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Growth of degrees of integrable mappings

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We study mappings obtained as **s**-periodic reductions of the lattice Korteweg–de Vries equation. For small $\mathbf{s} \in \mathbb{N}^2$, we establish upper bounds on the growth of the degree of the numerator of their iterates. These upper bounds appear to be exact. Moreover, we conjecture that for any s_1, s_2 that are co-prime, the growth is $\sim (2s_1s_2)^{-1}n^2$, except when $s_1 + s_2 = 4$, where the growth is linear $\sim n$. Also, we conjecture the degree of the nth iterate in projective space to be $\sim (s_1 + s_2)(2s_1s_2)^{-1}n^2$.

Keywords: integrable mappings; algebraic entropy; polynomial growth; lattice equations

1. Introduction

Integrable mappings are characterized by low complexity [2,19]. This idea culminated in the notion of algebraic entropy, introduced by Viallet and collaborators [4,6,8]. Low complexity means vanishing algebraic entropy which corresponds to polynomial growth of degrees of iterates of the mapping. A first proof of such a polynomial bound on the degrees was given in [5]. In [3] it was proven that foliation by invariant curves implies zero algebraic entropy. Examples show that degree growth is a better indication of integrability than singularity confinement [8,9], cf. the discussion in [13]. Recently, the notion has been extended to lattice equations [16,17] and used to find new integrable models [10].

In practice, one calculates the growth of degrees d_n of the first n iterates of a mapping. Then one guesses the pattern by fitting the generating function $g(x) = \sum d_n x^n$ with a rational function $p(x)/q(x) = g(x)$, and the algebraic entropy $\lim_{n\to\infty} \log(d_n)/n$ is obtained as the logarithm of the inverse of the smallest zero of $q(x)$, see [17]. We present an elementary method that enables one to derive upper bounds for the growth of degrees. Our formulae exactly produce all degrees that we have been able to calculate.

2. Outline

We will perform **s**-periodic reductions of the lattice Korteweg–de Vries equation

$$(u_{l,m} - u_{l+1,m+1})(u_{l+1,m} - u_{l,m+1}) = \alpha. \tag{1}$$

This corresponds to studying solutions that satisfy the periodicity condition $u_{l,m} = u_{l+s_1,m+s_2}$. We choose s_1 and $s_2 \leq s_1$ to be co-prime natural numbers. Under this assumption, the lattice equation reduces to a single ordinary difference equation (OΔE) of order $q := s_1 + s_2$ (or a q-dimensional mapping). For background on periodic reductions we refer to [11,14]. There are q initial values, which we denote by $x_1, x_2, \ldots, x_q$. The OΔE,

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or the mapping, can be used to generate a solution $x_{n\in\mathbb{Z}}$, which are rational functions in the initial values.

One aim is to find a formula for the degree of the numerator (or denominator) of x_n, as a function of n. We set $x_n = a_n/b_n$ and derive a system of two OΔEs for a_n and b_n, which are polynomials in the initial values. By choosing $b_n = 1$ for $n = 1, 2, \ldots, q$, the degree (i.e. total degree in the variables $x_1 = a_1, \ldots, x_q = a_q$) of the numerator of x_n is given by $d_n^a - d_n^g$. Here, d_n^p denotes the degree of a polynomial p_n and g_n is the greatest common divisor $g_n = \gcd(a_n, b_n)$. First, we obtain a recursive formula for $d_n^a = d_n^b + 1$. Then, we look at the growth of g_n. After a number of iterates, a miracle occurs: any divisor of b_n will divide $g_{n+q}(q \neq 4)$. This statement has been verified for a range of periodicities **s**, but seems to be difficult to prove in general. Next, we find a recurrence formula for the growth of the multiplicities of divisors: a divisor of g_n divides g_{n+i} with multiplicity t_i, where t is an integer sequence satisfying a linear recurrence relation. We define a new set of polynomials $c_n = b_n/f$, where f is the product of all divisors of $b_{i<n}$ with the right multiplicities as given by the integer sequence t. Multiplying by f (which is a product $c_{i<n}s$) and taking the degree on both sides of $c_n f = b_n$, we find that $d_n^c + (d^c * t)_n = d_n^b$, where $*$ denotes discrete convolution

$$(d * t)_{n+1} = d_1 t_n + d_2 t_{n-1} + \cdots + d_n t_1. \tag{2}$$

Using the recursive formulae for d^b and t, we find a recursive formula for d^c, which can be solved to find polynomial growth of degree 2. Moreover, we obtain the coefficient of the leading term: $(2s_1 s_2)^{-1}$.

