case of well-posedness, the periodic solutions are uniquely determined by iteration of single-valued mappings. Here, $\mathbf{s}$-periodicity on the band of initial values implies $\mathbf{s}$-periodicity of the solution on $\mathbb{Z} \times \mathbb{Z}$. In the case of nearly-well-posedness the periodic solutions are generated by correspondences. Here we impose $\mathbf{s}$-periodicity on $\mathbb{Z} \times \mathbb{Z}$.

In our construction the mappings are particularly simple as they can be obtained by using the equation only $r = \gcd(s_1, s_2)$ times, and no composition of functions is involved. To obtain explicit expressions for the correspondences one needs to solve a system of r equations in r unknowns. We also formulate a conjecture about the multi-valuedness of certain correspondences, which relates to the number of fixed points of a certain mapping.

We note that from well-posed $\mathbf{s}$-periodic initial value problems, one can construct non-periodic initial value problems, of Cauchy type, by taking the limit where r goes to infinity. For the nearly-well-posed $\mathbf{s}$-periodic initial value problems, if one takes the same limit, one has to add complementary lines of initial values. Therefore, this leads to discrete Goursat problems.

It will be instructive to first describe these (non-periodic) initial value problems. This will be done in geometric terms in the next section. Secondly, in section 3, we will impose periodicity conditions. Then, in section 4, we will arrive at the traveling wave interpretation of the periodic solutions. In section five, we conclude with a hint to applications in the area of discrete integrable systems. We provide explicit reductions of an integrable 5-point equation. And, we show how to pose initial value problems for the quotient-difference algorithm, which is an integrable *system* of equations.

2 Initial value problems

A point $\mathbf{s} = (s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}$ is a *direction* if $s_1 \in \mathbb{N} := \{0, 1, \ldots\}$, and s_1 is co-prime with $\epsilon s_2 \in \mathbb{N}$, where ϵ is the sign of $s_2 \in \mathbb{Z}$. Thus, every nonzero point $\mathbf{s} \in \mathbb{Z} \times \mathbb{Z}$ can be written uniquely as $\mathbf{s} = \pm r\hat{\mathbf{s}}$, where $\hat{\mathbf{s}}$ denotes the direction of $\mathbf{s}$, and $r \in \mathbb{N}$. A line piece L has direction $\hat{\mathbf{s}}$ if $L = \mathbf{a} + \mathbf{s}I$, for certain $\mathbf{a}, \mathbf{s} \in \mathbb{Z} \times \mathbb{Z}$ and interval $I \subset \mathbb{R}$.

A *stencil* S is a finite set of points of $\mathbb{Z} \times \mathbb{Z}$ such that not all points lie on the same line. Thinking of S as a subset of the real plane $\mathbb{R} \times \mathbb{R}$, the boundary of the convex hull of S is a closed polygonal line. We call this boundary the *S-polygon*, its vertices the *S-extreme points* (can be proved that they belong to S), its edges *S-edges* and the directions of S-edges *S-directions*.

Let a lattice equation on S be given. We suppose that the equation is multilinear, at least in the points that are on the edges of the S-polygon.

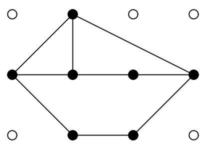

Figure 1: 7-point sailboat stencil

As a pedagogical example, throughout the text, we will exploit the 7-point stencil depicted in Figure 1. For this stencil the S-directions are

$$(1, 1), (2, -1), (1, -1), (1, 0). \qquad (5)$$

2.1 Well-posed Cauchy problems

We choose a point $\mathbf{s} \in \mathbb{Z} \times \mathbb{Z}$, whose direction is not an S-direction. Clearly, there are only finitely many directions in which $\mathbf{s}$ may not be chosen.

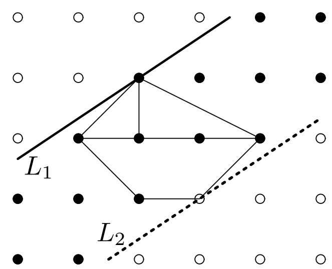

Figure 2: Squeezing the sailboat

For a sailboat equation we choose $\mathbf{s} = (3, 2)$. We squeeze its stencil S between two lines with direction $\hat{\mathbf{s}}$, denoted L_1 and L_2, see Figure 2. Because $\mathbf{s}$ is not an S-direction, the squeezing lines L_1 and L_2 intersect the stencil S at one S-extreme point. It is plain that if initial values are given at all lattice points between the two lines together with the values at the lattice points on one of the two lines, say L_1, then the values at the lattice points on the other line, in this case L_2, can be calculated using copies of the equation.
We can shift the lines further apart, in a parallel fashion, and are able to calculate the values at the shifted lines. Iterating this procedure determines a solution uniquely.

We now expand a little on how to shift lines further apart over the lattice. Let A be a line through the origin with direction $\mathbf{a}$. Let us shift the line in a parallel and continuous fashion until the line intersects the lattice $\mathbb{Z} \times \mathbb{Z}$ nontrivially again. The question we then ask is: what are the possible discrete translation vectors? Or, which lattice points are closest to A?

Choose $\mathbf{c} \in \mathbb{Z} \times \mathbb{Z}$. The component of $\mathbf{c}$ perpendicular to $\mathbf{a}$ is

$$\mathbf{c}_\perp = \mathbf{c} - \frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{a}}.\mathbf{a} = \det \begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix} .(-a_2, a_1).$$

Hence, the distance from $\mathbf{c}$ to A is $D(\mathbf{a}, \mathbf{c})/||\mathbf{a}||$, where $D : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{N}$ is given by

$$D(\mathbf{a}, \mathbf{c}) := |\det \begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix}| \tag{6}$$

As a fact from elementary number theory, since a_1 and a_2 are co-prime, there exist $\mathbf{c}$, such that $c_2 a_1 - c_1 a_2 = 1$. This is, after 0, the smallest value $D(\mathbf{a}, \mathbf{c})$ can acquire. Moreover, $\mathbf{c}$ can be obtained from the extended Euclidean algorithm. Clearly, once a particular $\mathbf{c}$ has been found, all lattice points that are closest to the line A are $\pm\mathbf{c} + \mathbf{a}\mathbb{Z}$.