We also consider the projective analogues of these mappings. We introduce homogeneous coordinates and derive a polynomial mapping in q-dimensional projective space. Here, the aim is to find a formula for the degrees of the components of this mapping. The strategy is similar as the above. Once one has a divisor c_i of certain components of the mapping, one can derive a recursive formula for the multiplicities at higher iterates of the mapping. At a certain point these multiplicities are (miraculously) higher than expected, after which the growth can be described recursively again. As before, a convolution formula provides us with a recurrence for the degrees of the divisors. In this case the degree of the nth iterate is given by the sum $1 + d_{n-1}^c + d_{n-2}^c + \cdots + d_{n-q}^c$. This growth can also be described recursively and the leading term is found to be $(s_1 + s_2)(2s_1 s_2)^{-1} n^2$.

The case $s = (3, 1)$ is exceptional. Here the growth is linear $\sim n$, and the mapping is linearizable. We provide its explicit solution in terms of an interesting sequence of polynomials, see Section 3.3 and the Appendix.

3. Growth of degrees of rational mappings

We first illustrate our approach by considering a low-dimensional example, taking $\mathbf{s} = (2, 1)$.

3.1 A low-dimensional example

We take initial values x_1, x_2, x_3 on a staircase as shown in Figure 1. The x_n are rational functions of x_1, x_2, x_3, α which can be calculated recursively using

$$x_n = P(x_{n-1}, x_{n-2}, x_{n-3}), \tag{3}$$

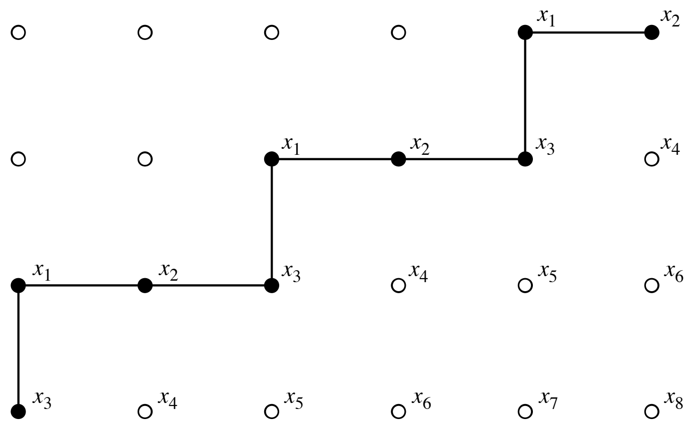

Figure 1. Staircase with $(1, 2)$ periodic initial values (x_1, x_2, x_3) solved to the right.

where P solves equation (1) for $u_{l+1,m}$,

$$u_{l+1,m} = P(u_{l,m}, u_{l+1,m+1}, u_{l,m+1}) := u_{l,m+1} + \frac{\alpha}{u_{l,m} - u_{l+1,m+1}}. \tag{4}$$

We write $x_n = a_n/b_n$. The recurrence (3) yields the following recurrences for a, b:

$$a_n = a_{n-3}w_n - \alpha b_{n-1}b_{n-2}b_{n-3}, \tag{5a}$$

$$b_n = b_{n-3}w_n, \tag{5b}$$

where $w_n = a_{n-2}b_{n-1} - a_{n-1}b_{n-2}$. We choose $b_1 = b_2 = b_3 = 1$, so that a_n and b_n are polynomials in the variables (initial values) a_1, a_2 and a_3. Their total degree will be denoted d_n^a and d_n^b, respectively. From (5) it follows that the degrees are at most

$$\begin{aligned} d_n^a &= \max\left(d_{n-1}^b + d_{n-2}^a + d_{n-3}^a, d_{n-1}^a + d_{n-2}^b + d_{n-1}^a, d_{n-1}^b + d_{n-2}^b + d_{n-3}^b\right), \\ d_n^b &= \max\left(d_{n-1}^b + d_{n-2}^a + d_{n-3}^b, d_{n-1}^a + d_{n-2}^b + d_{n-3}^b\right). \end{aligned}$$

Given the initial degrees $d_n^a = d_n^b + 1 = 1 \quad (n = 1, 2, 3)$, we find that

$$\begin{aligned} d_n^a &= d_{n-1}^a + d_{n-2}^a + d_{n-3}^a - 1, \\ d_n^b &= d_{n-1}^b + d_{n-2}^b + d_{n-3}^b + 1 \end{aligned} \tag{6}$$

are upper bounds for the degrees of a_n and b_n, and $d_n^a = d_n^b + 1$ $(n \in \mathbb{N})$. The sequence d^b comprises sums of tribonacci numbers, cf. [15, seq. A008937]. Certainly, these sequences grow exponentially. However, there will be a lot of cancellations in $x_n = a_n/b_n$ due to common factors of a_n, b_n. We will prove that the degree of the greatest common divisor

$$g_n = \gcd(a_n, b_n)$$

is sufficiently large to ensure that $d_n^a - d_n^g$ grows polynomially.