2.2 Well-posed Goursat problems

We could have chosen the point $\mathbf{s}$ such that one of the lines L_1 or L_2, or both, would intersect the S-polygon in an S-edge. When this happens the initial value problem is ill-posed; we have two options. One option is to add initial values on complementary lines, as in figure 3. The other option is to impose periodicity on $\mathbb{Z} \times \mathbb{Z}$. This will be done in the next section.

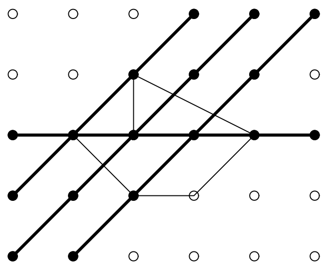

Figure 3: Goursat-problem for sailboat equations

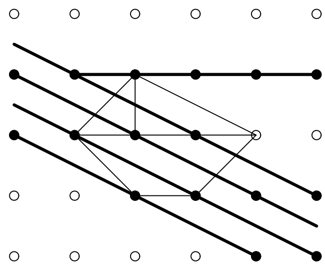

Figure 4: Another Goursat-problem for sailboat equations

When exactly one of the lines, say L_2, intersects the S-polygon in an S-edge, we only have to add initial values on complementary half-lines attached to L_2. This is the case for sailboat equations when $\hat{\mathbf{s}} = (2, -1)$, see Figure 4.

In general, the direction $\mathbf{c} \in \mathbb{Z} \times \mathbb{Z}$ of the complementary lines should be chosen such that $D(\hat{\mathbf{s}}, \mathbf{c}) = 1$. And, if P denotes the intersection of L_i with the stencil S, the number $n \in \mathbb{N}$ of complementary half lines that should be attached to the line L_i is $n = \max_{\mathbf{p}_1,\mathbf{p}_2 \in P}\{k : \mathbf{p}_1 - \mathbf{p}_2 = k.\hat{\mathbf{s}}\}$.

3 Periodic initial value problems

By imposing $\mathbf{s}$-periodicity with $\hat{\mathbf{s}}$ not an S-direction, we get a well-posed periodic initial value problem. If $\hat{\mathbf{s}}$ is an S-direction, we get a nearly-well-posed periodic initial value problem.

3.1 Well-posed periodic initial value problems

Imposing $\mathbf{s}$-periodicity amounts to identifying points $\mathbf{a} \sim \mathbf{c}$ if and only if there exists $k \in \mathbb{Z}$ such that $\mathbf{a} - \mathbf{c} = k.\mathbf{s}$. We would like to know how many inequivalent lattice points are either on L_1, or in between L_1 and L_2. This number gives us the dimension of the $\mathbf{s}$-periodic initial value problem.

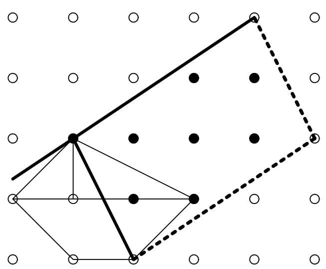

Figure 5: (3,2)-periodic initial value problem for sailboat equations

By imposing $(3, 2)$-periodicity the number of inequivalent initial values, cf. Figure 2, becomes 8. The reader may verify this, by counting the number of black dots in the parallelogram given in Figure 5, excluding the points on the dashed lines.

In general, let $\mathbf{d} \in \mathbb{Z} \times \mathbb{Z}$ represent a *difference* between two parallel lines L_1, L_2 with direction $\hat{\mathbf{s}}$, that is, let there be $\mathbf{p}_1 \in L_1$ and $\mathbf{p}_2 \in L_2$ such that $\mathbf{p}_1 - \mathbf{p}_2 = \pm d$.

We can now rephrase the question: What is the number of lattice points in the parallelogram $\mathbf{p} = a.\mathbf{d} + b.\mathbf{s}$, with $0 \leq a, b < 1 \in \mathbb{Q}$? The answer is quite appealing, it is $D(\mathbf{s}, \mathbf{d})$, with D defined by (6). The short proof would be that the number of lattice points is equal to the area of the parallelogram. But the reader may want to verify the statement by counting lattice points in rectangles, see Figure 6.

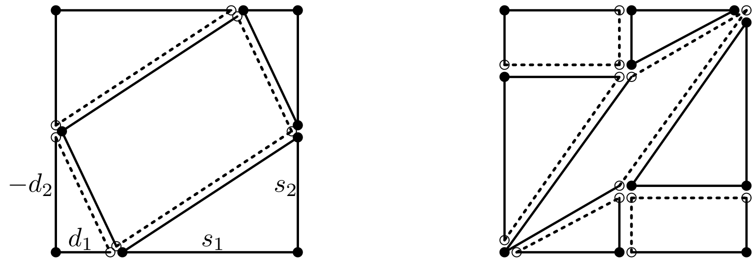

Figure 6: Counting lattice points in parallelograms. The number of lattice points in the parallelogram on the left is equal to the number of lattice point in a $s_1 + d_1 \times s_2 - d_2$ rectangle minus the number of lattice point in a $s_1 \times s_2$ rectangle minus the number of points in a $d_1 \times (-d_2)$ rectangle plus one.

Indeed, for the $\mathbf{s} = (3, 2)$ periodic initial value problem, depicted in Figure 5, taking $\mathbf{d} = (1, -2)$ yields $D(\mathbf{s}, \mathbf{d}) = 8$. Note that the number $D(\mathbf{s}, \mathbf{d})$ does not depend on the choice of $\mathbf{d}$. If $\mathbf{d}$ is a difference between L_1 and L_2, then the set Z of all differences is $Z = \{\pm\mathbf{d} + \hat{\mathbf{s}}\mathbb{Z}\}$. Clearly, $\mathbf{p} \in Z$ implies $D(\mathbf{s}, \mathbf{p}) = D(\mathbf{s}, \mathbf{d})$. Also note that the number $D(\hat{\mathbf{s}}, \mathbf{d})$ counts the number of discrete lines in between L_1 and L_2. Let $r \in \mathbb{N}$ be such that $\mathbf{s} = \pm r\hat{\mathbf{s}}$. Then, there are r inequivalent points on each line.