Suppose that f^{t_k} divides g_k with $k \in \{n-1, n-2, n-3\}$. Then from (5) it follows that f^{t_n} divides g_n, where

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}. \tag{7}$$

We define an integer sequence t by $t_1 = t_2 = t_3 - 2 = 0$ and the above recursion. Such numbers t are called tribonacci numbers, cf. [15, seq. A000073]. Thus, we have the following:

$$f^2 | g_n \Rightarrow f^{t_{3+i}} | g_{n+i}, \quad i \in \mathbb{N}.$$

By direct calculation, using Maple and the recurrences (5), we find that the polynomial w_n^2 divides g_{n+3} $(n > 3)$. This implies that $w_n^{t_i}$ divides g_{n+i}. We can now write symbolically

$$\begin{aligned}
b_i &= c_i = 1,\ i = 1, 2, 3, \\
b_i &= c_i,\ i = 4, 5, 6, \\
b_7 &= c_7 c_4^2, \\
b_8 &= c_8 c_4^2 c_5^2 = c_8 c_4^{t_4} c_5^{t_3}, \\
b_9 &= c_9 c_4^4 c_5^2 c_6^2 = c_9 c_1^{t_8} c_2^{t_7} c_3^{t_6} c_4^{t_5} c_5^{t_4} c_6^{t_3} c_7^{t_2} c_8^{t_1}, \\
&\vdots \\
b_n &= c_n \prod_{i=1}^{n-1} c_i^{t_{n-i}},
\end{aligned} \tag{8}$$

which defines polynomials c_n. Taking the degree on both sides of equation (8), we find $d_n^b = d_n^c + (d^c * t)_n$ where $*$ denotes discrete convolution, see (2). From this we infer, using the recurrence for t (7), that

$$\begin{aligned}
d_n^b - d_n^c &= d_1^c t_{n-1} + \cdots + d_{n-4}^c t_4 + d_{n-3}^c t_3 \\
&= d_1^c (t_{n-2} + t_{n-3} + t_{n-4}) + \cdots + d_{n-4}^c (t_3 + t_2 + t_1) + d_{n-3}^c t_3 \\
&= (d^c * t)_{n-1} + (d^c * t)_{n-2} + (d^c * t)_{n-3} + 2 d_{n-3}^c \\
&= d_{n-1}^b - d_{n-1}^c + d_{n-2}^b - d_{n-2}^c + d_{n-3}^b + d_{n-3}^c,
\end{aligned}$$

which, using the recursion for d^b (6), shows that

$$d_n^c = d_{n-1}^c + d_{n-2}^c - d_{n-3}^c + 1.$$

Together with $d_1^c = d_2^c = d_3^c = 0$, this gives a sequence of quarter squares, cf. [15, seq. A033638],

$$d_n^c = \lfloor \frac{(n-2)^2}{4} \rfloor.$$

Note that the $c_{i<n}$s in (8) are divisors of g_n. Thus the quantity $d_n^a - d_n^g$ is bounded from above by $d_n^b + 1 - (d^c * t)_n = d_n^c + 1$, which grows asymptotically $\sim n^2/4$.

3.2 More general periodic reductions

Next, we consider the mapping obtained from **s**-periodic reduction taking s_1 and s_2 to be co-prime. Without loss of generality we may assume $s_1 \geq s_2$. Remember we denote $s_1 + s_2 = q$. Initial values $x_1, x_2, \ldots, x_q$ are given on a standard staircase [14], see also [11] in which a general theory of periodic reductions for equations not necessarily defined on a square has been developed. The initial values are updated by a recurrence of order q:

$$x_n = P(x_{n-s_2}, x_{n-s_1}, x_{n-q}), \tag{9}$$

cf. equation (4). For example, when $\mathbf{s} = (3, 2)$, we pose initial values as in Figure 2. These are updated by shifting over (2, 1), e.g. $x_5 \mapsto x_6 = P(x_4, x_3, x_1)$.

By setting $x_n = a_n/b_n$, we derive for $n > q$ as follows:

$$a_n = a_{n-q}w_n - \alpha b_{n-s_1}b_{n-s_2}b_{n-q}, \tag{10}$$

$$b_n = b_{n-q}w_n, \tag{11}$$

where $w_n = a_{n-s_1}b_{n-s_2} - a_{n-s_2}b_{n-s_1}$. We choose $b_i = 1$, $i = 1, 2, \ldots, q$, so that a_n and b_n are polynomials in $a_1, a_2, \ldots, a_q$. As before, from initial degrees $d_n^a = d_n^b + 1 = 1$ ($n = 1, 2, \ldots, q$) we find that $d_n^a = d_n^b + 1$ ($n \in \mathbb{N}$), and that

$$d_n^a = d_{n-s_1}^a + d_{n-s_2}^a + d_{n-q}^a - 1, \quad d_n^b = d_{n-s_1}^b + d_{n-s_2}^b + d_{n-q}^b + 1$$

are upper bounds for the degrees of a_n and b_n. If f^{t_k} divides g_k with $k < n$, then f^{t_n} divides g_n, where

$$t_n = t_{n-s_1} + t_{n-s_2} + t_{n-q}. \tag{12}$$

If initially $t_i = 0$, $i = 1, 2, \ldots, q-1$, $t_q = 2$, then

$$f^2 | g_n \Rightarrow f^{t_{q+i}} | g_{n+i}, \quad i \in \mathbb{N}.$$

CONJECTURE 1. *The polynomial w_n^2 divides g_{n+q} (for $n > q$).*

It turns out that this conjecture is more difficult to verify for $s_2 \ll s_1$. We verified the conjecture in the following ranges of values $s_2 < s_1$: $s_2 = 1, \ldots, 5$ with $s_2 < s_1 \leq 9s_2$ and $s_1 = s_2 + 1$ with $s_2 = 6, 7, \ldots, 25, 50, 100, 150, 200, 250, 1000$.

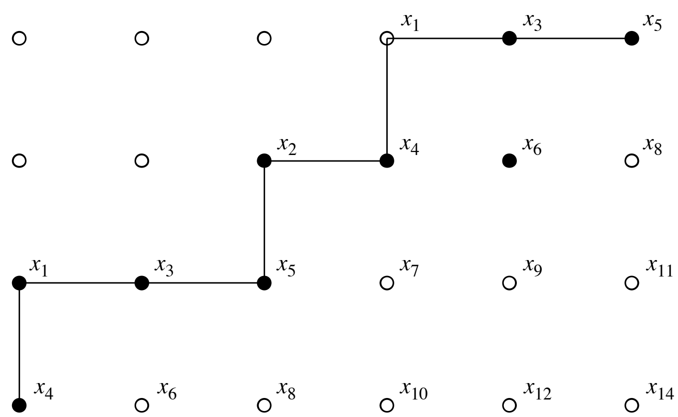

Figure 2. (3, 2) periodic initial value problem updated in direction (2, 1).

The conjecture would imply that $w_n^{t_i}$ divides g_{n+i}. Assuming it, we can define polynomials c_i by

$$b_n = c_n \prod_{i=1}^{n-1} c_i^{t_{n-i}},$$

which yields $d_n^b = d_n^c + (d^c * t)_n$. Using the recurrences for t and d^b, we find

$$d_n^c = d_{n-s_1}^c + d_{n-s_2}^c - d_{n-q}^c + 1.$$

In the case $\mathbf{s} = (3, 2)$, the sequence [15, seq. A001399]

$$0, 0, 0, 0, 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16, 19, 21, 24, \ldots$$

is given by

$$d_n^c = \frac{47}{72} + \frac{(-1)^n}{8} + \frac{\zeta^n + \zeta^{-n}}{9} - \frac{1}{2}n + \frac{1}{12}n^2, \quad \zeta^3 = 1.$$

In general, the quantity $d_n^a - d_n^g$ is bounded from above by $d_n^c + 1$, whose asymptotic growth is

$$\sim (2s_1 s_2)^{-1} n^2.$$

3.3 The exceptional case

The case $\mathbf{s} = (3, 1)$ is an exceptional case. Here the growth is linear, which resembles the fact that the mapping can be linearized. Introducing $h = (x_1 - x_3)(x_2 - x_4)$, the mapping

$$(x_1, x_2, x_3, x_4) \mapsto \left(x_2, x_3, x_4, x_1 + \frac{\alpha}{x_4 - x_2} \right)$$

reduces to $h \mapsto \alpha - h$, which is an involution.[1] Nevertheless, it is interesting to see what cancellations cause the growth to become linear.