Given a well-posed $\mathbf{s}$-periodic initial value problem a particular simple way of updating the $D(\mathbf{s}, \mathbf{d})$ initial values is to perform a shift over $\mathbf{c}^{\mathbf{s}}$, where $\mathbf{c}^{\mathbf{s}}$ satisfies $D(\hat{\mathbf{s}}, \mathbf{c}^{\mathbf{s}}) = 1$. For the initial value problem depicted in Figure 5 we take $\mathbf{c}^{\mathbf{s}} = (2, 1)$. By shifting over $\mathbf{c}^{\mathbf{s}}$ all the black dots will be shifted to a point between the two lines, except for the point closest to the dashed line, which is shifted to a point on the dashed line. This means that to write down the corresponding mapping we only have to use the sailboat equation once. In general, one has to use the equation r times, in order to calculate the r values at the lattice points closest to the initial values. We note that we only need to impose $\mathbf{s}$-periodicity on the band of initial values. Due to the multi-linearity this will guarantee the $\mathbf{s}$-periodicity of the solution on $\mathbb{Z} \times \mathbb{Z}$.

3.2 Nearly-well-posed periodic initial value problems

There is a finite number of directions where the line L_1 or L_2 intersects the S-polygon at an S-edge. In these cases we get nearly-well-posed periodic initial value problems by imposing $\mathbf{s}$-periodicity on $\mathbb{Z} \times \mathbb{Z}$.

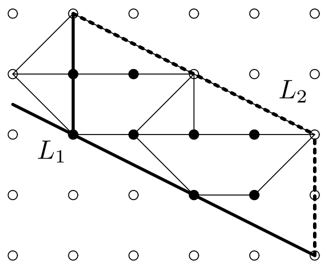

Figure 7: Nearly-well-posed (4,-2)-periodic initial value problem for sailboat equations

Let us impose $\mathbf{s} = (4, -2)$ periodicity and, once again, consider a sailboat equation, see Figure 7. A difference between the lines L_1 and L_2 is $\mathbf{d} = (0, 2)$. Therefore, the $\mathbf{s}$-periodic initial value problem has dimension $D(\mathbf{s}, \mathbf{d}) = 8$. With $\mathbf{c}^{\mathbf{s}} = (1, 0)$ we have $D(\hat{\mathbf{s}}, \mathbf{c}^{\mathbf{s}}) = 1$. We update to the left by shifting over $-\mathbf{c}^{\mathbf{s}}$. At both points on $L_1 - \mathbf{c}^{\mathbf{s}}$ we can determine its value uniquely by using a single sailboat equation. On the other side, at the dashed line L_2, we need to solve a system of two sailboat equations to obtain the two values at L_2.

If we were to consider $(2r, -r)$-reduction, with $r \in \mathbb{N}^+$ this would yield a system of r equations in r unknowns, which can be represented as

$$f(x_1, x_2) = f(x_2, x_3) = \cdots = f(x_{r-1}, x_r) = f(x_r, x_1) = 0 \tag{7}$$

Assuming the sailboat equation to be multi-linear in the points of intersection of L_2 and the stencil S, then Lemma 1 tells us that the correspondence that generates the $(2r, -r)$-periodic solution will, generically, be two-valued.

Lemma 1 *Let $f_i(x, y)$, $i = 1, 2, \ldots, n$ be generic polynomials of degree one in x, y. Then the system of n equations in n unknowns*

$$f_1(x_1, x_2) = f_2(x_2, x_3) = \cdots = f_{n-1}(x_{n-1}, x_n) = f_n(x_n, x_1) = 0$$

has 2 solutions.

Proof: Eliminating x_2 from $f_1(x_1, x_2) = f_2(x_2, x_3) = 0$ gives us a polynomial equation $g_1(x_1, x_3) = 0$, where g_1 has degree one in x_1, x_3. Similarly, eliminating x_3 from $g_1(x_1, x_3) = f_3(x_3, x_4) = 0$ yields $g_2(x_1, x_4) = 0$, where g_2 has degree one in x_1, x_4. Performing further elimination of $x_4, \ldots, x_{n-1}$ we find $g_{n-2}(x_1, x_n) = 0$, where g_{n-2} has degree one in x_1, x_n. Now, by eliminating x_n from the system $g_{n-2}(x_1, x_n) = f_n(x_n, x_1) = 0$ we get a quadratic for x_1. So x_1 can take two values. Once x_1 has been fixed, the values of $x_2, \ldots, x_n$ are determined by linear relations. $\square$

In general, let P be the intersection of a line L_i, with direction $\hat{\mathbf{s}}$, and a stencil S. The n lattice points in P can be written as $\mathbf{p} + g_i.\hat{\mathbf{s}}$, where $i = 1, 2, \ldots, n$, $\mathbf{p} \in \mathbb{Z} \times \mathbb{Z}$ and the integers $g_i \in \mathbb{N}$ form (part of) an arithmetic series with difference $d \in \mathbb{N}^+$. Let us impose $r.\hat{\mathbf{s}}$-periodicity. If d is not a divisor of r, we get a system of r equations in r unknowns, which can be written as

$$f(x_{g_1+k}, x_{g_2+k}, \ldots, x_{g_n+k}) = 0, \quad k = 0, 1, \ldots r-1 \tag{8}$$

where $x_k = x_l$ if $k \equiv l \bmod r$. Contrarily, if d divides r, we get d systems of r/d equations in r/d unknowns, similar to the system (8), dividing r by d.

When $n = 2$ we can always relabel the unknowns $x_i \mapsto x_{\sigma(i)}$, such that equation (8), with $n = 2$, equals equation (7), in which case we know there are (at most) two solutions. This is independent of the parameter r, which is proportional to the dimension. When $n > 2$ these facts change drastically. Taking $n = 3$, we have studied the following system of r equations in r unknowns:

$$f(x_1, x_2, x_3) = f(x_2, x_3, x_4) = \cdots = f(x_{r-1}, x_r, x_1) = f(x_r, x_1, x_2) = 0, \quad (9)$$

where $f(x, y, z)$, $i = 1, 2, \ldots, n$ is a generic polynomial of degree one in x, y, z. We define a_r to be the number of solutions to the system (9). In fact, the number a_r is equal to the number of fixed points of the r-th iterate of the mapping $H : (a, b) \mapsto (b, c)$, $f(a, b, c) = 0$. Based on Gröbner basis calculations in Maple we conjecture the following, cf. [14, Seq. A000211].