We set $x_n = a_n / b_n$ to find

$$a_n = a_{n-4}(a_{n-3}b_{n-1} - a_{n-1}b_{n-3}) - \alpha b_{n-1}b_{n-3}b_{n-4}, \tag{13}$$

$$b_n = b_{n-4}(a_{n-3}b_{n-1} - a_{n-1}b_{n-3}). \tag{14}$$

Taking initial values $(a_1, a_2, a_3, a_4) = (x - w, y + z, -w, z)$ and $b_1 = b_2 = b_3 = b_4 = 1$, we have found that (see Appendix)

$$a_n = y^{t_{n-2}}(\alpha - xy)^{t_{n-3}} x^{t_{n-4}} c_n, \tag{15}$$

$$b_n = y^{s_{n+1}}(\alpha - xy)^{s_n} x^{s_{n-1}}, \tag{16}$$

where $t_0 = t_1 = s_0 = s_1 = 0$ and

$$t_{n+2} = t_{n+1} + t_n + \lfloor \frac{n}{4} \rfloor, \tag{17}$$

$$s_{n+2} = s_{n+1} + s_n + (-1)^n \lfloor \frac{n}{4} \rfloor.$$

Define $r_n = s_{n+4} - t_{n+1}$. One can show that $r_n = n(1 + (-1)^n)/4$, which is non-negative. It follows that the a_n/c_n is a common divisor of a_n and b_n. Dividing out this factor we are left with denominator growth ($n \geq 4$)

$$d_n^b - d_n^{a/c} = r_{n-3} + 2r_{n-4} + r_{n-5} = n - 4.$$

Note that in this case the common divisor of a_n and b_n consists of three different factors only, whereas for other values of $\mathbf{s}$ the number of common divisors grows linearly with n. Here, the multiplicity grows faster than what can be expected from the form of the recurrence. In other words, a 'miracle' happens at every iterate: from (17) one can derive

$$t_{n+4} = t_{n+3} + t_{n+1} + t_n + \lfloor \frac{n}{2} \rfloor,$$

which should be compared to (12), taking $s_1 = 1, s_2 = 3$, $q = 4$.

4. Growth of degrees of projective mappings

The entropy of a rational mapping has also been defined in terms of the growth of the degree of its equivalent in projective space [4]. Again we first consider the case $\mathbf{s} = (2, 1)$.

4.1 A low-dimensional example

The 3D mapping is

$$(x_1, x_2, x_3) \mapsto \left(x_2, x_3, x_1 + \frac{\alpha}{x_3 - x_2} \right).$$

We set $x_i = a_i/a_4$, $i = 1, 2, 3$. If we denote the image by b_i/b_4, then the homogenized mapping is $a \mapsto b$:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \mapsto \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} a_2(a_3 - a_2) \\ a_3(a_3 - a_2) \\ a_1(a_3 - a_2) + \alpha a_4^2 \\ a_4(a_3 - a_2) \end{pmatrix}. \tag{18}$$

Note that the first, second and fourth components of the image share a common divisor. We are interested in the growth of the multiplicities of such a divisor. Suppose that c divides a_1, a_2 and a_4. From (18) it follows that c is a common divisor of b_1, b_3 and b_4. We continue the argument

$$c|(a_1, a_3, a_4) \Rightarrow c|(b_2, b_3, b_4),$$

and

$$c|(a_2,a_3,a_4) \Rightarrow c^2|(b_1,b_2,b_4), c|b_3.$$

However, if we denote the common divisor of b_1, b_2, b_4 by c, then miraculously c^3 divides all four components of the fourth iterate of $a \mapsto b$. At the next iterates the multiplicities double. Denoting the multiplicity of c in the fourth component of the ith iterate by t_i, we have $t_1 = t_2 = t_3 = 1$, $t_4 = 3$ and (at least) $t_{n>4} = 2t_{n-1}$. We now introduce two sets of polynomials c_i, d_i as follows:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \mapsto \begin{pmatrix} a_2c_1 \\ a_3c_1 \\ d_1 \\ a_4c_1 \end{pmatrix} \mapsto \begin{pmatrix} a_3c_1c_2 \\ d_1c_2 \\ d_2c_1 \\ a_4c_1c_2 \end{pmatrix} \mapsto \begin{pmatrix} d_1c_2c_3 \\ d_2c_1c_3 \\ d_3c_1c_2 \\ a_4c_1c_2c_3 \end{pmatrix} \mapsto$$
$$\begin{pmatrix} d_2c_1^3c_3c_4 \\ d_3c_1^3c_2c_4 \\ d_4c_1^3c_2c_3 \\ a_4c_1^3c_2c_3c_4 \end{pmatrix} \mapsto \begin{pmatrix} d_3c_1^6c_2^3c_4c_5 \\ d_4c_1^6c_2^3c_3c_5 \\ d_5c_1^6c_2^3c_3c_4 \\ a_4c_1^6c_2^3c_3c_4c_5 \end{pmatrix} \mapsto \cdots \mapsto$$
$$\begin{pmatrix} d_{n-2}\prod_{i=1}^{n-3} c_i^{t_{n+1-i}} c_{n-1}c_n \\ d_{n-1}\prod_{i=1}^{n-2} c_i^{t_{n+1-i}} c_n \\ d_n\prod_{i=1}^{n-1} c_i^{t_{n+1-i}} \\ a_4\prod_{i=1}^{n} c_i^{t_{n+1-i}} \end{pmatrix} \mapsto \cdots. \tag{19}$$