Conjecture 2 *We have* $a_1 = 3, a_2 = 5, a_r = a_{r-1} + a_{r-2} - 2$.

The number c_r of orbits of H with length r is then given by the Möbius inversion formula

$$c_r = 1/r \sum_{d|r} \mu(d) a_{r/d}.$$

These integer sequences, see Table 1, are almost the same as the fixed point and orbit counts for the golden cat map [2], cf. [14, Seq. A060280].

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_r	3	5	6	9	13	20	31	49	78	125	201	324	523	845
c_r	3	1	1	1	2	2	4	5	8	11	18	25	40	58

Table 1: Conjectured fixed point and orbit counts for the mapping H.

3.3 On the dimension of periodic initial value problems

In this section we show that the dimension of $\mathbf{s}$-periodic initial value problems is a piecewise linear function of $\mathbf{s}$.

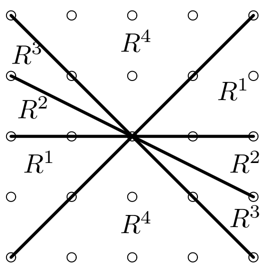

Figure 8: Distinguished regions for sailboat equations

For a given stencil S, we distinguish pie-piece shaped regions of $\mathbb{Z}\times\mathbb{Z}$, cut out by S-direction lines through the origin. For sailboat equations we denote the four S-directions (5) by

$$\hat{\mathbf{s}}^1 = (1, 1), \quad \hat{\mathbf{s}}^2 = (1, 0)$$
$$\hat{\mathbf{s}}^3 = (2, -1), \quad \hat{\mathbf{s}}^4 = (1, -1).$$

They give rise to four different regions, see Figure 8. To each region R^i we may associate a point $\mathbf{d}^i \in \mathbb{Z} \times \mathbb{Z}$ as follows.

Take $\mathbf{s} \in R^i$. Let L_1 and L_2 be lines with direction $\mathbf{s}$, which squeeze the stencil S and intersect the S-polygon in points $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively. Choose $d^i = \pm(p_1 - p_2)$. Clearly, d^i is a difference between the lines L_1 and L_2, and hence the dimension of the $\mathbf{s}$-periodic initial value problem is $D(\mathbf{s}, \mathbf{d}^i)$. For sailboat equations we may take

$$\mathbf{d}^1 = (1, -2), \quad \mathbf{d}^2 = (0, 2), \quad \mathbf{d}^3 = (2, 1), \quad \mathbf{d}^4 = (3, 0).$$

For $\mathbf{s}$ on the boundary between two regions, since the associated points are not uniquely defined one can choose any point $\mathbf{p}_1$ in the intersection of L_1 and S and any point $\mathbf{p}_2$ in the intersection of L_2 and S and take a difference, $\mathbf{d}^i = \pm(\mathbf{p}_1 - \mathbf{p}_2)$. In particular, to the boundary between R^i and R^{i+1}, one may associate $\mathbf{d}^i$ or $\mathbf{d}^{i+1}$. These choices are all equivalent: if $\mathbf{d}^1$ and $\mathbf{d}^2$ are two possible choices on a boundary with direction $\mathbf{s}$, then $\mathbf{d}^1 \pm \mathbf{d}^2 = k.\hat{\mathbf{s}}$, for some $k \in \mathbb{Z}$. For example, for sailboat equations we have

$$\mathbf{d}^1 + \mathbf{d}^2 = \hat{\mathbf{s}}^2, \quad \mathbf{d}^2 - \mathbf{d}^3 = -\hat{\mathbf{s}}^3, \quad \mathbf{d}^3 - \mathbf{d}^4 = -\hat{\mathbf{s}}^4, \quad \mathbf{d}^4 - \mathbf{d}^1 = 2\hat{\mathbf{s}}^1.$$

We can write down a single formula for the dimension of the $\mathbf{s}$-periodic initial value problem. Suppose that there are n distinct S-directions. These define n different regions R^i, which have n associated differences $\mathbf{d}^i$, $i = 1, 2, \ldots, n$. The dimension of the $\mathbf{s}$-periodic initial value problem is then given by the piece-wise linear expression

$$\max\{D(\mathbf{s}, \mathbf{d}^i), i = 1, 2, \ldots, n\}. \tag{10}$$

To see this, we prove that $\mathbf{s} \in \mathbb{R}^j$ implies $\max\{D(\mathbf{s}, \mathbf{d}^i), i = 1, 2, \ldots, n\} = D(\mathbf{s}, \mathbf{d}^j)$. Suppose that $\mathbf{s} \in \mathbb{R}^j$. Then $\mathbf{d}^j$ is a difference between the two lines L_1 and L_2 with direction $\mathbf{s}$, which squeeze the stencil S. Any other $\mathbf{d}^i$, with $i \neq j$, is a difference between two points which lie in between the two lines. Therefore, $\mathbf{d}^i$ itself fits in between the two lines and we find that $D(\mathbf{s}, \mathbf{d}^i) \leq D(\mathbf{s}, \mathbf{d}^j)$, which establishes the result. For sailboat equations the dimension of the $\mathbf{s}$-periodic solution is given by

$$\max\{|2s_1|, |3s_2|, |2s_1 + s_2|, |2s_2 - s_1|\}.$$

4 Discrete traveling wave reduction

In this section we explain how the periodic solutions arise as traveling wave reductions.

To explicitly write down the $D(\mathbf{s}, \mathbf{d})$-dimensional mappings, that generate the $\mathbf{s}$-periodic solutions, it is convenient to perform a change of variables $(l, m) \mapsto (n, p)$. One variable, n, will tell us on which line with direction $\hat{\mathbf{s}}$ the point is and the other, $\mathbf{p}$, will distinguish the r inequivalent points on each line.

Let ϵ denote the sign of $\hat{s}_2$, when $\hat{s}_2 = 0$ we set $\epsilon = -1$. We define

$$n_{\mathbf{s}}(\mathbf{a}) = \epsilon \det \begin{pmatrix} \mathbf{a} \\ \hat{\mathbf{s}} \end{pmatrix}. \tag{11}$$

Thus $n_{\mathbf{s}}$ only depends on $\hat{\mathbf{s}}$: for all $k \in \mathbb{Z}$ we have $n_{k.\hat{\mathbf{s}}}(\mathbf{a}) = n_{\hat{\mathbf{s}}}(\mathbf{a})$. Also, the function $n_{\mathbf{s}}(\mathbf{a}) \in \mathbb{Z}$ is invariant under shifting $\mathbf{a}$ by $\hat{\mathbf{s}}$: for all $k \in \mathbb{Z}$ we have

$$n_{\mathbf{s}}(\mathbf{a} + k.\hat{\mathbf{s}}) = n_{\mathbf{s}}(\mathbf{a}).$$

We have build in an antisymmetry of reflections in the s_1-axis: $n_{(s_1,-s_2)}(\mathbf{a}) = -n_{\mathbf{s}}(\mathbf{a})$. And, the new variable n increases to the right: $n_{\mathbf{s}}(\mathbf{a} + (1,0)) > n_{\mathbf{s}}(\mathbf{a})$, unless $\hat{\mathbf{s}} = (1,0)$. Next, we fix a point $\mathbf{c}^{\mathbf{s}} \in \mathbb{Z}\times\mathbb{Z}$ which will tell us how to update the initial values. Also we will use $\mathbf{c}^{\mathbf{s}}$ to define the variable p. If $\hat{\mathbf{s}} = (1,0)$, we take $\mathbf{c}^{\mathbf{s}} = (0,1)$. If $\hat{s}_2 = \epsilon$, we take $\mathbf{c}^{\mathbf{s}} = (1,0)$. And otherwise, we let $\mathbf{c}^{\mathbf{s}}$ be the unique lattice point inside the parallelogram spanned by $\hat{\mathbf{s}}$ and $(1,0)$, such that $D(\hat{\mathbf{s}}, \mathbf{c}^{\mathbf{s}}) = 1$. In this way, shifting over $\mathbf{c}^{\mathbf{s}}$ raises the index n by 1:

$$n_{\mathbf{s}}(\mathbf{a} + \mathbf{c}^{\mathbf{s}}) = n_{\mathbf{s}}(\mathbf{a}) + 1.$$

Clearly, lines with direction $\mathbf{s}$ are shifted to the next line on the right, except when $\hat{\mathbf{s}} = (1,0)$, where the lines are shifted upwards. Using the point $\mathbf{c}^{\mathbf{s}}$ we define

$$p_{\mathbf{s}}(\mathbf{a}) = \epsilon \det \begin{pmatrix} \mathbf{c}^{\mathbf{s}} \\ \mathbf{a} \end{pmatrix} \bmod r, \tag{12}$$

and we take $p \in \mathbb{N}_r := \{0, 1, \ldots, r-1\}$. The function $p_{\mathbf{s}}(\mathbf{a}) \in \mathbb{N}_q$ is invariant under shifting $\mathbf{a}$ by $\mathbf{c}^{\mathbf{s}}$: for all $k \in \mathbb{Z}$ we have

$$p_{\mathbf{s}}(\mathbf{a} + k.\mathbf{c}^{\mathbf{s}}) = p_{\mathbf{s}}(\mathbf{a}).$$

The antisymmetry of reflections in the s_1-axis is present: $p_{(s_1,-s_2)}(\mathbf{a}) = -p_{\mathbf{s}}(\mathbf{a})$. And, the function p increases by one when shifting $\mathbf{a}$ in the direction $\hat{\mathbf{s}}$:

$$p_{\mathbf{s}}(\mathbf{a} + \hat{\mathbf{s}}) = p_{\mathbf{s}}(\mathbf{a}) + 1.$$

The $\mathbf{s}$-periodic traveling wave reduction is now given by

$$u_{l,m} \rightarrowtail v_n^p, \quad n = n_{\mathbf{s}}(l,m), \ p = p_{\mathbf{s}}(l,m),$$

which should be compared to (2). The lattice equation (3) reduces to the system of r OΔEs, with $p = 0, 1, \ldots, r-1$,

$$F(v_n^p, v_{n+\epsilon\hat{s}_2}^{p-\epsilon c_2^{\mathbf{s}}}, v_{n-\epsilon\hat{s}_1}^{p+\epsilon c_1^{\mathbf{s}}}, v_{n+2\epsilon\hat{s}_2}^{p-2\epsilon c_2^{\mathbf{s}}}, v_{n+\epsilon(\hat{s}_2-\hat{s}_1)}^{p-\epsilon(c_2^{\mathbf{s}}-c_1^{\mathbf{s}})}, v_{n-2\epsilon\hat{s}_1}^{p+2\epsilon c_1^{\mathbf{s}}}, \ldots) = 0. \tag{13}$$

Suppose that $\mathbf{s}$ is taken in one of the regions where one has a well-posed initial value problem. In terms of the reduced variables, the set of initial value can be specified as $\{v_n^p, \ n \in \mathbb{N}_{D/q}, \ p \in \mathbb{N}_q\}$, where D is the dimension as given by equation (10). The corresponding mapping is

$$\begin{aligned} v_n^p &\mapsto v_{n+1}^p, && n \in \mathbb{N}_{D/q-1}, \ p \in \mathbb{N}_q \\ v_{D/q-1}^p &\mapsto v_{D/q}^p, && p \in \mathbb{N}_q, \end{aligned}$$

where the $v_{D/q}^p$ can be found by solving one of the equations (13), which we assumed to be linear in order to have uniqueness.

5 Applications, examples, extensions

Recently, in [1], a criterion was given for the well-posedness of Cauchy problems for *integrable* equations defined on the square, on a so called quad-graph, that is, a *planar graph* with quadrilateral faces. In contrast, we considered lattice equations not necessarily defined on a square, which live on the regular lattice (section 2), or on regular *cylindrical lattices* (section 3,4). Also, for our results we did not require the equations to be integrable. An interesting open problem would be to consider (integrable) lattice equations defined on stencils different from the square, on non-regular quad-graphs.