As an ordinary polynomial map, the degree of the nth iterate is

$$2^n = 1 + (d^c * t)_{n+1}.$$

Subtracting $2^n = 2 + 2(d^c * t)_n$ from this equation, and using the recursion for t, we find that

$$d_n^c = d_{n-1}^c + d_{n-2}^c - d_{n-3}^c + 1.$$

Projectively, the nth iterate (with $n > 2$) is

$$(d_{n-2}c_{n-1}c_n, d_{n-1}c_{n-2}c_n, d_nc_{n-2}c_{n-1}, a_4c_{n-2}c_{n-1}c_n),$$

after division by the common factor $\prod_{i=1}^{n-3} c_i^{t_{n+1-i}}$. We define

$$p_n := a_4 \prod_{i=\max(1,n-3)}^{n-1} c_i.$$

The projective degree is

$$d^p_{n>3} = 1 + d^c_{n-1} + d^c_{n-2} + d^c_{n-3}.$$

The recursion for d^c yields $d^p_n = d^p_{n-1} + d^p_{n-2} - d^p_{n-3} + 3$. Together with initial values $d^p_i = 2^{i-1}$, $i = 1, 2, 3$, this gives the sequence [15, seq. A084684]

$$1, 2, 4, 8, 13, 20, 28, 38, 49, 62, \ldots,$$

which agrees with computations in projective space. The growth

$$d^p_n = \frac{15}{8} + \frac{(-1)^n}{8} - \frac{3}{2}n + \frac{3}{4}n^2$$

is the same as for a mapping connected to the discrete Painlevé I equation [4,9].

4.2 More general periodic reductions

Now we consider the projective mapping that corresponds to $\mathbf{s}$-periodic reduction with s_1 and s_2 co-prime. We take $s_1 \leq s_2$, and $q = s_1 + s_2$. It is convenient to take initial values $x_0, x_2, \ldots, x_{q-1}$. They are updated using the recurrence (9), or equivalently, the q-dimensional mapping

$$(x_0, x_1, \ldots, x_{q-1}) \mapsto (x_1, \ldots, x_{q-1}, P(x_{s_1}, x_{s_2}, x_0)).$$

Denoting the image of $x_i = a_i/a_q$ by b_i/b_q, we find a mapping $a \mapsto b$ in q-dimensional projective space

$$\begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_{q-1} \\ a_q \end{pmatrix} \mapsto \begin{pmatrix} a_1(a_{s_1} - a_{s_2}) \\ a_2(a_{s_1} - a_{s_2}) \\ \cdots \\ a_0(a_{s_1} - a_{s_2}) + a_q^2 \\ a_q(a_{s_1} - a_{s_2}) \end{pmatrix}.$$

As in the case $\mathbf{s} = (2, 1)$, there is a common factor dividing all components but 1. When $s_1 > 1$, we have

$$c|(a_0, a_1, \ldots, a_{q-2}, a_q) \Rightarrow \begin{cases} c^2|(b_0, b_1, \ldots, b_{q-3}, b_{q-1}, b_q) \\ c|b_{q-2}. \end{cases}$$

When $s_1 > 2$, we have

$$\begin{cases} c^2|(a_0, a_1, \ldots, a_{q-3}, a_{q-1}, a_q), \\ c|a_{q-2} \end{cases} \Rightarrow \begin{cases} c^4|(b_0, b_1, \ldots, b_{q-4}, b_{q-2}, b_{q-1}, b_q) \\ c^3|b_{q-3}. \end{cases}$$

This doubling in most components continues until after $s_1 - 1$ iterations we are led to (if $s_2 > s_1 + 1$)

$$\begin{cases} c^{2^{s_1-1}}|(a_0, a_1, \ldots, a_{s_2-1}, a_{s_2+1}, \ldots, a_q), \\ c^{2^{s_1-1}-1}|a_{s_2} \end{cases} \Rightarrow \begin{cases} c^{2^{s_1}-1}|(b_0, b_1, \ldots, b_{s_2-2}, b_{s_2}, \ldots, b_q) \\ c^{2^{s_1}-2}|b_{s_2-1}. \end{cases}$$

Then we have doubling again, until after $s_2 - 1$ iterations where the growth is similar to the above. Doubling continues until ...