The main application of our results we have in mind is in the area of discrete integrable systems. It is thought that reductions of integrable lattice equations are integrable. A common property of integrable lattice equations is that they posses a *Lax-pair*. Recently, it has been shown that their $\mathbf{s}$-periodic reductions also posses a Lax-pair [13]. An open question is whether the reductions are *completely integrable* in the sense of Liouville-Arnold-Veselov, i.e. whether there are sufficiently many functionally independent integrals in involution [4, 16].

The staircase method provides integrals for mappings and correspondences that are obtained as traveling wave reductions of integrable lattice equations that exhibit a Lax-pair. The integrals are obtained by taking the trace of powers of a *monodromy matrix* $\mathcal{L}$, which is an ordered product of Lax-matrices along a staircase. The method was originally developed for scalar equations on the square [11, 15, 13], but applies to more general situations [9, 12]. In [13] it is shown that the monodromy matrix actually is one of the Lax-matrices for the reduced system of OΔEs. In [8] we give a short, but general, proof of the invariance of $\mathrm{tr}(\mathcal{L}^i)$, in the spirit of the original work [15]. The proof is solely based on the existence of $\mathbf{s}$-periodic solutions and the presence of a Lax-pair for the PΔE. In this paper we have shown the existence of $\mathbf{s}$-periodic solutions for any (non-zero) $\mathbf{s} \in \mathbb{Z} \times \mathbb{Z}$ for scalar lattice equations defined on any stencil. Therefore we may conclude that the staircase method can be applied to all integrable scalar lattice equations. The question whether this provides sufficiently many integrals remains open, see [8].

5.1 Reductions of a 5-point equation

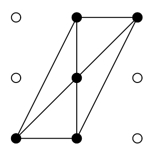

Figure 9: A five point stencil

In this section we perform different reductions for an integrable 5-point equation found in [5], namely the five-point equation

$$\frac{(u_{l,m} - u_{l,m-1})(u_{l,m} - u_{l-1,m-1})}{(u_{l,m} - u_{l,m+1})(u_{l,m} - u_{l+1,m+1})} = 1,$$

see [5, Equation (4a)], with $\alpha^{(\nu)} = 0$. The stencil on which the equation is defined is depicted in Figure 9. We perform a few different reductions, writing down explicitly the associated mapping, or correspondence.

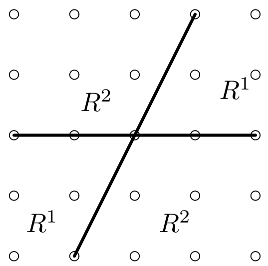

Figure 10: Distinguished regions for the 5-point stencil in Figure 9

The directions of the convex hull of the stencil are $(1,2)$ and $(1,0)$. Therefore, we distinguish two different regions, as in Figure 10. The distances we associate to the different regions are

$$d^1 = (0,2), \quad d^2 = (2,2).$$

Hence, the dimension of the $\mathbf{s}$-periodic initial value problems is given by

$$2\max(|(s_1 - s_2)|, |s_1|).$$

- **(-1,2)-reduction.** We have $\mathbf{s} = -\hat{\mathbf{s}}$, $\hat{\mathbf{s}} = (1,-2) \in R^2$, and $\mathbf{c}^\mathbf{s} = (1,-1)$. The dimension is $D(\hat{\mathbf{s}}, \mathbf{d}^2) = 6$. We change variables $n = 2l+m$ and $p \equiv 0$. In Figure 11 we display the positions of the initial values $\{v_0^0, v_1^0, \ldots, v_5^0\}$ in part of the lattice.

$$\begin{array}{ccccc}
v_0^0 & v_2^0 & v_4^0 & . & . \\
. & v_1^0 & v_3^0 & v_5^0 & . \\
. & v_0^0 & v_2^0 & v_4^0 & . \\
. & . & v_1^0 & v_3^0 & v_5^0 \\
. & . & v_0^0 & v_2^0 & v_4^0
\end{array}$$

Figure 11: (-1,2)-reduction

$$\begin{array}{ccccc}
. & v_0^0 & v_1^0 & . & . \\
. & v_0^1 & v_1^1 & . & . \\
. & v_0^0 & v_1^0 & . & . \\
. & v_0^1 & v_1^1 & . & . \\
. & v_0^0 & v_1^0 & . & .
\end{array}$$

Figure 12: (0,2)-reduction

The mapping which generates the $(-1,2)$-periodic solution is

$$(v_0^0, v_1^0, v_2^0, v_3^0, v_4^0, v_5^0) \mapsto \left(v_1^0, v_2^0, v_3^0, v_4^0, v_5^0, \frac{v_3^0 v_0^0 + v_2^0 v_3^0 - v_2^0 v_0^0 - v_3^0 v_4^0}{v_3^0 - v_4^0} \right)$$

- **(0,2)-reduction.** We have $\mathbf{s} = 2\hat{\mathbf{s}}$ with $\hat{\mathbf{s}} = (0,1) \in R^2$ and $\mathbf{c}^\mathbf{s} = (1,0)$. The dimension is $D(\hat{\mathbf{s}}, \mathbf{d}^2) = 4$. We change variables $n = l$ and $p \equiv m \mod 2$. In Figure 12 we display the positions of the initial values $\{v_0^0, v_1^0, v_0^1, v_1^1\}$ in part of the lattice. The mapping which generates the $(0,2)$-periodic solution is a permutation

$$(v_0^0, v_1^0, v_0^1, v_1^1) \mapsto (v_1^0, v_0^0, v_1^1, v_0^1).$$

- **(1,2)-reduction.** We have $\hat{\mathbf{s}} = (1,2)$ which is on the boundary of R^1 and R^2. The shift is $\mathbf{c}^\mathbf{s} = (1,1)$ and the dimension is $D(\hat{\mathbf{s}}, \mathbf{d}^1) = D(\hat{\mathbf{s}}, \mathbf{d}^2) = 2$. We change variables $n = 2l - m$ and $p \equiv 0$. In Figure 13 we display the positions of the initial values $\{v_0^0, v_1^0\}$ in part of the lattice. In this case the equation vanishes and infinitely many values for v_i^0, $i \in \mathbb{Z}$, may be chosen arbitrarily. This initial value problem is not nearly-well-posed.