CONJECTURE 2. *The 'miracle' happens after q iterations where suddenly the multiplicity is one higher than double the previous one.*

Thus, we have only verified for a couple of small values of s_1, s_2. Conjecture 2 is harder to verify, using direct calculation, than Conjecture 1. We will assume it in the sequel. We define integer sequences by $t_1 = 1$ and

$$t_{n+1} = \begin{cases} 2t_n - 1, & n = s_1, s_2, \\ 2t_n + 1, & n = s_1 + s_2, \\ 2t_n, & \text{otherwise.} \end{cases}$$

We now introduce two sets of polynomials c_i, d_i as follows:

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{q-2} \\ a_{q-1} \\ a_q \end{pmatrix} \mapsto \begin{pmatrix} a_1 c_1^{t_1} \\ a_2 c_1^{t_1} \\ \vdots \\ a_{q-1} c_1^{t_1} \\ d_1 c_1^{t_1 - 1} \\ a_q c_1^{t_1} \end{pmatrix} \mapsto \begin{pmatrix} a_2 c_1^{t_2} c_2^{t_1} \\ a_3 c_1^{t_2} c_2^{t_1} \\ \vdots \\ d_1 c_1^{t_2 - 1} c_2^{t_1} \\ d_2 c_1^{t_2} c_2^{t_1 - 1} \\ a_q c_1^{t_2} c_2^{t_1} \end{pmatrix} \mapsto \cdots \mapsto$$
$$\begin{pmatrix} d_1 c_1^{t_q - 1} c_2^{t_{q-1}} \cdots c_q^{t_1} \\ d_2 c_1^{t_q} c_2^{t_{q-1} - 1} \cdots c_q^{t_1} \\ \vdots \\ d_{q-1} c_1^{t_q} \cdots c_{q-1}^{t_2 - 1} c_q^{t_1} \\ d_q c_1^{t_q} \cdots c_{q-1}^{t_2} c_q^{t_1 - 1} \\ a_q c_1^{t_q} \cdots c_{q-1}^{t_2} c_q^{t_1} \end{pmatrix} \mapsto \begin{pmatrix} d_2 c_1^{t_{q+1}} c_2^{t_q - 1} \cdots c_q^{t_1} \\ d_3 c_1^{t_{q+1}} c_2^{t_q} c_3^{t_{q-1} - 1} \cdots c_q^{t_1} \\ \vdots \\ d_q c_1^{t_{q+1}} \cdots c_q^{t_2 - 1} c_{q+1}^{t_1} \\ d_{q+1} c_1^{t_{q+1}} \cdots c_q^{t_2} c_{q+1}^{t_1 - 1} \\ a_q c_1^{t_{q+1}} \cdots c_q^{t_2} c_{q+1}^{t_1} \end{pmatrix} \mapsto \cdots \mapsto$$
$$\begin{pmatrix} d_{n-q+1} \prod_{i=1}^{n} c_i^{t_{n+1-i}} / c_{n-q+1} \\ d_{n-q+2} \prod_{i=1}^{n} c_i^{t_{n+1-i}} / c_{n-q+2} \\ \vdots \\ d_{n-1} \prod_{i=1}^{n} c_i^{t_{n+1-i}} / c_{n-1} \\ d_n \prod_{i=1}^{n} c_i^{t_{n+1-i}} / c_n \\ a_q \prod_{i=1}^{n} c_i^{t_{n+1-i}} \end{pmatrix} \mapsto \cdots . \tag{20}$$

As an ordinary polynomial map, the degree of the nth iterate is

$$2^n = 1 + (d^c * t)_{n+1}.$$

Subtracting $2^n = 2 + 2(d^c * t)_n$ from this equation and using the recursion for t, we find

$$d_n^c = \begin{cases} 1, & 1 \le n \le s_1, \\ d_{n-s_1}^c + 1, & s_1 < n \le s_2, \\ d_{n-s_1}^c + d_{n-s_2}^c + 1, & s_2 < n \le q, \\ d_{n-s_1}^c + d_{n-s_2}^c - d_{n-q}^c + 1, & q < n, \end{cases}$$

or $d_n^c = d_{n-s_1}^c + d_{n-s_2}^c - d_{n-q}^c + 1$ for all n, taking $d_{n<1}^c = 0$.