$$\begin{array}{ccccc}
\cdot & \cdot & v_0^0 & \cdot & \cdot \\
\cdot & \cdot & v_1^0 & \cdot & \cdot \\
\cdot & v_0^0 & \cdot & \cdot & \cdot \\
\cdot & v_1^0 & \cdot & \cdot & \cdot \\
v_0^0 & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Figure 13: (1,2)-reduction

$$\begin{array}{ccccc}
\cdot & \cdot & \cdot & v_0^0 & v_1^0 \\
\cdot & \cdot & v_0^1 & v_1^1 & \cdot \\
\cdot & v_0^0 & v_1^0 & \cdot & \cdot \\
v_0^1 & v_1^1 & \cdot & \cdot & \cdot \\
v_1^0 & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Figure 14: (2,2)-reduction

- **(2,2)-reduction.** Specifying initial values as in Figure 14 we find the same permutation as for (0,2)-reduction.
- **(2,1)-reduction.** We specify initial values as in Figure 15 which we update by shifting over $\mathbf{c}^\mathbf{s} = (1,0)$. The 4-dimensional mapping is

$$(v_0^0, v_1^0, v_2^0, v_3^0) \mapsto \left(v_1^0, v_2^0, v_3^0, \frac{v_2^0 v_1^0 + v_2^0 v_0^0 - v_2^0 v_3^0 - v_0^0 v_1^0}{v_2^0 - v_3^0} \right).$$

$$\begin{array}{ccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & v_0^0 & v_1^0 \\
\cdot & v_0^0 & v_1^0 & v_2^0 & v_3^0 \\
v_1^0 & v_2^0 & v_3^0 & \cdot & \cdot \\
v_3^0 & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Figure 15: (2,1)-reduction

$$\begin{array}{ccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
v_1^1 & v_1^0 & v_1^1 & v_1^0 & v_1^1 \\
v_0^1 & v_0^0 & v_0^1 & v_0^0 & v_0^1 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Figure 16: (2,0)-reduction

- **(2,0)-reduction.** We specify initial values as in Figure 16 which we update by shifting over $\mathbf{c}^\mathbf{s} = (0,1)$. The 4-dimensional correspondence comprises two permutations:

$$\begin{array}{c}
(v_0^0, v_1^0, v_0^1, v_1^0) \mapsto (v_1^0, v_0^0, v_1^1, v_0^1) \\
(v_0^0, v_1^0, v_0^1, v_1^0) \mapsto (v_1^0, v_0^1, v_1^1, v_0^0)
\end{array}$$

- **(2,-1)-reduction.** We specify initial values as in Figure 17 which we update by shifting over $\mathbf{c}^\mathbf{s} = (0,1)$. The 6-dimensional mapping is

$$(v_0^0, v_1^0, v_2^0, v_3^0, v_4^0, v_5^0) \mapsto \left(v_1^0, v_2^0, v_3^0, v_4^0, v_5^0, \frac{v_3^0 v_0^0 + v_1^0 v_3^0 - v_1^0 v_0^0 - v_3^0 v_5^0}{v_3^0 - v_5^0} \right)$$

$$\begin{array}{ccccc} v_3^0 & v_4^0 & v_5^0 & . & . \\ v_1^0 & v_2^0 & v_3^0 & v_4^0 & v_5^0 \\ . & v_0^0 & v_1^0 & v_2^0 & v_3^0 \\ . & . & . & v_0^0 & v_1^0 \\ . & . & . & . & . \end{array}$$

Figure 17: (2,-1)-reduction

$$\begin{array}{ccccc} v_1^0 & v_2^0 & v_3^0 & . & . \\ v_0^1 & v_1^1 & v_2^1 & v_3^1 & . \\ . & v_0^0 & v_1^0 & v_2^0 & v_3^0 \\ . & . & v_0^1 & v_1^1 & v_2^1 \\ . & . & . & v_0^0 & v_1^0 \end{array}$$

Figure 18: (2,-2)-reduction

- **(2,-2)-reduction.** We specify initial values as in Figure 18 which we update by shifting over $\mathbf{c^s} = (0, 1)$. The 8-dimensional mapping is

$$\begin{aligned}(v_0^0, v_1^0, v_2^0, v_3^0, v_0^1, v_1^1, v_2^1, v_3^1) \mapsto \Bigg(& v_1^0, v_2^0, v_3^0, \frac{v_2^1 v_0^0 + v_1^0 v_2^1 - v_1^0 v_0^0 - v_3^0 v_2^1}{v_2^1 - v_3^0}, \\ & v_1^1, v_2^1, v_3^1, \frac{v_2^0 v_0^1 + v_1^1 v_2^0 - v_1^1 v_0^1 - v_3^1 v_2^0}{v_2^0 - v_3^1}\Bigg)\end{aligned}$$

Because we have considered low dimensional periodic reductions, in a few cases, $\mathbf{s} \in \{(0, 2), (1, 2), (2, 2), (2, 0)\}$, the mappings/correspondences obtained are fairly trivial. This has to do with symmetries of the particular equation we considered. If we would consider $\mathbf{s} = \pm r\hat{\mathbf{s}}$ with larger $r \in \mathbb{N}$, we would find non-trivial mappings/correspondences.

5.2 Reductions of QD-type systems

Now that we know how to pose initial values for scalar equations, the question arises whether something similar can be done for systems of equations. Here we will present an example where this is the case indeed. Note that also for integrable *systems* the staircase method does provide integrals as long as there is a Lax pair and a well-posed (or nearly-well-posed) initial value problem.

The quotient-difference (QD) algorithm, see for example [10, equations 7,8],

$$\begin{aligned} e_{l,m+1} + q_{l+1,m+1} &= q_{l+1,m} + e_{l+1,m} \\ e_{l,m+1} q_{l,m+1} &= q_{l+1,m} e_{l,m}, \end{aligned} \tag{14}$$

is used to construct continued fractions whose convergents form ordered sequences in a normal Padé table [6], and to find the zeros of a polynomial [7]. It is an integrable two-component equation defined on the stencil as depicted in Figure 19.