Projectively, the last component of the $(n-1)$st iterate is

$$p_n := a_q \prod_{i=\max(1,n-q)}^{n-1} c_i,$$

which has degree

$$d_n^p = 1 + \sum_{i=\max(1,n-q)}^{n-1} d_i^c.$$

We find

$$d_n^p = \begin{cases} n, & 1 \le n \le s_1 + 1, \\ d_{n-s_1}^p + n - 1, & s_1 + 1 < n \le s_2 + 1, \\ d_{n-s_1}^p + d_{n-s_2}^p + n - 2, & s_2 + 1 < n \le q, \\ d_{n-s_1}^p + d_{n-s_2}^p - d_{n-q}^p + q, & q < n. \end{cases} \tag{21}$$

For example, in the case $\mathbf{s} = (2, 3)$ the sequence of degrees

$$1, 2, 3, 5, 8, 12, 16, 22, 28, 35, 43, 52, 61, 72, 83, 95, 108, 122, 136, \ldots$$

is given by

$$d_n^p = \frac{127}{72} + \frac{(-1)^n}{8} - \frac{\zeta^{n-1} + \zeta^{1-n}}{9} - \frac{5}{6}n + \frac{5}{12}n^2, \quad \zeta^3 = 1.$$

In general, the recursion (21) yields asymptotic growth

$$\sim (s_1 + s_2)(2s_1s_2)^{-1}n^2.$$

5. Conclusion

In [18] Viallet discussed two approaches: the *heuristic method*, where no proofs are obtained, and *serious singularity analysis*, which is limited to 2D maps, or some exceptional higher dimensional cases. The question was raised, how can we go further, in particular to high dimensions? The *arithmetical approach* was given as one suggestion, cf. [1,7].

In this paper, we have presented a different approach and showed that it works for high dimensions, at least for (most) mappings obtained as reductions from an integrable lattice equation. The only condition on the dimension is that one has to be able to iterate the q-dimensional map q times to verify Conjecture 1 or 2. The scope of this approach is left open for future research, e.g. to consider other reductions, other lattice equations and non-integrable or almost integrable maps.

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Note

1. The function h is a 2-integral of the mapping. In [12] k-symmetries are used to perform explicit dimensional reduction of mappings related to $(s_1, 1)$ periodic reductions of lattice Korteweg–de Vries. The dimension $s_1 + 1$ is reduced to s_1 or $s_1 - 2$, when s_1 is even or odd, respectively.

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Appendix: Solution of the (3,1)-map

We prove that the recurrences (13,14) yield expressions (15,16), with

$$c_{2n+1} = y(x-w)(xy)^{n-2} - P_n, \quad c_{2n+2} = (z-y)(\alpha - xy)^{n-1} - yP_n, \tag{22}$$

where

$$P_n := \sum_{k=0}^{n-1} T^n_{n-k}(xy)^k \alpha^{n-1-k},$$

with

$$T^{n+1}_{k+1} = T^n_k - T^{n+1}_k, \quad T^n_0 = T^n_n = 1,$$

that is [15, seq. A112468],

$$T^n_k = \sum_{i=k}^{n} (-1)^{n-i} \binom{n+k-i-1}{n-i}.$$

Proof. Substituting (15,16) in (14) yields

$$c_{2n} = (\alpha - xy)c_{2n-2} - y(xy)^{n-2}, \quad c_{2n+1} = xyc_{2n-1} - (\alpha - xy)^{n-1}. \tag{23}$$

Substituting (22) in (23) yields

$$P_n = (\alpha - xy)P_{n-1} + (xy)^{n-1}, \quad P_n = (xy)P_{n-1} + (\alpha - xy)^{n-1},$$

which can be verified using the definition of P and T. Substituting (15,16) in (13) yields

$$(xy)^{i-4}\, y(c_{2i+1} - (\alpha - xy)^{i-2}) = -c_{2i-3}(c_{2i} - (\alpha - xy)c_{2i-2})$$

and

$$(\alpha - xy)^{i-4}(c_{2i} - \alpha(xy)^{i-3}\, y) = -c_{2i-4}(c_{2i-1} - xyc_{2i-3}),$$

which follows as a consequence of (23). □

Remark 1. The expressions for $x_n = a_n/b_n$ can be simplified as follows. Let

$$(x_1, x_2, x_3, x_4) = (x-w, y+z, -w, z), \quad x_{n>4} = x_{n-4} + \frac{\alpha}{x_{n-1} - x_{n-3}}. \tag{24}$$

Then

$$x_{2n+1} = x - w - x\frac{P_n}{(xy)^{n-1}}, \quad x_{2n+2} = y + z - y\frac{P_n}{(\alpha - xy)^{n-1}}. \tag{25}$$