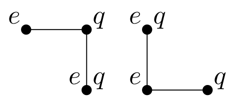

Figure 19: The stencil of the QD-system

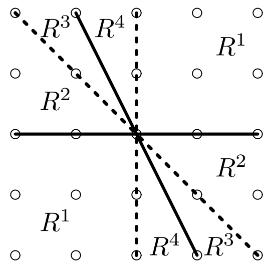

Figure 20: Distinguished regions for the QD-stencil

We will denote the $\mathbf{s}$-reduced fields with the same symbols, that is, we write $(e,q)_{l,m} \rightarrowtail (e,q)^p_n$, with $n = n_{\mathbf{s}}(l,m)$, and $p = p_{\mathbf{s}}(l,m)$. We distinguish four regions as in Figure 20. For each region we will describe how to pose particular simple well-posed initial value problems. To write down the generating mappings, one has to calculate r e-values and r q-values, where $\mathbf{s} = \pm r\hat{\mathbf{s}}$.

Proposition 3 *A multi-linear equation of QD-type admits a well-posed $\mathbf{s}$-periodic initial value problem if $\hat{\mathbf{s}}$ is not equal to $(1,0)$ or $(1,-2)$.*

1. *When $\mathbf{s} \in R^1$, or $\hat{\mathbf{s}} = (0,1)$, the set $\{e^p_n, q^p_n : n \in \mathbb{N}_{\hat{s}_1+\hat{s}_2}, p \in \mathbb{N}_r\}$ provides a well-posed initial value problem of dimension $2|s_1+s_2|$.*
2. *When $\mathbf{s} \in R^2$, or $\hat{\mathbf{s}} = (1,-1)$, the set $\{e^p_n, q^p_m : n \in \mathbb{N}_{\hat{s}_1}, m + \hat{s}_2 \in \mathbb{N}_{\hat{s}_1}, p \in \mathbb{N}_r\}$ provides a well-posed initial value problem of dimension $2|s_1|$.*
3. *When $\mathbf{s} \in R^3$, the set $\{e^p_n, q^p_m : n \in \mathbb{N}_{\hat{s}_1}, m - \hat{s}_1 \in \mathbb{N}_{\hat{s}_1}, p \in \mathbb{N}_r\}$ provides a well-posed initial value problem of dimension $2|s_1|$.*
4. *When $\mathbf{s} \in R^4$, the set $\{e^p_n, q^p_m : n \in \mathbb{N}_{-\hat{s}_1-\hat{s}_2}, m - \hat{s}_1 \in \mathbb{N}_{-\hat{s}_1-\hat{s}_2}, p \in \mathbb{N}_r\}$ provides a well-posed initial value problem of dimension $2|s_1 - s_2|$.*

Proof: Any multi-linear system that is defined on the stencil in Figure 19 can be written as $f_{l,m} = g_{l,m} = 0$, where

$$f_{l,m} = f(e_{l,m+1}, e_{l+1,m}, q_{l+1,m}, q_{l+1,m+1}), \quad g_{l,m} = g(e_{l,m}, e_{l,m+1}, q_{l+1,m}, q_{l,m+1}).$$

Performing the reduction we get $f^p_n = g^p_n = 0$, with

$$f^p_n = f(e^{p+\epsilon c^{\mathbf{s}}_1}_{n-\epsilon\hat{s}_1}, e^{p-\epsilon c^{\mathbf{s}}_2}_{n+\epsilon\hat{s}_2}, q^{p-\epsilon c^{\mathbf{s}}_2}_{n+\epsilon\hat{s}_2}, q^{p-\epsilon(c^{\mathbf{s}}_2-c^{\mathbf{s}}_1)}_{n+\epsilon(\hat{s}_2-\hat{s}_1)}), \quad g^p_n = g(e^p_n, e^{p+\epsilon c^{\mathbf{s}}_1}_{n-\epsilon\hat{s}_1}, q^{p-\epsilon c^{\mathbf{s}}_2}_{n+\epsilon\hat{s}_2}, q^{p+\epsilon c^{\mathbf{s}}_1}_{n-\epsilon\hat{s}_1}).$$

1. We have $\epsilon = 1$. If $\mathbf{s} \in R^1$ then $\hat{s}_1, \hat{s}_2 \in \mathbb{N}$. We have to show that one can calculate $(e,q)^p_{\hat{s}_1+\hat{s}_2}$. Indeed, we can calculate $q^p_{\hat{s}_1+\hat{s}_2}$ from $g^{p+c^{\mathbf{s}}_2}_{\hat{s}_1} = 0$, after which $e^p_{\hat{s}_1+\hat{s}_2}$ is determined by solving $f^{p+c^{\mathbf{s}}_2}_{\hat{s}_1} = 0$.
2. When $\mathbf{s} \in R^2$, we have $\epsilon = -1$, and $0 < -\hat{s}_2 \leq \hat{s}_1$. We can calculate $e^p_{\hat{s}_1}$ from $g^{p+c^{\mathbf{s}}_1}_0 = 0$ and $q^p_{\hat{s}_1-\hat{s}_2}$ from $f^{p+c^{\mathbf{s}}_1-c^{\mathbf{s}}_2}_0 = 0$.

3. When $\mathbf{s} \in R^3$, we have $\epsilon = -1$, and $\hat{s}_1 < -\hat{s}_2 < 2\hat{s}_1$. We can calculate $e^p_{\hat{s}_1}$ from $g_0^{p+c_1^{\mathbf{s}}} = 0$ and $q^p_{2\hat{s}_1}$ from $f^{p+c_1^{\mathbf{s}}-c_2^{\mathbf{s}}}_{\hat{s}_1+\hat{s}_2} = 0$.
4. When $\mathbf{s} \in R^4$, we have $\epsilon = -1$, and $0 < 2\hat{s}_1 < -\hat{s}_2$. We can calculate $q^p_{-\hat{s}_2}$ from $g_0^{p-c_2^{\mathbf{s}}} = 0$ and $e^p_{-\hat{s}_1-\hat{s}_2}$ from $f^{p-c_2^{\mathbf{s}}}_{-\hat{s}_1} = 0$.

One expects correspondences instead of mappings when $\hat{\mathbf{s}}$ equals $(1,0)$ or $(1,-2)$. In [8] we provide examples of one parameter families of reductions of the QD-system (14), together with a sufficient number of integrals for the mappings and correspondences.

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